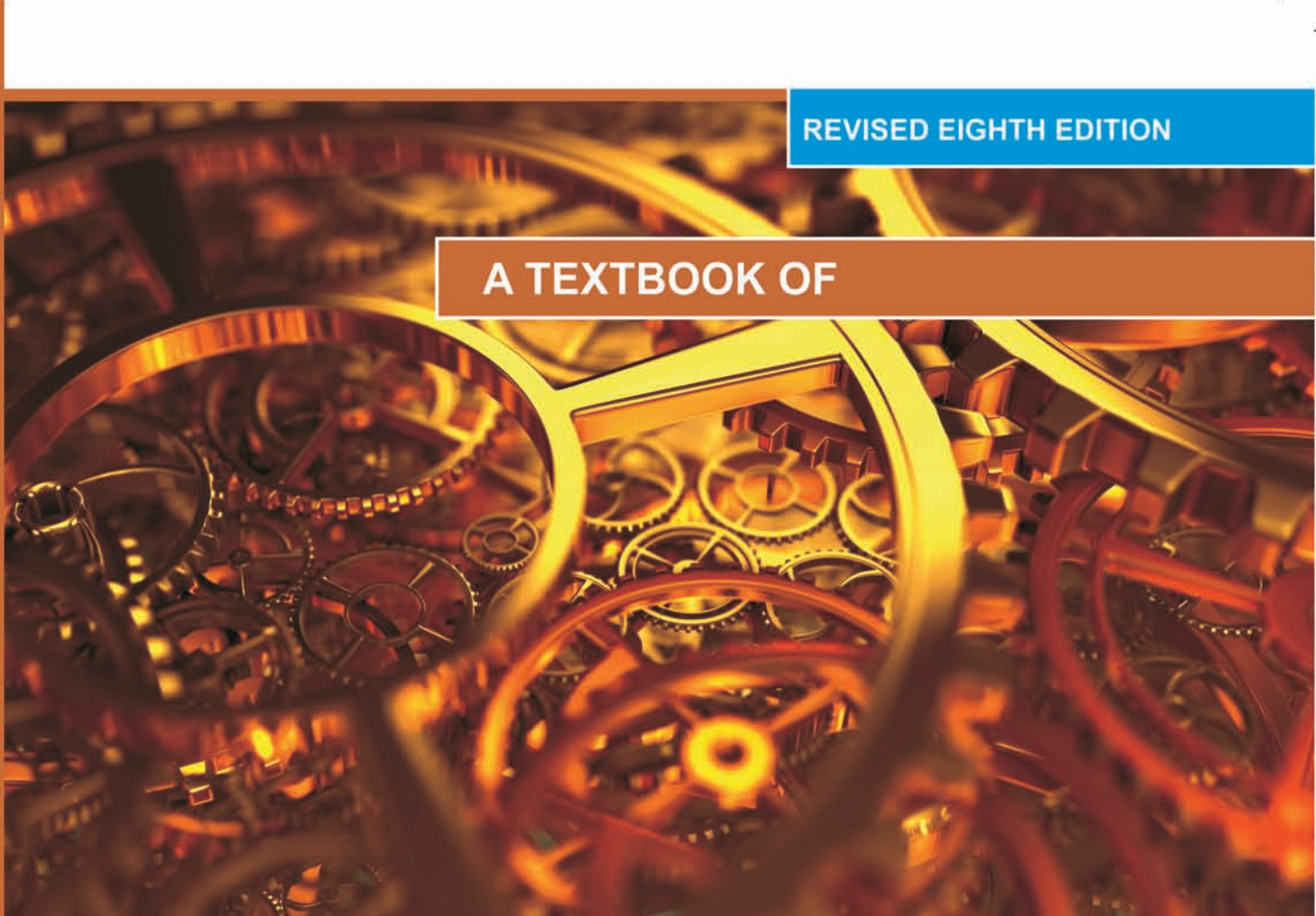




REVISED EIGHTH EDITION

A TEXTBOOK OF

ENGINEERING MECHANICS

Strictly as per the latest syllabus prescribed by
Gautam Buddh Technical University, Lucknow

Dr. R.K. Bansal



A TEXTBOOK OF
ENGINEERING MECHANICS

A TEXTBOOK OF ENGINEERING MECHANICS

(In SI Units)

For

B.E./B.Tech. Ist YEAR

*Strictly as per the latest syllabus prescribed by
Gautam Buddh Technical University, Lucknow*

By

Dr. R.K. BANSAL

B.Sc. Engg. (Mech.), M. Tech., Hons. (I.I.T., Delhi),
Ph.D., M.I.E. (India)

Formerly

*Professor of Mechanical Engineering
Department of Mechanical Engineering
Delhi College of Engineering
Delhi*

AND

SANJAY BANSAL

B.E. (Computer)



LAXMI PUBLICATIONS (P) LTD

(An ISO 9001:2008 Company)

BENGALURU • CHENNAI • GUWAHATI • HYDERABAD • JALANDHAR
KOCHI • KOLKATA • LUCKNOW • MUMBAI • RANCHI • NEW DELHI
BOSTON (USA) • NAIROBI (KENYA)

A TEXTBOOK OF ENGINEERING MECHANICS

Compiled by **Smt. Nirmal Bansal**

Copyright © by the Authors and the Laxmi Publications Pvt. Ltd.

All rights reserved including those of translation into other languages. In accordance with the Copyright (Amendment) Act, 2012, no part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise. Any such act or scanning, uploading, and or electronic sharing of any part of this book without the permission of the publisher constitutes unlawful piracy and theft of the copyright holder's intellectual property. If you would like to use material from the book (other than for review purposes), prior written permission must be obtained from the publishers.

Printed and bound in India

Typeset at Excellent Graphics, Delhi

First Edition : 2011; Reprint : 2012, 2013, 2014, Eight Edition : 2014, 2016

ISBN 978-81-318-0855-9

Limits of Liability/Disclaimer of Warranty: The publisher and the author make no representation or warranties with respect to the accuracy or completeness of the contents of this work and specifically disclaim all warranties. The advice, strategies, and activities contained herein may not be suitable for every situation. In performing activities adult supervision must be sought. Likewise, common sense and care are essential to the conduct of any and all activities, whether described in this book or otherwise. Neither the publisher nor the author shall be liable or assumes any responsibility for any injuries or damages arising herefrom. The fact that an organization or Website if referred to in this work as a citation and/or a potential source of further information does not mean that the author or the publisher endorses the information the organization or Website may provide or recommendations it may make. Further, readers must be aware that the Internet Websites listed in this work may have changed or disappeared between when this work was written and when it is read.

All trademarks, logos or any other mark such as Vibgyor, USP, Amanda, Golden Bells, Firewall Media, Mercury, Trinity, Laxmi appearing in this work are trademarks and intellectual property owned by or licensed to Laxmi Publications, its subsidiaries or affiliates. Notwithstanding this disclaimer, all other names and marks mentioned in this work are the trade names, trademarks or service marks of their respective owners.

PUBLISHED IN INDIA BY



Laxmi Publications (P) Ltd.

(An ISO 9001:2008 Company)

113, GOLDEN HOUSE, DARYAGANJ,

NEW DELHI - 110002, INDIA

Telephone : 91-11-4353 2500, 4353 2501

Fax : 91-11-2325 2572, 4353 2528

www.laxmipublications.com info@laxmipublications.com

Branches		Bengaluru	080-26 75 69 30	
		Chennai	044-24 34 47 26,	24 35 95 07
		Guwahati	0361-254 36 69,	251 38 81
		Hyderabad	040-27 55 53 83,	27 55 53 93
		Jalandhar	0181-222 12 72	
		Kochi	0484-237 70 04,	405 13 03
		Kolkata	033-22 27 43 84	
		Lucknow	0522-220 99 16	
		Mumbai	022-24 93 12 61	
		Ranchi	0651-220 44 64	

C—

Printed at:

Contents

Chapters

Pages

UNIT-I: FORCE SYSTEMS AND FRICTION

1. Basic Concepts of Engineering Mechanics	3–27
1.1. Definitions	...
1.2. Scalar and Vector Quantities	...
1.3. Free, Forced and Fixed Vectors	...
1.4. Law of Parallelogram of Forces	...
1.5. Lami's Theorem	...
1.6. System of Units	...
1.7. Resolution of a Force	...
1.8. Moment of a Force	...
1.9. Laws of Mechanics	...
1.10. The Principle of Transmissibility of Forces	...
<i>Highlights</i>	...
<i>Exercise 1</i>	...
2. Force System and Classification	28–48
2.1. Coplanar Collinear and Concurrent Forces	...
2.2. Classification of a Force System	...
2.3. Resultant of Several Forces	...
2.4. Resultant of Coplanar Forces	...
2.5. Resultant of Collinear Coplanar Forces	...
2.6. Resultant of Concurrent Coplanar Forces	...
2.7. Resultant of Non-concurrent Coplanar Forces	...
<i>Highlights</i>	...
<i>Exercise 2</i>	...
3. Moment of a Force and Varignon's Theorem	49–66
3.1. Parallel Forces	...
3.2. Moment of a Force	...
3.3. Varignon's Theorem (or Principle of Moments)	...
3.4. Types of Parallel Forces	...
3.5. Resultant of Two Parallel Forces	...
3.6. Resolution of a Force into a Force and a Couple	...
3.7. General Case of Parallel Forces in a Plane	...
<i>Highlights</i>	...
<i>Exercise 3</i>	...

4. Equilibrium of Coplanar Force System and Free Body Diagram	67–102
4.1. Equilibrium of Coplanar Force System	... 67
4.2. Action and Reaction	... 78
4.3. Free Body Diagrams	... 79
4.4. Equilibrium of a Body Under Three Forces	... 97
4.5. Lami's Theorem	... 98
<i>Highlights</i>	... 100
<i>Exercise 4</i>	... 100
5. Determination of Support Reactions	103–127
5.1. Introduction	... 103
5.2. Types of Supports	... 103
5.3. Types of Loading	... 104
5.4. Determination of Reactions of a Beam	... 105
<i>Highlights</i>	... 125
<i>Exercise 5</i>	... 125
6. Friction	128–173
6.1. Introduction	... 128
6.2. Co-efficient of Friction (μ)	... 128
6.3. Angle of Friction (ϕ)	... 129
6.4. Cone of Friction	... 130
6.5. Types of Friction	... 130
6.6. Coulomb's Laws of Friction	... 130
6.7. Angle of Repose	... 135
6.8. Equilibrium of a Body Lying on a Rough Inclined Plane	... 136
6.9. Applications of Friction	... 148
6.10. Wedge Friction	... 149
6.11. Belt Friction	... 153
6.12. Brakes and Screws Friction	... 160
6.13. Friction in a Square Threaded Screw	... 164
<i>Highlights</i>	... 171
<i>Exercise 6</i>	... 172

UNIT-II: BASIC STRUCTURAL ANALYSIS

7. Plane Truss	177–214
7.1. Difference between Truss and Frame	... 177
7.2. Perfect and Imperfect Truss	... 178
7.3. Assumptions and Analysis of Plane Truss	... 179
7.4. Method of Joints	... 179
7.5. Method of Sections	... 200

<i>Chapters</i>	<i>Pages</i>
7.6. Zero Force Members	... 207
<i>Highlights</i>	... 208
<i>Exercise 7</i>	... 208
8. Beams (Shear Force and Bending Moment)	215–265
8.1. Types of Beams	... 215
8.2. Types of Load	... 216
8.3. Statically Determinate Beams	... 216
8.4. Shear Force and Bending Moment in Beams	... 217
8.5. Shear Force and Bending Moment Diagrams	... 217
8.6. Shear Force and Bending Moment Diagrams for a Cantilever	... 219
8.7. Shear Force and Bending Moment Diagrams for a Simply Supported Beam...	234
8.8. Shear Force and Bending Moment Diagrams for Overhanging Beams	... 247
8.9. Shear Force and Bending Moment Diagrams for Beams Subjected to Couples	... 255
8.10. Relations between Load, Shear Force and Bending Moment	... 259
8.11. Support Reactions from given Shear Force Diagram or given Bending Moment Diagram	... 260
<i>Highlights</i>	... 262
<i>Exercise 8</i>	... 263
UNIT-III: CENTROID AND MOMENT OF INERTIA	
9. Centroid and Moment of Inertia	269–325
9.1. Centre of Gravity	... 269
9.2. Centre of Mass	... 269
9.3. Centroid or Centre of Gravity of Simple Plane Figures	... 269
9.4. Centroid (or Centre of Gravity) of Areas of Plane Figures by the Method of Moments	... 269
9.5. Centroid of Composite Bodies	... 271
9.6. Determination of Centroid by Integration	... 278
9.7. Centroid of Volume	... 290
9.8. Definition of Moment of Inertia of Area	... 293
9.9. Perpendicular Axis Theorem and Polar Moment of Inertia	... 295
9.10. Parallel Axis Theorem	... 295
9.11. Moment of Inertia of Simple Areas by Integration	... 296
9.12. Moment of Inertia of Composite Areas	... 304
9.13. Polar Moment of Inertia of Masses	... 311
9.14. Parallel Axis Theorem for Mass Moment of Inertia	... 312
9.15. Mass Moment of Inertia of Simple Bodies by Integration	... 312
9.16. Mass Moment of Inertia of Composite Bodies	... 318
<i>Highlights</i>	... 319
<i>Exercise 9</i>	... 321

UNIT-IV: KINEMATICS AND KINETICS OF RIGID BODIES

10. Kinematics of Rigid Bodies	329–394
10.1. Introduction	... 329
10.2. Absolute Motion	... 329
10.3. Plane Rectilinear Motion of Rigid Body	... 329
10.4. Plane Curvilinear Motion of Rigid Body	... 347
10.5. Rotation of Rigid Bodies	... 350
10.6. $x - y$ and $n - t$ Components	... 358
10.7. Relative Motion	... 362
10.8. Plane Motion of Rigid Bodies	... 366
10.9. Instantaneous Centre of Zero Velocity	... 382
<i>Highlights</i>	... 390
<i>Exercise 10</i>	... 391
11. Kinetics of Rigid Bodies	395–468
11.1. Introduction	... 395
11.2. Force, Mass and Acceleration	... 395
11.3. Newton's Laws of Motion	... 396
11.4. D' Alembert's Principle and Dynamic Equilibrium	... 401
11.5. Laws of Motion Applied to Planer Translation	... 407
11.6. Momentum and Angular Momentum (or Moment of Momentum)	... 428
11.7. Laws of Motion Applied to Rotary Motion	... 432
11.8. Laws of Motion Applied to Plane Motion	... 441
11.9. Work and Energy	... 442
11.10. Kinetic Energy	... 448
11.11. Principle of Work and Energy	... 449
11.12. Conservative Forces	... 451
11.13. Law of Conservation of Energy	... 451
11.14. Linear Impulse and Momentum	... 459
11.15. Conservation of Linear Momentum	... 462
<i>Highlights</i>	... 465
<i>Exercise 11</i>	... 466

UNIT-V: MECHANICS OF DEFORMABLE SOLIDS

12. Simple Stresses and Strains	471–497
12.1. Introduction	... 471
12.2. Simple Stress and Strain	... 471
12.3. Normal and Shear Stresses	... 472
12.4. Elasticity and Elastic Limit	... 475
12.5. One Dimensional Loading	... 476

<i>Chapters</i>		<i>Pages</i>
12.6.	Stress on an Inclined Plane	... 481
12.7.	Members of Varying Cross-Sections	... 486
12.8.	Bars in Series	... 491
12.9.	Tensile Test Diagrams for Ductile and Brittle Materials	... 493
	<i>Highlights</i>	... 494
	<i>Exercise 12</i>	... 495
13.	Elastic Constants	498–516
13.1.	Introduction	... 498
13.2.	Longitudinal Strain	... 498
13.3.	Lateral Strain	... 498
13.4.	Poisson's Ratio	... 499
13.5.	Volumetric Strain	... 501
13.6.	Volumetric Strain of a Cylindrical Rod	... 507
13.7.	Bulk Modulus	... 508
13.8.	Expression for Young's Modulus in Terms of Bulk Modulus	... 508
13.9.	Relationship between Modulus of Elasticity and Modulus of Rigidity	... 510
	<i>Highlights</i>	... 513
	<i>Exercise 13</i>	... 514
14.	Strain Energy and Impact Loading	517–526
14.1.	Introduction	... 517
14.2.	Some Definitions	... 517
14.3.	Expression for Strain Energy Stored in a Body when the Load is Applied Gradually	... 517
14.4.	Expression for Strain Energy Stored in a Body when the Load is Applied Suddenly	... 519
14.5.	Expression for Strain Energy Stored in a Body when the Load is Applied with Impact	... 521
	<i>Highlights</i>	... 525
	<i>Exercise 14</i>	... 525
15.	Bending of Beams	527–543
15.1.	Introduction	... 527
15.2.	Theory of Pure Bending	... 527
15.3.	Neutral Surface and Neutral Axis	... 530
15.4.	Stresses in Beams of Different Cross-Sections	... 532
	<i>Highlights</i>	... 541
	<i>Exercise 15</i>	... 542
16.	Theory of Torsion	544–559
16.1.	Torque and Twist	... 544
16.2.	Shear Stress Produced in a Circular Shaft Subjected to Torsion	... 544

<i>Chapters</i>	<i>Pages</i>
16.3. Maximum Torque Transmitted by a Circular Solid Shaft	... 546
16.4. Torque Transmitted by a Hollow Circular Shafts	... 548
16.5. Power Transmitted by Shafts	... 549
<i>Highlights</i>	... 557
<i>Exercise 16</i>	... 558
Objective Type Questions	561–593
Examination Paper with Solutions	595–605

Preface to the Eighth Edition

The course contents of the eighth edition of the book entitled, '**A Textbook of Engineering Mechanics**' are planned in such a way that the book covers the complete course of first year students of all disciplines of Gautam Buddha Technical University, Lucknow (Formerly known as U.P. Technical University, Lucknow) strictly as per the latest syllabus prescribed by G.B. Technical University, Lucknow. This edition has been thoroughly revised and made up-to-date.

The book contains following five units:

- Unit I : Force Systems, Basic Concepts, Resultant of a Force System, Free Body Diagrams, Equilibrium and Equation of Equilibrium and Friction.
- Unit II : Basic Structural Analysis, Plane Truss and Beam (Shear Force and Bending Moment Diagrams).
- Unit III : Centroid and Moment of Inertia.
- Unit IV : Kinematics and Kinetics of Rigid Body.
- Unit V : Mechanics of Deformable Solids, Simple Stress and Strain, Bending of Beams and Theory of Torsion.

The following additions have been made in this edition:

1. Free, Forced and Fixed Vectors. Application of Friction, Wedge, Belt, Brakes and Screws.
2. Zero Force Member and Statically determinate Beams.
3. Mass Moment of Inertia of Composite Bodies.
4. Relative Velocity, Plane Motion of Rigid Bodies and Conservative Forces.
5. Tensile Test Diagrams for Ductile and Brittle Materials.

Also the **latest solved papers of U.P. Technical University** have been added at proper places. These additions makes the book more useful to the students.

The book is written in a simple and easy-to-follow languages, so that even an average student can grasp the subject by self-study. At the end of each chapter, highlights, theoretical questions and many unsolved problems with answers are given for the students to solve them.

Though every care has been taken in checking the manuscripts and proofreading, yet claiming perfection is very difficult. We shall be very grateful to the readers and users of this book for pointing any mistake that might have crept in. Suggestions for improvement are most welcome and would be incorporated in the next edition with a view to make the book more useful.

—AUTHORS

SYLLABUS

B. Tech. Ist Year

NME-102 : ENGINEERING MECHANICS

L T P [310]

UNIT I: FORCE SYSTEMS

9

- Basic concepts: Definitions, Basic assumptions, Scalar and Vector quantities, Free, Forced and fixed vectors.
- Force System; Force, Classification and Representation, Force as a Vector, Composition of forces, Parallelogram Law, Resolution, Principle of Transmissibility of forces.
- Moment of a force, Vector representation, Moment for coplanar force system, Varignon's theorem.
- Couple, Vector representation, Resolution of a force into a force and a couple.
- Force Systems: Coplanar Concurrent Force system and Coplanar Non-concurrent force systems, Resultant of coplanar force system.
- Equilibrium of coplanar force system, Free body diagrams, Determination of reactions, Equilibrium of a body under three forces, Lami's theorem.

Friction

- Introduction, Wet and Dry friction, Theory of Dry friction, Angle of friction, Angle of Repose, Cone of friction, Coulomb's laws of friction, Applications of friction: Wedge, Belt, Brakes and Screws.

UNIT II: BASIC STRUCTURAL ANALYSIS

8

- **Plane Truss:** Difference between truss and frame, Perfect and imperfect truss, Assumptions and Analysis of Plane Truss, Method of joints, Method of section, Zero force members.
- Beams, Types of beams, Statically Determinate Beams, Shear force and bending moment in beams, Shear force and bending moment diagrams, Relationships between load, shear and bending moment.

UNIT III: CENTROID AND MOMENT OF INERTIA

8

- Center of Gravity, Center of Mass and Centroid of curves, areas, volumes, Determination of centroid by integration, Centroid of composite bodies.
- Definition of Moment of inertia of area, Perpendicular axis theorem and Polar moment of Inertia, Parallel axis theorem, Moment of inertia of simple areas by integration, Moment of Inertia of Composite Areas.
- Moment of Inertia of masses, Parallel axis theorem for mass moment of inertia, Mass moment of inertia of simple bodies by integration, Mass moment of inertia of composite bodies.

UNIT IV: KINEMATICS OF RIGID BODY

9

- Introduction, Absolute motion, Plane rectilinear motion of rigid body, Plane curvilinear Motion of rigid body, x - y and n - t components, Rotation of rigid bodies, Relative Motion, Plane Motion of rigid bodies, Instantaneous center of zero velocity

Kinetics of Rigid Body

- Introduction, Force, Mass and Acceleration, Newton's law of motion, D' Alembert's Principles and Dynamic Equilibrium, Laws of motion applied to planar translation, rotation and plane motion.
- Work and Energy, Kinetic energy, Principle of work and energy, Conservative forces, Law of conservation of energy.
- Linear Impulse and Momentum, Conservation of linear momentum.

UNIT V: MECHANICS OF DÉFORMABLE SOLIDS

8

- **Simple Stress and Strain:** Normal and shear stresses. One Dimensional Loading; Stress on an inclined plane, members of varying cross-section, bars in series. Tensile Test diagram for ductile and brittle materials, Elastic constants, Strain energy, resilience.
- **Bending of Beams:** Theory of pure bending, neutral surface and neutral axis, stresses in beams of different cross-sections.
- **Theory of Torsion:** Torque and twist, Shear stress due to torsion circular sections.

UNIT I

FORCE SYSTEMS AND FRICTION

Chapters

1. Basic Concepts of Engineering Mechanics
2. Force System and Classification
3. Moment of a Force and Varignon's Theorem
4. Equilibrium of Coplanar Force System and Free Body Diagram
5. Determination of Support Reactions
6. Friction

Basic Concepts of Engineering Mechanics

1.1. DEFINITIONS

Engineering mechanics is that branch of science which deals with the behaviour of a body when the body is at rest or in motion. The engineering mechanics may be divided into Statics and Dynamics. The branch of science, which deals with the study of a body when the body is at rest, is known as *Statics* while the branch of science which deals with the study of a body when the body is in motion, is known as *Dynamics*. Dynamics is further divided into kinematics and kinetics. The study of a body in motion, when the forces which cause the motion are not considered, is called *kinematics* and if the forces are also considered for the body in motion, that branch of science is called *kinetics*. The classification of Engineering Mechanics are shown in Fig. 1.1 below.

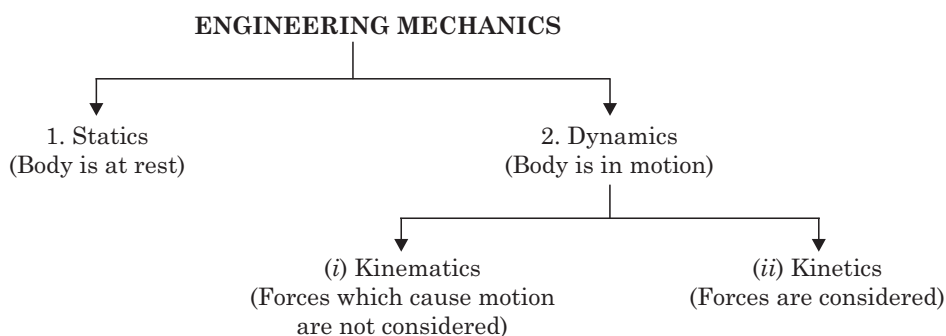


Fig. 1.1

Note. Statics deals with equilibrium of bodies at rest, whereas dynamics deals with the motion of bodies and the forces that cause them.

1.2. SCALAR AND VECTOR QUANTITIES

1.2.1. Vector Quantity. A quantity which is completely specified by magnitude and direction, is known as a vector quantity. Some examples of vector quantities are : velocity, acceleration, force and momentum. A vector quantity is represented by means of a straight line with an arrow as shown in Fig. 1.2. The length of the straight line (*i.e.*, AB) represents the magnitude and arrow represents the direction of the vector. The symbol $\overrightarrow{AB}$ also represents this vector, which means it is acting from A to B .



Fig. 1.2. Vector Quantity.

1.2.2. Scalar Quantity. A quantity, which is completely specified by magnitude only, is known as a scalar quantity. Some examples of scalar quantity are : mass, length, time and temperature (4 kg mass, 3.2 m length, 2 second etc.)

1.2.3. A Particle. A particle is a body of infinitely small volume (or a particle is a body of negligible dimensions) and the mass of the particle is considered to be concentrated at a point. Hence a particle is assumed to a point and the mass of the particle is concentrated at this point.

1.3. FREE, FORCED AND FIXED VECTORS

1.3.1. Free Vector. A vector which can be moved parallel to its position anywhere in space provided its magnitude, direction and sense remain the same, is known as **free vector**. Fig. 1.3 (a) shows free vector.

1.3.2. Fixed Vector. A vector whose initial point is fixed, is known as **fixed vector**. Fig. 1.3 (b) shows fixed vector.

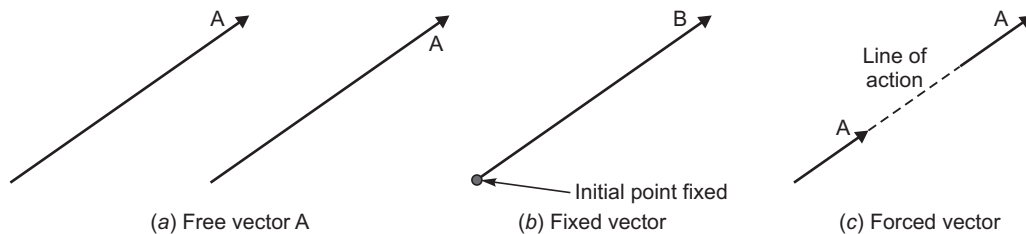


Fig. 1.3

1.3.3. Forced Vector. A vector which can be applied anywhere along its line of action is known as forced vector. Fig. 1.3 (c) shows a forced vector.

1.4. LAW OF PARALLELOGRAM OF FORCES

The law of parallelogram of forces is used to determine the resultant* of two forces acting at a point in a plane. It states, “If two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.”

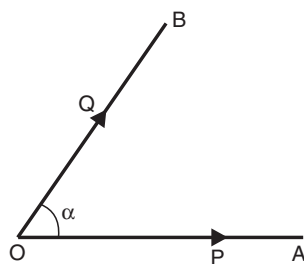


Fig. 1.4

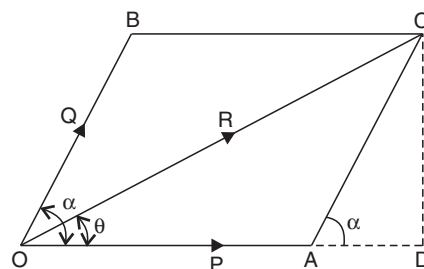


Fig. 1.4 (a)

Let two forces P and Q act at a point O as shown in Fig. 1.4. The force P is represented in magnitude and direction by OA whereas the force Q is presented in magnitude and direction

*The resultant of a system of forces may be defined as a single force which has the same effect as system of forces acting on the body.

by OB . Let the angle between the two forces be ' α '. The resultant of these two forces will be obtained in magnitude and direction by the diagonal (passing through O) of the parallelogram of which OA and OB are two adjacent sides. Hence draw the parallelogram with OA and OB as adjacent sides as shown in Fig. 1.4 (a). The resultant R is represented by OC in magnitude and direction.

1.4.1. Magnitude of Resultant (R)

From C draw CD perpendicular to OA produced.

Let $\alpha = \text{Angle between two forces } P \text{ and } Q = \angle AOB$

Now $\angle DAC = \angle AOB$ (Corresponding angles)
 $= \alpha$

In parallelogram $OACB$, AC is parallel and equal to OB .

$\therefore AC = Q$.

In triangle ACD ,

$$AD = AC \cos \alpha = Q \cos \alpha$$

and

$$CD = AC \sin \alpha = Q \sin \alpha.$$

In triangle OCD ,

$$OC^2 = OD^2 + DC^2.$$

But $OC = R$, $OD = OA + AD = P + Q \cos \alpha$

and

$$DC = Q \sin \alpha.$$

$$\begin{aligned} \therefore R^2 &= (P + Q \cos \alpha)^2 + (Q \sin \alpha)^2 = P^2 + Q^2 \cos^2 \alpha + 2PQ \cos \alpha + Q^2 \sin^2 \alpha \\ &= P^2 + Q^2 (\cos^2 \alpha + \sin^2 \alpha) + 2PQ \cos \alpha \\ &= P^2 + Q^2 + 2PQ \cos \alpha \end{aligned} \quad (\because \cos^2 \alpha + \sin^2 \alpha = 1)$$

$$\therefore R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} \quad \dots(1.1)$$

Equation (1.1) gives the magnitude of resultant force R .

1.4.2. Direction of Resultant

Let $\theta = \text{Angle made by resultant with } OA$.

Then from triangle OCD ,

$$\tan \theta = \frac{CD}{OD} = \frac{Q \sin \alpha}{P + Q \cos \alpha}$$

$$\therefore \theta = \tan^{-1} \left(\frac{Q \sin \alpha}{P + Q \cos \alpha} \right) \quad \dots(1.2)$$

Equation (1.2) gives the direction of resultant (R).

The direction of resultant can also be obtained by using sine rule [In triangle OAC , $OA = P$, $AC = Q$, $OC = R$, angle $OAC = (180 - \alpha)$, angle $ACO = 180 - [\theta + 180 - \alpha] = (\alpha - \theta)$]

$$\begin{aligned} \frac{\sin \theta}{AC} &= \frac{\sin (180 - \alpha)}{OC} = \frac{\sin (\alpha - \theta)}{OA} \\ \frac{\sin \theta}{Q} &= \frac{\sin (180 - \alpha)}{R} = \frac{\sin (\alpha - \theta)}{P} \end{aligned}$$

Two cases are important.

1st Case. If the two forces P and Q act at right angles, then

$$\alpha = 90^\circ$$

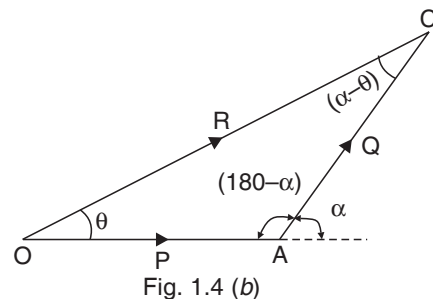


Fig. 1.4 (b)

From equation (1.1), we get the magnitude of resultant as

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} = \sqrt{P^2 + Q^2 + 2PQ \cos 90^\circ} \\ &= \sqrt{P^2 + Q^2} \quad (\because \cos 90^\circ = 0) \quad \dots(1.2 A) \end{aligned}$$

From equation (1.2), the direction of resultant is obtained as

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{Q \sin \alpha}{P + Q \cos \alpha} \right) \\ &= \tan^{-1} \left(\frac{Q \sin 90^\circ}{P + Q \cos 90^\circ} \right) = \tan^{-1} \frac{Q}{P} \quad (\because \sin 90^\circ = 1 \text{ and } \cos 90^\circ = 0) \end{aligned}$$

2nd Case. The two forces P and Q are equal and are acting at an angle α between them. Then the magnitude and direction of resultant is given as

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} = \sqrt{P^2 + P^2 + 2P \times P \cos \alpha} \quad (\because P = Q) \\ &= \sqrt{2P^2 + 2P^2 \cos \alpha} = \sqrt{2P^2(1 + \cos \alpha)} \\ &= \sqrt{2P^2 \times 2 \cos^2 \frac{\alpha}{2}} \quad \left(\because 1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2} \right) \\ &= \sqrt{4P^2 \cos^2 \frac{\alpha}{2}} = 2P \cos \frac{\alpha}{2} \quad \dots(1.3) \end{aligned}$$

and

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{Q \sin \alpha}{P + Q \cos \alpha} \right) = \tan^{-1} \frac{P \sin \alpha}{P + P \cos \alpha} \quad (\because P = Q) \\ &= \tan^{-1} \frac{P \sin \alpha}{P(1 + \cos \alpha)} = \tan^{-1} \frac{\sin \alpha}{1 + \cos \alpha} \\ &= \tan^{-1} \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} \quad \left(\because \sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right) \\ &= \tan^{-1} \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \tan^{-1} \left(\tan \frac{\alpha}{2} \right) = \frac{\alpha}{2} \quad \dots(1.4) \end{aligned}$$

It is not necessary that one of two forces, should be along x -axis. The forces P and Q may be in any direction as shown in Fig. 1.5. If the angle between the two forces is ' α ', then their resultant will be given by equation (1.1). The direction of the resultant would be obtained from equation (1.2). But angle θ will be the angle made by resultant with the direction of P .

1.4.3. Law of Triangle of Forces. It states that, "if three forces acting at a point be represented in magnitude and direction by the three sides of a triangle, taken in order, they will be in equilibrium."

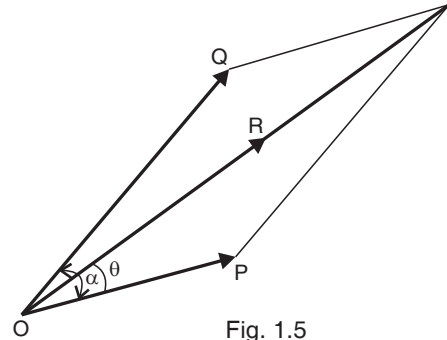


Fig. 1.5

1.5. LAMI'S THEOREM

It states that, "If three forces acting at a point are in equilibrium, each force will be proportional to the sine of the angle between the other two forces."

Suppose the three forces P , Q and R are acting at a point O and they are in equilibrium as shown in Fig. 1.6.

Let α = Angle between force P and Q .

β = Angle between force Q and R .

γ = Angle between force R and P .

Then according to Lami's theorem,

$P \propto \sin$ of angle between Q and $R \propto \sin \beta$.

$$\therefore \frac{P}{\sin \beta} = \text{constant}$$

$$\text{Similarly } \frac{Q}{\sin \gamma} = \text{constant and } \frac{R}{\sin \alpha} = \text{constant}$$

or

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha} \quad \dots(1.5)$$

Proof of Lami's Theorem. The three forces acting on a point, are in equilibrium and hence they can be represented by the three sides of the triangle taken in the same order. Now draw the force triangle as shown in Fig. 1.6 (a).

Now applying sine rule, we get

$$\frac{P}{\sin (180 - \beta)} = \frac{Q}{\sin (180 - \gamma)} = \frac{R}{\sin (180 - \alpha)}$$

This can also be written

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha}$$

This is same equation as equation (1.5).

Note. All the three forces should be acting either towards the point or away from the point.

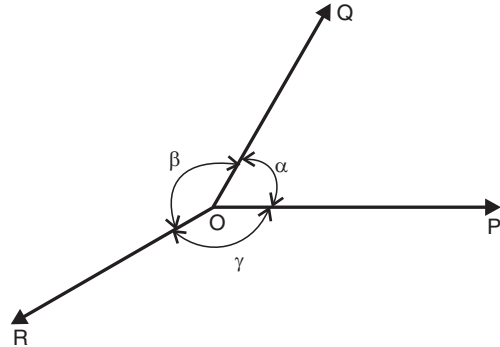


Fig. 1.6

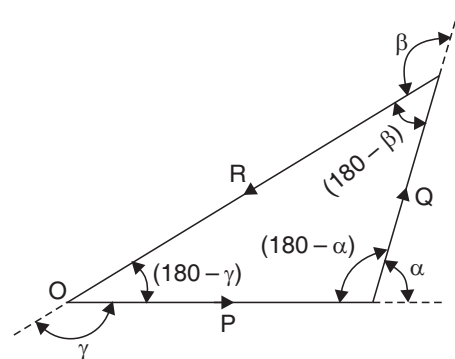


Fig. 1.6 (a)

1.6. SYSTEM OF UNITS

The following system of units are mostly used :

1. C.G.S. (*i.e.*, Centimetre-Gram-Second) system of units.
2. M.K.S. (*i.e.*, Metre-Kilogram-Second) system of units.
3. S.I. (*i.e.*, International) system of units.

1.6.1. C.G.S. System of Units. In this system, length is expressed in centimetre, mass in gram and time in second. The unit of force in this system is *dyne*, which is defined as the force acting on a mass of one gram and producing an acceleration of one centimetre per second square.

1.6.2. M.K.S. System of Units. In this system, length is expressed in metre, mass in kilogram and time in second. The unit of force in this system is expressed as kilogram force and is represented as kgf.

1.6.3. S.I. System of Units. S.I. is abbreviation for 'The System International d'Units'. It is also called the International System of Units. In this system length is expressed in metre mass in kilogram and time in second. The unit of force in this system is Newton and is represented N. Newton is the force acting on a mass of one kilogram and producing an acceleration of one metre per second square. The relation between newton (N) and dyne is obtained as

$$\begin{aligned}
 \text{One Newton} &= \text{One kilogram mass} \times \frac{\text{One metre}}{\text{s}^2} \\
 &= 1000 \text{ gm} \times \frac{100 \text{ cm}}{\text{s}^2} \quad (\because \text{one kg} = 1000 \text{ gm}) \\
 &= 1000 \times 100 \times \frac{\text{gm} \times \text{cm}}{\text{s}^2} \\
 &= 10^5 \text{ dyne} \quad \left\{ \because \text{dyne} = \frac{\text{gm} \times \text{cm}}{\text{s}^2} \right\}
 \end{aligned}$$

When the magnitude of forces is very large, then the unit of force like kilo-newton and mega-newton is used. Kilo-newton is represented by kN.

$$\begin{aligned}
 \text{One kilo-newton} &= 10^3 \text{ newton} \\
 \text{or} \quad 1 \text{ kN} &= 10^3 \text{ N} \\
 \text{and} \quad \text{One mega newton} &= 10^6 \text{ Newton}
 \end{aligned}$$

The large quantities are represented by kilo, mega, giga and tera. They stand for :

$$\begin{aligned}
 \text{Kilo} &= 10^3 \text{ and represented by k} \\
 \text{Mega} &= 10^6 \text{ and represented by M} \\
 \text{Giga} &= 10^9 \text{ and represented by G} \\
 \text{Tera} &= 10^{12} \text{ and represented by T}
 \end{aligned}$$

Thus mega newton means 10^6 newton and is represented by MN. Similarly, giga newton means 10^9 N and is represented by GN. The symbol TN stands for 10^{12} N.

The small quantities are represented by milli, micro, nano and pico. They are equal to

$$\begin{aligned}
 \text{Milli} &= 10^{-3} \text{ and represented by m} \\
 \text{Micro} &= 10^{-6} \text{ and represented by}\mu \\
 \text{Nano} &= 10^{-9} \text{ and represented by n} \\
 \text{Pico} &= 10^{-12} \text{ and represented by p.}
 \end{aligned}$$

Thus milli newton means 10^{-3} newton and is represented by mN. Micro newton means 10^{-6} N and is represented by μN .

Table 1.1 shows the multiples and sub-multiples of the S.I. units prefixes.

Table 1.1. S.I. Prefixes

Prefix	Symbol	Multiplying factor
Tera	T	$10^{12} = 1\,000\,000\,000\,000$
Giga	G	$10^9 = 1\,000\,000\,000$
Mega	M	$10^6 = 1\,000\,000$

Kilo	k	$10^3 =$	1 000
Milli	m	$10^{-3} =$	0.001
Micro	μ	$10^{-6} =$	0.000 001
Nano	n	$10^{-9} =$	0.000 000 001
Pico	p	$10^{-12} =$	0.000 000 000 001

The relation between kilogram force (kgf) and newton (N) is given by one kgf = 9.81 N

Weight of a body is the force with which the body is attracted towards earth. If W = weight of a body, m = mass in kg, then $W = m \times g$ Newtons

If mass, m of the body is 1 kg, then its weight will be,

$$W = 1 \text{ (kg)} \times 9.81 \frac{\text{m}}{\text{s}^2} = 9.81 \text{ N.} \quad \left(\because \text{N} = \text{kg} \frac{\text{m}}{\text{s}^2} \right)$$

1.6.4. Trigonometric Formulae and Expressions. The following are the trigonometric formulae in a right-angled triangle ABC of Fig. 1.7.

$$(i) \sin \theta = \frac{AC}{BC} \quad (ii) \cos \theta = \frac{AB}{BC}$$

$$(iii) \tan \theta = \frac{AC}{AB}$$

$$(iv) \sin (A + B) = \sin A \cos B + \cos A \sin B$$

$$(v) \sin (A - B) = \sin A \cos B - \cos A \sin B$$

$$(vi) \cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$(vii) \cos (A - B) = \cos A \cos B + \sin A \sin B$$

$$(viii) \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$(ix) \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$(x) \sin 2A = 2 \sin A \cos A$$

$$(xi) \sin^2 \theta + \cos^2 \theta = 1.$$

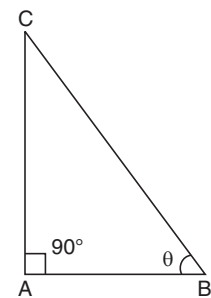


Fig. 1.7

1.6.5. Differentiation and Integration

1. Differentiations. (i) Differentiation of a quantity (say A) with respect to x is written

as $\frac{d}{dx}(A)$ or $\frac{dA}{dx}$

$$(ii) \frac{d}{dx}(x^4) = 4x^3, \quad \frac{d}{dx}(x^n) = n x^{n-1} \quad \text{and} \quad \frac{d}{dx}(x) = 1$$

$$(iii) \frac{d}{dx}(8x + 5)^4 = 4(8x + 5)^3 \times 8$$

$$(iv) \frac{d}{dx}(4) = 0 \text{ as differentiation of constant is zero.}$$

$$(v) \frac{d}{dx}(u.v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \quad [\text{when } u \text{ and } v \text{ are functions of } x]$$

(vi) Differentiation of trigonometrical functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\tan x) = \sec^2 x.$$

2. Integrations. (i) Integration of a quantity (say A) with respect to x is written as $\int A dx$.

$$(ii) \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$(iii) \int 4 dx = 4x$$

$$(iv) \int (8x+5)^4 dx = \frac{(8x+5)^{4+1}}{(4+1) \times 8}.$$

Problem 1.1. Two forces of magnitude 10 N and 8 N are acting at a point. If the angle between the two forces is 60° , determine the magnitude of the resultant force.

Sol. Given :

Force $P = 10$ N

Force $Q = 8$ N

Angle between the two forces, $\alpha = 60^\circ$

The magnitude of the resultant force (R) is given by equation (1.1)

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} = \sqrt{10^2 + 8^2 + 2 \times 10 \times 8 \times \cos 60^\circ} \\ &= \sqrt{100 + 64 + 2 \times 10 \times 8 \times \frac{1}{2}} \quad (\because \cos 60^\circ = \frac{1}{2}) \\ &= \sqrt{100 + 64 + 80} = \sqrt{244} = 15.62 \text{ N. } \quad \text{Ans.} \end{aligned}$$

Problem 1.2. Two equal forces are acting at a point with an angle of 60° between them.

If the resultant force is equal to $20 \times \sqrt{3}$ N, find magnitude of each force.

Sol. Given : Angle between the force, $\alpha = 60^\circ$

Resultant, $R = 20 \times \sqrt{3}$

The forces are equal. Let P is the magnitude of each force.

Using equation (1.3), we have

$$\begin{aligned} R &= 2P \cos \frac{\alpha}{2} \quad \text{or} \quad 20 \times \sqrt{3} = 2P \times \cos \left(\frac{60^\circ}{2} \right) = 2P \cos 30^\circ \\ &= 2P \times \frac{\sqrt{3}}{2} = P \times \sqrt{3} \quad \left(\because \cos 30^\circ = \frac{\sqrt{3}}{2} \right) \end{aligned}$$

$$\therefore P = \frac{20 \times \sqrt{3}}{\sqrt{3}} = 20 \text{ N.}$$

$\therefore$ Magnitude of each force = 20 N. **Ans.**

Problem 1.3. The resultant of the two forces, when they act at an angle of 60° is 14 N. If the same forces are acting at right angles, their resultant is $\sqrt{136}$ N. Determine the magnitude of the two forces.

Sol. Given :

Case I

Resultant, $R_1 = 14$ N

Angle, $\alpha = 60^\circ$

Case II

Resultant, $R_2 = \sqrt{136}$ N

Angle, $\alpha = 90^\circ$

Let the magnitude of the two forces are P and Q .

Using equation (1.1) for case I.

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$$

or $14 = \sqrt{P^2 + Q^2 + 2PQ \times \cos 60^\circ} = \sqrt{P^2 + Q^2 + 2PQ \times \frac{1}{2}}$

or $14 = \sqrt{P^2 + Q^2 + PQ}$

Squaring, $196 = P^2 + Q^2 + PQ$... (i)

Using equation (1.2 A) for case II,

$$R = \sqrt{P^2 + Q^2} \quad \text{or} \quad \sqrt{136} = \sqrt{P^2 + Q^2}$$

or $136 = P^2 + Q^2$ (Squaring both sides) ... (ii)

Subtracting equation (ii) from equation (i), we get

$$196 - 136 = P^2 + Q^2 + PQ - (P^2 + Q^2)$$

or $60 = PQ$... (iii)

Multiplying the above equation by two, we get $120 = 2PQ$... (iv)

Adding equation (iv) to equation (ii), we get $136 + 120 = P^2 + Q^2 + 2PQ$

or $256 = P^2 + Q^2 + 2PQ$ or $(16)^2 = (P + Q)^2$

or $16 = P + Q$

$\therefore P = (16 - Q)$... (v)

Substituting the value of P in equation (iii), we get

$$60 = (16 - Q) \times Q = 16Q - Q^2 \quad \text{or} \quad Q^2 - 16Q + 60 = 0$$

$\therefore$ This is a quadratic equation.

$$\therefore Q = \frac{16 \pm \sqrt{(-16)^2 - 4 \times 60}}{2} = \frac{16 \pm \sqrt{256 - 240}}{2} = \frac{16 \pm 4}{2}$$

$$= \frac{16 + 4}{2} \quad \text{and} \quad \frac{16 - 4}{2} = 10 \quad \text{and} \quad 6.$$

Substituting the value of Q in equation (v), we get

$$P = (16 - 10) \quad \text{or} \quad (16 - 6) = 6 \quad \text{or} \quad 10.$$

$\therefore$ Hence the two forces are **10 N** and **6 N**. **Ans.**

Problem 1.4. Two forces are acting at a point O as shown in Fig. 1.8. Determine the resultant in magnitude and direction.

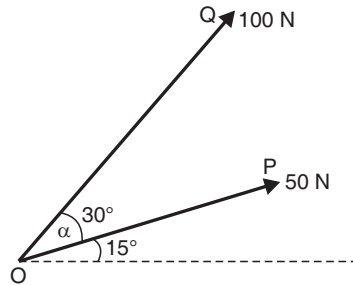


Fig. 1.8

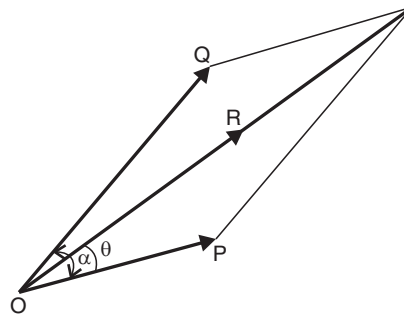


Fig. 1.9

Sol. Given :

Force $P = 50 \text{ N}$, Force $Q = 100 \text{ N}$

Angle between the two forces, $\alpha = 30^\circ$

The magnitude of the resultant R is given by equation (1.1) as

$$\begin{aligned} R &= \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} = \sqrt{50^2 + 100^2 + 2 \times 50 \times 100 \times \cos 30^\circ} \\ &= \sqrt{2500 + 10000 + 8660} = \sqrt{21160} = 145.46 \text{ N. Ans.} \end{aligned}$$

The resultant R is shown in Fig. 1.9.

The angle made by the resultant with the direction of P is given by equation (1.2) as

$$\tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}$$

or

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{Q \sin \alpha}{P + Q \cos \alpha} \right) = \tan^{-1} \left(\frac{100 \times \sin 30^\circ}{50 + 100 \cos 30^\circ} \right) \\ &= \tan^{-1} 0.366 = 20.10^\circ \end{aligned}$$

$\therefore$ Angle made by resultant with x -axis $= \theta + 15^\circ = 20.10 + 15 = 35.10^\circ$. **Ans.**

Problem 1.5. The resultant of two concurrent forces is 1500 N and the angle between the forces is 90° . The resultant makes an angle of 36° with one of the force. Find the magnitude of each force.

Sol. Given :

Resultant, $R = 1500 \text{ N}$

Angle between the forces, $\alpha = 90^\circ$

Angle made by resultant with one force, $\theta = 36^\circ$

Let P and Q are two forces.

Using equation (1.2), $\tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}$

$$\text{or} \quad \tan 36^\circ = \frac{Q \sin 90^\circ}{P + Q \cos 90^\circ} = \frac{Q \times 1}{P + Q \times 0} = \frac{Q}{P} \quad \text{or} \quad 0.726 = \frac{Q}{P}$$

$$\text{or} \quad Q = 0.726 P \quad \dots(i)$$

Using equation (1.1), $R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$

$$\text{or} \quad R^2 = P^2 + Q^2 + 2PQ \cos \alpha$$

$$\text{or} \quad 1500^2 = P^2 + (0.726P)^2 + 2P(0.726P) \times \cos 90^\circ \quad (\because Q = 0.726P)$$

$$1500^2 = P^2 + 0.527P^2 + 0 \quad (\because \cos 90^\circ = 0)$$

$$= 1.527 P^2$$

$$\therefore P = \sqrt{\frac{1500^2}{1.527}} = \frac{1500}{1.2357} = 1213.86 \text{ N}$$

Substituting the value of P in equation (i), we get

$$Q = 0.726 \times 1213.86 = \mathbf{881.26 \text{ N. Ans.}}$$

Alternate Method. Refer to Fig. 1.9 (a). Consider triangle OAC .

Using sine rule, we get

$$\frac{\sin 90^\circ}{R} = \frac{\sin 36^\circ}{Q} = \frac{\sin 54^\circ}{P}$$

$$\text{or } \frac{\sin 90^\circ}{R} = \frac{\sin 36^\circ}{Q}$$

$$\text{or } Q = \frac{R \sin 36^\circ}{\sin 90^\circ} \quad (\text{where } R = 1500 \text{ N})$$

$$= \frac{1500 \times 0.5877}{1} = \mathbf{881.67 \text{ N. Ans.}}$$

$$\text{Also, we have } \frac{\sin 90^\circ}{R} = \frac{\sin 54^\circ}{P}$$

$$\therefore P = \frac{R \sin 54^\circ}{\sin 90^\circ} = \frac{1500 \times 0.8090}{1}$$

$$= \mathbf{1213.52 \text{ N. Ans.}}$$

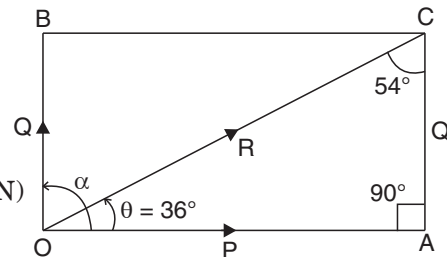


Fig. 1.9(a)

Problem 1.6. The sum of two concurrent forces P and Q is 270 N and their resultant is 180 N . The angle between the force P and resultant R is 90° . Find the magnitude of each force and angle between them.

Sol. Given :

Sum of two concurrent forces = 270 N or $P + Q = 270 \text{ N}$

Resultant, $R = 180 \text{ N}$

Angle between force P and resultant $R = 90^\circ$

This means $\theta = 90^\circ$

Find : (i) Magnitude of P and Q

(ii) Angle between P and Q (i.e., angle α)

$$\text{Using equation (1.2), } \tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}$$

$$\text{or } \tan 90^\circ = \frac{Q \sin \alpha}{P + Q \cos \alpha}$$

But $\tan 90^\circ = \infty$ (i.e., infinity). This is only possible when $P + Q \cos \alpha = 0$

$$\therefore P = -Q \cos \alpha \quad \dots(i)$$

The above result can also be obtained by using alternate method.

Alternate Method. Refer to Fig. 1.9 (b). Consider triangle OAC in which $\theta = 90^\circ$, $\angle OAC = 180 - \alpha$, $\angle ACO = \alpha - \theta = \alpha - 90^\circ$

$$\text{Using sine rule, we get } \frac{\sin 90^\circ}{Q} = \frac{\sin (180 - \alpha)}{R} = \frac{\sin (\alpha - 90^\circ)}{P}$$

From first and last terms, we get $\frac{\sin 90^\circ}{Q} = \frac{\sin (\alpha - 90)}{P}$

or $\frac{1}{Q} = \frac{-\cos \alpha}{P}$ $[\because \sin (\alpha - 90) = \sin [-(90 - \alpha)] = -\sin (90 - \alpha) = -\cos \alpha]$

$\therefore P = -Q \cos \alpha$

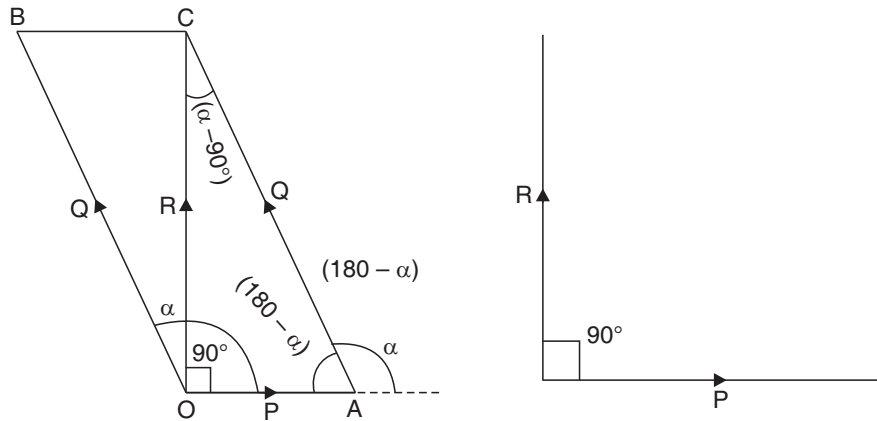


Fig. 1.9(b)

This is the same result as given by equation (i) above.

Using equation (1.1), $R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$

Squaring to both sides, we get $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$
 $= P^2 + Q^2 + 2P(-P)$

$(\because \text{From equation (i), } Q \cos \alpha = -P)$

$= P^2 + Q^2 - 2P^2 = Q^2 - P^2 = (Q + P)(Q - P)$

or $180^2 = 270(Q - P)$ $(\because R = 180, Q + P = 270)$

$32400 = 270(Q - P)$

$\therefore Q - P = \frac{32400}{270} = 120$

But $P + Q = 270$ (given)

Adding the above two equations, we get $2Q = 270 + 120 = 390$

$\therefore Q = 195 \text{ N. Ans.}$

and $P = 270 - Q = 270 - 195 = 75 \text{ N. Ans.}$

Value of angle α

Substituting the values of P and Q in equation (i),

$$P = -Q \cos \alpha \quad \text{or} \quad 75 = -195 \cos \alpha \quad \text{or} \quad \cos \alpha = \frac{-75}{195} = -0.3846$$

$\therefore \alpha = \cos^{-1}(-0.3846) = 112.618^\circ. \text{ Ans.}$

Problem 1.7. A weight of 1000 N is supported by two chains as shown in Fig. 1.10. Determine the tension in each chain.

Sol. Given : Weight at C = 1000 N

$$\angle CAB = 30^\circ$$

$$\angle CBA = 60^\circ$$

$$\angle ACB = 90^\circ$$

In right-angled triangle ADC ,

$$\angle ACD = 90^\circ - 30^\circ = 60^\circ$$

In right-angled triangle BDC ,

$$\angle BCD = 90^\circ - 60^\circ = 30^\circ$$

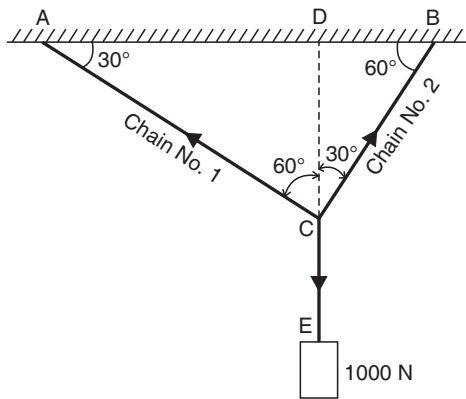


Fig. 1.10

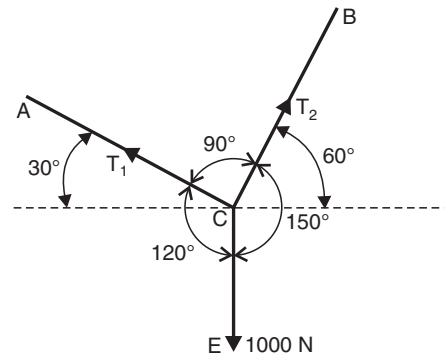


Fig. 1.11

$$\therefore \angle ACE = 180^\circ - 60^\circ = 120^\circ$$

$$\angle BCE = 180^\circ - 30^\circ = 150^\circ$$

Let T_1 = Tension in chain No. 1

T_2 = Tension in chain No. 2.

Applying Lami's theorem at point C (Refer Fig. 1.11).

$$\frac{T_1}{\sin 150^\circ} = \frac{T_2}{\sin 120^\circ} = \frac{1000}{\sin 90^\circ}$$

$$\text{or} \quad \frac{T_1}{\sin 150^\circ} = \frac{T_2}{\sin 120^\circ} = 1000 \quad (\because \sin 90^\circ = 1)$$

$$\therefore T_1 = 1000 \sin 150^\circ = 1000 \times .5 = \mathbf{500 \text{ N. Ans.}}$$

$$\text{and} \quad T_2 = 1000 \sin 120^\circ = 1000 \times .866 = \mathbf{866 \text{ N. Ans.}}$$

Problem 1.8. A weight of 900 N is supported by two chains of lengths 4 m and 3 m as shown in Fig. 1.12. Determine the tension in each chain.

Sol. Given : Weight at C = 900 N

Length, $AC = 4 \text{ m}$

Length, $BC = 3 \text{ m}$

Length, $AB = 5 \text{ m}$

In triangle ABC ,

$$AC^2 + BC^2 = 4^2 + 3^2 = 16 + 9 = 25$$

$$AB^2 = 5^2 = 25$$

$$\therefore AB^2 = AC^2 + BC^2$$

$\therefore$ Triangle ABC is a right-angled triangle in

which

$$\angle ACB = 90^\circ$$

$$\sin \alpha = \frac{BC}{AB} = \frac{3}{5} = 0.6$$

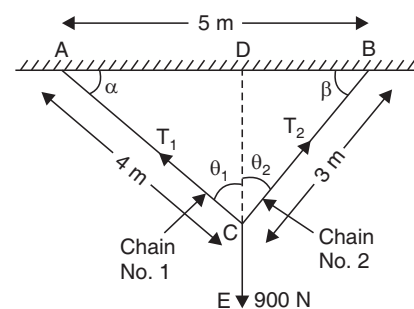


Fig. 1.12

$$\therefore \alpha = 36^\circ 52' \quad \text{and} \quad \alpha + \beta = 90^\circ$$

$$\therefore \beta = 90^\circ - \alpha = 90^\circ - (36^\circ 52') = 53^\circ 8'.$$

Let T_1 = Tension in chain AC
 T_2 = Tension in chain BC

In right-angled triangle ADC,

$$\theta_1 = 90^\circ - \alpha = 90^\circ - 36^\circ 52' = 53^\circ 8'$$

In right-angled triangle BDC,

$$\theta_1 = 90^\circ - \beta = 90^\circ - 53^\circ 8' = 36^\circ 52'$$

$$\therefore \angle ACE = 180^\circ - \theta_1 = 180^\circ - 53^\circ 8' = 126^\circ 52'$$

$$\angle BCE = 180^\circ - \theta_2 = 180^\circ - 36^\circ 52' = 143^\circ 8' \quad \text{and} \quad \angle ACB = 90^\circ.$$

Applying Lami's theorem at C

$$\frac{T_1}{\sin \text{ of } \angle BCE} = \frac{T_2}{\sin \text{ of } \angle ACE} = \frac{900}{\sin 90^\circ}$$

$$\text{or} \quad \frac{T_1}{\sin 143^\circ 8'} = \frac{T_2}{\sin 126^\circ 52'} = \frac{900}{1} \quad (\because \sin 90^\circ = 1)$$

$$\therefore T_1 = 900 \times \sin 143^\circ 8' = 900 \times .597 = 537.44 \text{ N. Ans.}$$

$$T_2 = 900 \times \sin 126^\circ 52' = 720 \text{ N. Ans.}$$

Problem 1.9. An electric light fixture weighing 15 N hangs from a point C, by two strings AC and BC. AC is inclined at 60° to the horizontal and BC at 45° to the vertical as shown in Fig. 1.13. Using Lami's theorem or otherwise determine the forces in the strings AC and BC. (AMIE Summer, 1975)

Sol. Given : Weight at C = 15 N

$$\angle OAC = 60^\circ$$

$$\angle CBD = 45^\circ$$

Let T_1 = Force in string BC
 T_2 = Force in string AC

1st Method

Using Lami's theorem at C

$$\frac{15}{\sin \text{ of } \angle BCA} = \frac{T_1}{\sin \text{ of } \angle ACE} = \frac{T_2}{\sin \text{ of } \angle BCE}$$

$$\text{But} \quad \angle BCA = 45^\circ + 30^\circ = 75^\circ$$

$$\angle ACE = 180^\circ - 30^\circ = 150^\circ$$

$$\angle BCE = 180^\circ - 45^\circ = 135^\circ$$

$$\therefore \frac{15}{\sin 75^\circ} = \frac{T_1}{\sin 150^\circ} = \frac{T_2}{\sin 135^\circ}$$

$$\therefore T_1 = \frac{15 \times \sin 150^\circ}{\sin 75^\circ} = 7.76 \text{ N. Ans.}$$

$$\text{and} \quad T_2 = \frac{15 \times \sin 135^\circ}{\sin 75^\circ} = 10.98 \text{ N. Ans.}$$

2nd Method

The point C is in the equilibrium. The forces acting at C are 15 N, T_1 and T_2 .

Resolving all forces at C in the horizontal direction

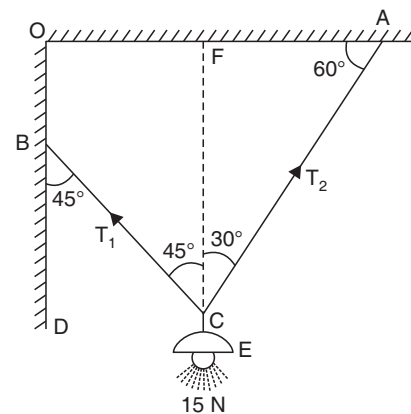


Fig. 1.13

$$T_1 \sin 45^\circ = T_2 \sin 30^\circ \quad \text{or} \quad T_1 \times \frac{1}{\sqrt{2}} = T_2 \times \frac{1}{2}$$

$$\therefore T_2 = T_1 \times \frac{2}{\sqrt{2}} = \sqrt{2} \times T_1 \quad \dots(i)$$

Resolving all forces at C in the vertical direction,

$$T_1 \cos 45^\circ + T_2 \cos 30^\circ = 15$$

$$\text{or} \quad T_1 \times \frac{1}{\sqrt{2}} + T_2 \times \frac{\sqrt{3}}{2} = 15 \quad \dots(ii)$$

Substituting the value of T_2 from equation (i) into equation (ii),

$$T_1 \times \frac{1}{\sqrt{2}} + \sqrt{2} \times T_1 \times \frac{\sqrt{3}}{2} = 15$$

$$\text{or} \quad \frac{T_1}{\sqrt{2}} + \frac{\sqrt{3}T_1}{\sqrt{2}} = 15 \quad \left(\because \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{1}{\sqrt{2}} \right)$$

$$\text{or} \quad T_1 + \sqrt{3}T_1 = 15 \times \sqrt{2} \quad \text{or} \quad T_1(1 + \sqrt{3}) = 15 \times \sqrt{2}$$

$$\therefore T_1 = \frac{15 \times \sqrt{2}}{1 + \sqrt{3}} = 7.76 \text{ N. Ans.}$$

Substituting this value of T_1 in equation (i), we get

$$T_2 = \sqrt{2} \times T_1 = \sqrt{2} \times 7.76 = 10.98 \text{ N. Ans.}$$

1.7. RESOLUTION OF A FORCE

Resolution of a force means “finding the components of a given force in two given directions.”

Let a given force be R which makes an angle θ with X-axis as shown in Fig. 1.14. It is required to find the components of the force R along X-axis and Y-axis.

Components of R along X-axis = $R \cos \theta$.

Components of R along Y-axis = $R \sin \theta$.

Hence, the resolution of forces is the process of finding components of forces in specified directions.

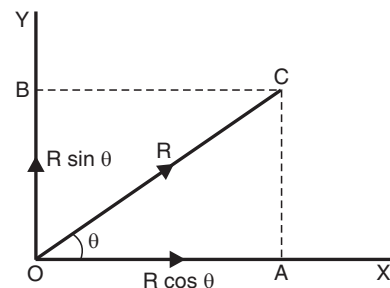


Fig. 1.14

1.7.1. Resolution of a Number of Coplanar Forces. Let a number of coplanar forces (forces acting in one plane are called co-planar forces) $R_1, R_2, R_3, \dots$ are acting at a point as shown in Fig. 1.15.

Let θ_1 = Angle made by R_1 with X-axis

θ_2 = Angle made by R_2 with X-axis

θ_3 = Angle made by R_3 with X-axis

H = Resultant component of all forces along X-axis

V = Resultant component of all forces along Y-axis

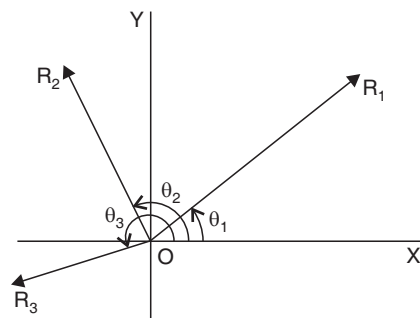


Fig. 1.15

R = Resultant of all forces

θ = Angle made by resultant with X -axis.

Each force can be resolved into two components, one along X -axis and other along Y -axis.

Component of R_1 along X -axis = $R_1 \cos \theta_1$

Component of R_1 along Y -axis = $R_1 \sin \theta_1$.

Similarly, the components of R_2 and R_3 along X -axis and Y -axis are $(R_1 \cos \theta_2, R_2 \sin \theta_2)$ and $(R_3 \cos \theta_3, R_3 \sin \theta_3)$ respectively.

Resultant components along X -axis

= Sum of components of all forces along X -axis.

$$\therefore H = R_1 \cos \theta_1 + R_2 \cos \theta_2 + R_3 \cos \theta_3 + \dots \quad \dots(1.6)$$

Resultant component along Y -axis.

= Sum of components of all forces along Y -axis.

$$\therefore V = R_1 \sin \theta_1 + R_2 \sin \theta_2 + R_3 \sin \theta_3 + \dots \quad \dots(1.7)$$

$$\text{Then resultant of all the forces, } R = \sqrt{H^2 + V^2} \quad \dots(1.8)$$

$$\text{The angle made by } R \text{ with } X\text{-axis is given by, } \tan \theta = \frac{V}{H} \quad \dots(1.9)$$

Problem 1.10. Two forces are acting at a point O as shown in Fig. 1.16. Determine the resultant in magnitude and direction.

Sol. The above problem has been solved earlier. Hence it will be solved by resolution of forces.

Force $P = 50$ N and force $Q = 100$ N.

Let us first find the angles made by each force with X -axis.

Angle made by P with x -axis = 15°

Angle made by Q with x -axis = $15 + 30 = 45^\circ$

Let H = Sum of components of all forces along X -axis.

V = Sum of components of all forces along Y -axis.

The sum of components of all forces along X -axis is given by,

$$H = P \cos 15^\circ + Q \cos 45^\circ \\ = 50 \times \cos 15^\circ + 100 \cos 45^\circ = 119 \text{ N}$$

The sum of components of all forces along Y -axis is given by,

$$V = P \sin 15^\circ + Q \sin 45^\circ \\ = 50 \sin 15^\circ + 100 \sin 45^\circ = 83.64 \text{ N}$$

The magnitude of the resultant force is given by equation (1.8),

$$R = \sqrt{H^2 + V^2} = \sqrt{119^2 + 83.64^2} = 145.46 \text{ N. Ans.}$$

The direction of the resultant force is given by equation (1.9), $\tan \theta = \frac{V}{H} = \frac{83.64}{119}$

$$\therefore \theta = \tan^{-1} \frac{83.64}{119} = 35.10^\circ. \text{ Ans.}$$

Here θ is the angle made by resultant R with x -axis.

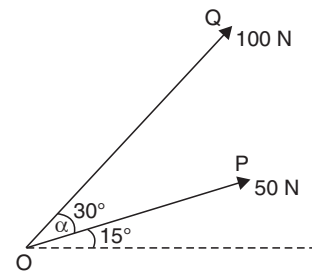


Fig. 1.16

Problem 1.11. Three forces of magnitude 40 kN, 15 kN and 20 kN are acting at a point O as shown in Fig. 1.17. The angles made by 40 kN, 15 kN and 20 kN forces with X-axis are 60° , 120° and 240° respectively. Determine the magnitude and direction of the resultant force.

Sol. Given :

$$R_1 = 40 \text{ kN}, \theta_1 = 60^\circ$$

$$R_2 = 15 \text{ kN}, \theta_2 = 120^\circ$$

$$R_3 = 20 \text{ kN}, \theta_3 = 240^\circ$$

The sum of components of all forces along X-axis is given by equation (1.6) as

$$\begin{aligned} H &= R_1 \cos \theta_1 + R_2 \cos \theta_2 + R_3 \cos \theta_3 \\ &= 40 \times \cos 60^\circ + 15 \times \cos 120^\circ + 20 \times \cos 240^\circ \\ &= 40 \times \frac{1}{2} + 15 \times \left(-\frac{1}{2}\right) + 20 \times \left(-\frac{1}{2}\right) \\ &= 20 - 7.5 - 10 = 2.5 \text{ kN.} \end{aligned}$$

The resultant component along Y-axis is given by equation (1.7) as

$$\begin{aligned} V &= R_1 \sin \theta_1 + R_2 \sin \theta_2 + R_3 \sin \theta_3 \\ &= 40 \times \sin (60^\circ) + 15 \times \sin (120^\circ) + 20 \times \sin (240^\circ) \\ &= 40 + \frac{\sqrt{3}}{3} + 15 \times \frac{\sqrt{3}}{2} + 20 \times \left(\frac{-\sqrt{3}}{2}\right) \\ &= 20 \times \sqrt{3} + 7.5 \times \sqrt{3} - 10 \times \sqrt{3} = 17.5 \times \sqrt{3} \text{ kN} = 30.31 \text{ kN.} \end{aligned}$$

The magnitude of the resultant force is given by equation (1.8)

$$R = \sqrt{H^2 + V^2} = \sqrt{2.5^2 + 30.31^2} = 30.41 \text{ kN. Ans.}$$

The direction of the resultant force is given by equation (1.9)

$$\tan \theta = \frac{V}{H} = \frac{30.31}{2.5} = 12.124 = \tan 85.28^\circ$$

$$\therefore \theta = 85.28^\circ \text{ or } 85^\circ 16.8'. \text{ Ans.}$$

Problem 1.12. Four forces of magnitude 10 kN, 15 kN, 20 kN and 40 kN are acting at a point O as shown in Fig. 1.18. The angles made by 10 kN, 15 kN, 20 kN and 40 kN with X-axis are 30° , 60° , 90° and 120° respectively. Find the magnitude and direction of the resultant force.

Sol. Given :

$$R_1 = 10 \text{ kN and } \theta_1 = 30^\circ$$

$$R_2 = 15 \text{ kN and } \theta_2 = 60^\circ$$

$$R_3 = 20 \text{ kN and } \theta_3 = 90^\circ$$

$$R_4 = 40 \text{ kN and } \theta_4 = 120^\circ$$

The resultant components along X-axis is given by (1.6) as

$$\begin{aligned} H &= R_1 \cos \theta_1 + R_2 \cos \theta_2 + R_3 \cos \theta_3 + R_4 \cos \theta_4 \\ &= 10 \times \cos 30^\circ + 15 \cos 60^\circ + 20 \cos 90^\circ \\ &\quad + 40 \cos 120^\circ \end{aligned}$$

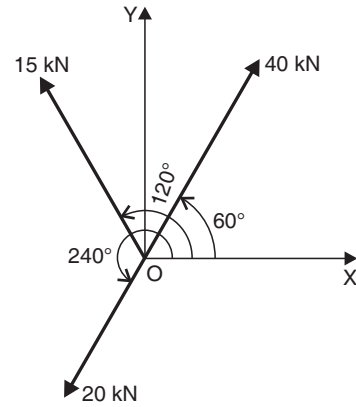


Fig. 1.17

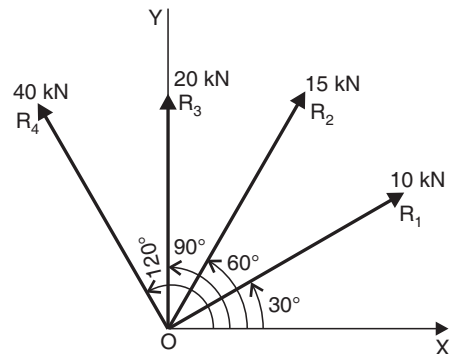


Fig. 1.18

$$\begin{aligned}
 &= 10 \times \frac{\sqrt{3}}{2} + 15 \times \frac{1}{2} + 20 \times 0 + 40 \times \left(-\frac{1}{2}\right) \quad (\because \cos 90^\circ = 0 \text{ and } \cos 120^\circ = -\frac{1}{2}) \\
 &= 5 \times \sqrt{3} + 7.5 - 20 = 8.66 + 7.5 - 20 = -3.84 \text{ kN.}
 \end{aligned}$$

Negative sign means that H is acting along OX' as shown in Fig. 1.19.

The resultant component along Y -axis is given by equation (1.7) as

$$\begin{aligned}
 V &= R_1 \sin \theta_1 + R_2 \sin \theta_2 + R_3 \sin \theta_3 + R_4 \sin \theta_4 \\
 &= 10 \sin 30^\circ + 15 \sin 60^\circ + 20 \sin 90^\circ + 40 \sin 120^\circ \\
 &= 10 \times \frac{1}{2} + 15 \times \frac{\sqrt{3}}{2} + 20 \times 1 + 40 \times \frac{\sqrt{3}}{2} \\
 &= 5 + 7.5 \times \sqrt{3} + 20 + 20 \times \sqrt{3} \\
 &= 25 + 27.5 \times \sqrt{3} = 72.63 \text{ kN.}
 \end{aligned}$$

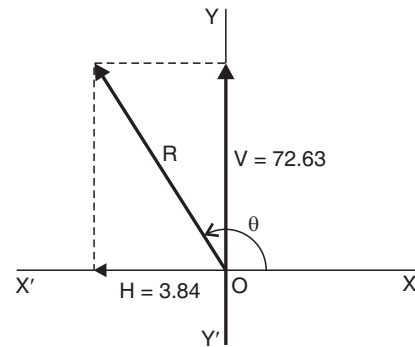


Fig. 1.19

Positive sign means that V is acting along OY as shown in Fig. 1.19.

The magnitude of the resultant force is given by equation (1.8) as

$$\begin{aligned}
 R &= \sqrt{H^2 + V^2} = \sqrt{(-3.84)^2 + 72.63^2} \\
 &= \sqrt{14.745 + 5275.117} = 72.73 \text{ kN. Ans.}
 \end{aligned}$$

The direction of the resultant force is given by equation (1.9) as

$$\tan \theta = \frac{V}{H} = \frac{72.63}{-3.84} = -18.91.$$

From Fig. 1.19 it is clear that θ lies between 90° and 180° .

The angle whose tangent is 18.91° is 86.97° .

$\therefore \theta = (180^\circ - 86.97^\circ) = 93.03^\circ$. Ans.

1.8. MOMENT OF A FORCE

The product of a force and the perpendicular distance of the line of action of the force from a point is known as moment of the force about that point.

P = A force acting on a body as shown in Fig. 1.20.

r = Perpendicular distance between the point O and line of action of the force P .

The moment of the force P about $O = P \times r$

The tendency of the moment $P \times r$ is to rotate the body in the clockwise direction about O .

Hence this moment is called clockwise moment. If the tendency of rotation is anti-clockwise, the moment is called anti-clockwise moments.

1.8.1. Units of Moment. In M.K.S. system the moment is expressed as kgf m whereas in S.I. system, moment is expressed as newton metre (Nm).

1.8.2. Effect of Force and Moment on a Body. The force acting on a body causes linear displacement while moment causes an angular displacement. Hence a body when acted by a number of coplanar forces will be in equilibrium if :

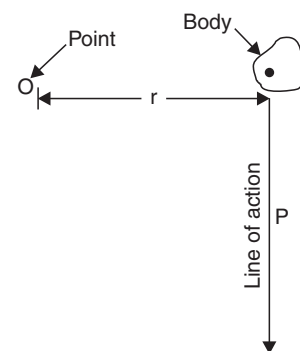


Fig. 1.20

(i) Resultant component of forces along any direction is zero *i.e.*, resultant component of forces in the direction of x , in the direction of y and in the direction of z are zero.

(ii) Resultant moments of the forces about any point in the plane of the forces is zero or clockwise moment is equal to anti-clockwise moments.

Note. If the resultant component of forces acting on a body along any direction is zero but the net moment of the forces about any point is not zero the body will not be in equilibrium. The body will have the tendency to rotate about the point.

Problem 1.13. A beam of span 10 m is carrying a point load of 200 N at a distance 4 m from A. Determine the beam reactions.

Sol. Given :

Span $AB = 10$ m

Load at C, $W = 200$ N

Distance, $AC = 4$ m

Distance, $BC = 10 - 4 = 6$ m

Let $R_A =$ Reaction at A and

$R_B =$ Reaction at B

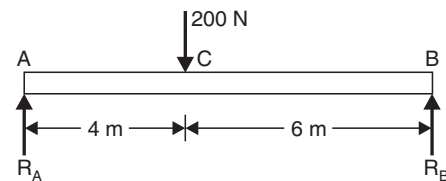


Fig. 1.21

As the beam is in equilibrium, the clockwise moments of all forces about any point must be equal to anti-clockwise moments about that point. Also the resultant force in any direction must be zero.

Taking moments about A,

Clockwise moment = Anti-clockwise moments

$$200 \times 4 = R_B \times 10$$

$$\therefore R_B = \frac{200 \times 4}{10} = 80 \text{ N. Ans.}$$

$$\text{Also } R_A + R_B = 200 \text{ N}$$

$$\therefore R_A = 200 - R_B = 200 - 80 = 120 \text{ N. Ans.}$$

Problem 1.14. Four forces of magnitudes 10 N, 20 N, 30 N and 40 N are acting respectively along the four sides of a square ABCD as shown in Fig. 1.22. Determine the magnitude, direction and position of the resultant force.

Sol. Given :

Force along $AB = 10$ N

Force along $BC = 20$ N

Force along $CD = 30$ N

Force along $DA = 40$ N

(i) *Magnitude and direction of the resultant force*

The net force in the horizontal direction is given as,

$$H = 10 - 30 = -20 \text{ N}$$

The net force in the vertical direction given as,

$$V = 20 - 40 = -20 \text{ N}$$

The resultant force is given by equation (1.8) as

$$R = \sqrt{H^2 + V^2} = \sqrt{(-20)^2 + (-20)^2}$$

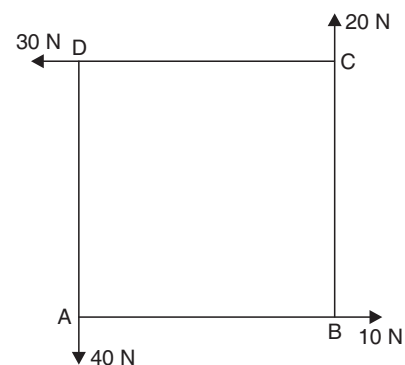


Fig. 1.22

$$= \sqrt{400 + 400} = \sqrt{2 \times 400}$$

$$= 20 \times \sqrt{2} \text{ N. Ans.}$$

The direction of the resultant force is given by equation (1.9) as

$$\tan \theta = \frac{V}{H} = \frac{-20}{-20} = 1$$

$$\therefore \theta = 45^\circ.$$

Since H and V are -ve, hence θ lies between 180° and 270° . Hence from Fig. 1.23, it is clear that actual

$$\theta = 180 + 45 = 225^\circ. \text{ Ans.}$$

(ii) *Position of the resultant force*

The position of the resultant force is obtained by equating the clockwise moments and anti-clockwise moments about A.

Let L = Perpendicular distance between A and line of action of the resultant force and

a = Side of the square ABCD.

Taking moments of all forces about A,

$$40 \times 0 + 10 \times 0 + 20 \times a + 30 \times a$$

$$= R \times \text{perpendicular distance of } R \text{ from A}$$

$$= 20 \times \sqrt{2} \times L \quad (\because R = 20 \times \sqrt{2})$$

$$\text{or} \quad 20a + 30a = 20 \times \sqrt{2} \times L$$

$$\text{or} \quad L = \frac{50a}{20 \times \sqrt{2}} = \frac{5a}{2 \times \sqrt{2}}. \text{ Ans.}$$

The position of the resultant force is shown in Fig. 1.23 (a).

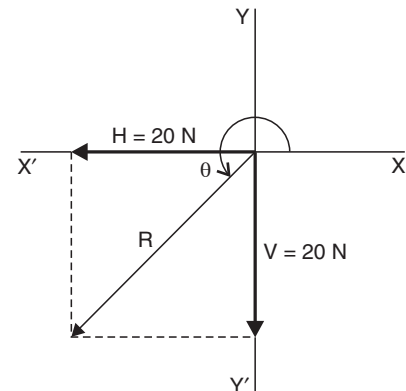


Fig. 1.23

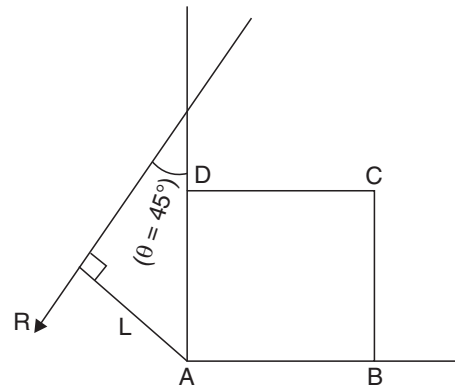


Fig. 1.23(a)

1.9. LAWS OF MECHANICS

The following basic laws and principles are considered to be the foundation of mechanics :

- (i) Newton's first and second laws of motion
- (ii) Newton's third law
- (iii) The gravitational law of attraction
- (iv) The parallelogram law
- (v) The Principle of Transmissibility of forces.

1.9.1. Newton's First and Second Laws of Motion. Newton's first law states, "Every body continues in a state of rest or uniform motion in a straight line unless it is compelled to change that state by some external force acting on it."

Newton's second law states, "The net external force acting on a body in a direction is directly proportional to the rate of change of momentum in that direction."

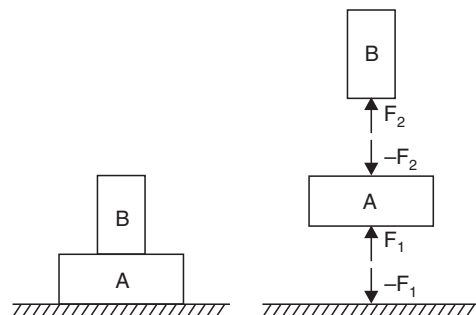


Fig. 1.24

1.9.2. Newton's Third Law. Newton's third law states, "To every action there is always equal and opposite reaction."

Fig. 1.24 shown two bodies A and B which are placed one above the other on a horizontal surface.

Here F_1 = Force exerted by horizontal surface on body A (action)
 $-F_1$ = Force exerted by body A on horizontal surface (reaction)
 F_2 = Force exerted by body A on body B (action)
 $-F_2$ = Force exerted by body B on body A (reaction)

1.9.3. The Gravitational Law of Attraction. It states that two bodies will be attracted towards each other along their connecting line with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.

Refer to Fig. 1.24 (a).

Let m_1 = Mass of first body
 m_2 = Mass of second body
 r = Distance between the centre of bodies
 F = Force of attraction between the bodies.

Then according to the law of gravitational attraction

$$F \propto m_1 \cdot m_2$$

$$\propto \frac{1}{r^2}$$

or
$$F \propto \frac{m_1 m_2}{r^2}$$

or
$$F = G \frac{m_1 m_2}{r^2} \quad \dots(1.10)$$

where G = Universal gravitational constant of proportionality.

The value of G is $6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$ or $\text{m}^3/\text{kg s}^2$.

In equation (1.10), if $m_1 = 1 \text{ kg}$, $m_2 = 1 \text{ kg}$ and $r = 1 \text{ m}$, then $F = G$. This means that the force of attraction between two bodies of mass 1 kg each when they are at a distance of 1 m apart, will be $6.67 \times 10^{-11} \text{ N}$ i.e., $0.0000000000667 \text{ N}$. This force is very very small.

Weight

The weight of a body is defined with the help of law of gravitation. Weight is defined as the **force** with which a body is attracted towards the centre of earth.

Let M = Mass of the body

M_E = Mass of the earth = $5.9761 \times 10^{24} \text{ kg}$

r = Distance between the centres of the earth and the body
 $= 6.371 \times 10^6 \text{ m}$ (i.e., radius of earth)

G = Universal gravitational constant = $6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$

F = Force of attraction which is equal to weight (W)

Substituting these values in equation (1.10), we get

$$W = G \frac{M_E \times M}{r^2} \quad (\because F = W, m_1 = M, m_2 = M_E)$$

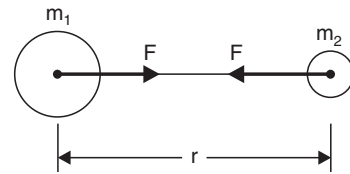


Fig. 1.24(a)

$$\begin{aligned}
 &= \frac{6.67 \times 10^{-11} \left(\frac{\text{Nm}^2}{\text{kg}^2} \right) \times 5.9761 \times 10^{24} (\text{kg}) \times M(\text{kg})}{(6.371 \times 10^6 \text{ m})^2} \\
 &= \frac{6.67 \times 10^{-11} \times 5.9761 \times 10^{24} M}{6.371^2 \times 10^{12} \times \text{m}^2} \frac{\text{Nm}^2}{\text{kg}^2} \times \text{kg} \times \text{kg} = (9.81 \times M) \text{ N}
 \end{aligned}$$

where 9.81 is acceleration due to gravity and is denoted by 'g'.

$$\therefore W = g \times M \quad \text{or} \quad M \times g$$

Actually the term $\frac{GM_E}{r^2}$ is equal to $9.81 \frac{\text{m}}{\text{s}^2}$, which is represented by 'g'.

1.9.4. The Parallelogram Law. This law has been already defined. It states that if two forces acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.

1.10. THE PRINCIPLE OF TRANSMISSIBILITY OF FORCES

It states that if a force, acting at a point on a rigid* body, is shifted to any other point which is on the line of action of the force, the external effect of the force on the body remains unchanged.

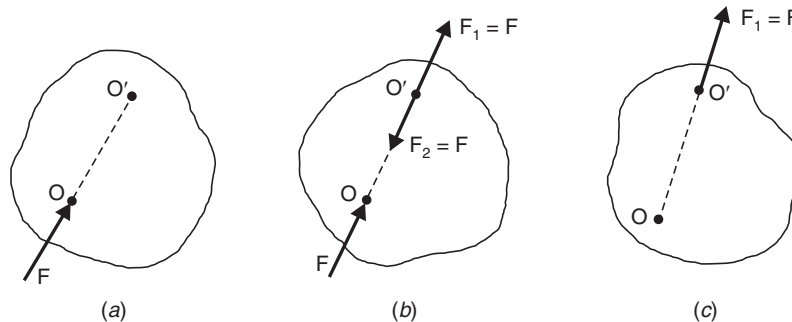


Fig. 1.25

For example, consider a force F acting at point O on a rigid body as shown in Fig. 1.25 (a). On this rigid body, "there is another point O' in the line of action of the force F . Suppose at this point O' , two equal and opposite forces F_1 and F_2 (each equal to F and collinear with F) are applied as shown in Fig. 1.25 (b). The force F and F_2 , being equal and opposite, will cancel each other, leaving a force F_1 at point O' as shown in Fig. 1.25 (c). But force F_1 is equal to force F .

The original force F acting at point O , has been transferred to point O' which is along the line of action of F without changing the effect of the force on the rigid body. Hence any force acting at a point on a rigid body can be transmitted to act at any other point along its line of action without changing its effect on the rigid body. This proves the principle of transmissibility of a force.

*A body which does not deform under the action of loads or external forces is known as rigid body. Hence in case of a rigid body, the relative movement between the various points of the body are negligible or the distance between any two points remains the same for all the times.

HIGHLIGHTS

1. Engineering mechanics is divided into statics and dynamics. The study of a body at rest is known as statics whereas the study of a body in motion is known as dynamics.
2. A quantity which is completely specified by magnitude and direction is known as vector quantity.
3. A particle is a body of infinitely small volume and is considered to be concentrated at a point.
4. Law of parallelogram of forces states that “If two forces, acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram passing through that point.
5. If two forces P and Q act at a point and the angle between the two forces be α , then the resultant is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$$

and the angle made by the resultant with the direction of force P is expressed as

$$\tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}.$$

6. If the two forces P and Q are equal and are acting at an angle α between them, then the resultant is given by

$$R = 2P \cos \frac{\alpha}{2}$$

and angle made by the resultant is expressed as $\theta = \frac{\alpha}{2}$.

7. According to Lami's theorem, “If three forces acting at a point are equilibrium, each force will be proportional to the sine of the angle between the other two forces.”
8. The relation between newton and dyne is given by One newton = 10^5 dyne.
9. Moment of a force about a point = Force $\times$ perpendicular distance of the line of action of the force from that point.
10. The force causes linear displacement while moment causes angular displacement. A body will be in equilibrium if (i) resultant force in any direction is zero and (ii) the net moment of the forces about any point is zero.
11. Gravitational law of attraction is given by,

$$F = G \frac{m_1 \times m_2}{r^2}$$

where G = Universal gravitational constant

m_1, m_2 = Mass of bodies

r = Distance between the bodies

F = Force of attraction between the bodies.

EXERCISE 1

(A) Theoretical Questions

1. What do you mean by scalar and vector quantities ?
2. Define the law of parallelogram of forces. What is the use of this law ?
3. State triangle law of forces and Lami's theorem.
4. Two forces P and Q are acting at a point in a plane. The angle between the forces is ' α '. Prove that the resultant (R) of the two forces is given by $R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$.

5. Define the following terms : dyne, newton, meganewton and moment of a force.
6. Prove that one newton is equal to 10^5 dyne.
7. Explain the terms : clockwise moments and anti-clockwise moments.
8. What is the effect of force and moment on a body ?
9. Indicate whether the following statement is true or false.
 "The resultant components of the forces acting on a body along any direction is zero but the net moment of the forces about any point is not zero, the body will be in equilibrium". [Ans. False]
10. Write the S.I. units of : Force, moment and velocity.
11. What do you mean by resolution of a force ?
12. A number of coplanar forces are acting at a point making different angles with x -axis. Find an expression for the resultant force. Find also the angle made by the resultant force with x -axis.
13. State and explain the principle of transmissibility of forces. What are its applications ?
 (U.P. Tech. University, 2000–2001)
14. State and explain the following laws :
 (i) Newton's laws of motion.
 (ii) The gravitational law of attraction.
15. Choose correct answer for the following part :
 In order to determine the effects of a force acting on a body, we must know :
 (a) Its magnitude (b) Its direction
 (c) Position or line of action (d) all of these. [Ans. (d)]
 (U.P. Tech. University, May 2009)
16. Explain fully the following terms :
 (i) Resolved part of a given force in a given direction.
 (ii) Lami's theorem.

(B) Numerical Problems

1. Determine the magnitude of the resultant of the two forces of magnitude 12 N and 9 N acting at a point, if the angle between the two forces is 30° . [Ans. 20.3 N]
2. Find the magnitude of two equal forces acting at a point with an angle of 60° between them, if the resultant is equal to $30 \times \sqrt{3}$ N. [Ans. 30 N]
3. The resultant of two forces when they act at right angles is 10 N, whereas when they act at an angle of 60° the resultants is $\sqrt{148}$. Determine the magnitude of the two forces.
 [Ans. 8 N and 6 N]
4. Three forces of magnitude 30 kN, 10 kN and 15 kN are acting at a point O . The angles made by 30 kN force, 10 kN force and 15 kN force with x -axis are 60° , 120° and 240° respectively. Determine the magnitude and direction of the resultant force. [Ans. 21.79 kN, $83^\circ 24'$]
5. A weight of 800 N is supported by two chains as shown in Fig. 1.26. Determine the tension in each chain. [Ans. 273.5 N, 751.7 N]
6. An electric light fixture weighing 20 N hangs from a point C , by two strings AC and BC . AC is inclined at 70° to the horizontal and BC at 30° to the vertical as shown in Fig. 1.27. Using Lami's theorem or otherwise determine the forces in the strings AC and BC . [Ans. 8.929 N, 13.05 N]

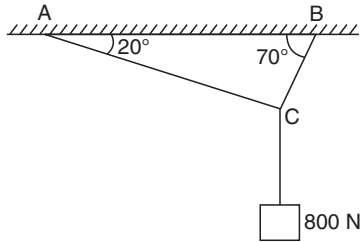


Fig. 1.26

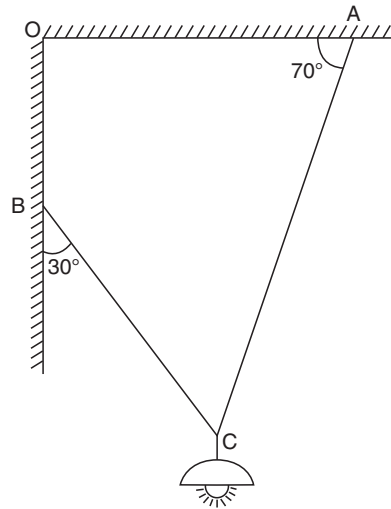


Fig. 1.27

7. A beam AB of span 6 m carries a point load of 100 N at a distance 2 m from A . Determine the beam reaction.
[Ans. $R_A = 66.67$ N ; $R_B = 33.33$ N]
8. Four forces of magnitudes 20 N, 30 N, 40 N and 50 N are acting respectively along the four sides of a square taken in order. Determine the magnitude, direction and position of the resultant force.
[Ans. $20 \times \sqrt{2}$ N, 225° , $\frac{7a}{2 \times \sqrt{2}}$]
9. Two forces magnitude 15 N and 12 N are acting at a point. If the angle between the two forces is 60° , determine the resultant of the forces in magnitude and direction. [Ans. 23.43 N, 26.3°]
10. Four forces of magnitude P , $2P$, $3 \times \sqrt{3}P$ and $4P$ are acting at a point O . The angles made by these forces with x -axis are 0° , 60° , 150° and 300° respectively. Find the magnitude and direction of the resultant force.
[Ans. P , 1200°]

2

Force System and Classification

2.1. COPLANAR COLLINEAR AND CONCURRENT FORCES

Coplanar forces means the forces in a plane. The word collinear stands for the forces which are having common lines of action whereas the word concurrent stands for the forces which intersect at a common point. When several forces act on a body, then they are called *a force system* or *a system of forces*. In a system in which all the forces lie in the same plane, it is known as *coplanar force system*. Hence this chapter deals with a system of forces which are acting in the same plane and the forces are either having a common line of action or intersecting at a common point.

2.2. CLASSIFICATION OF A FORCE SYSTEM

A force system may be coplanar or non-coplanar. If in a system all the forces lie in the same plane then the force system is known as coplanar. But if in a system all the forces lie in different planes, then the force system is known as non-coplanar. Hence a force system is classified as shown in Fig. 2.1.

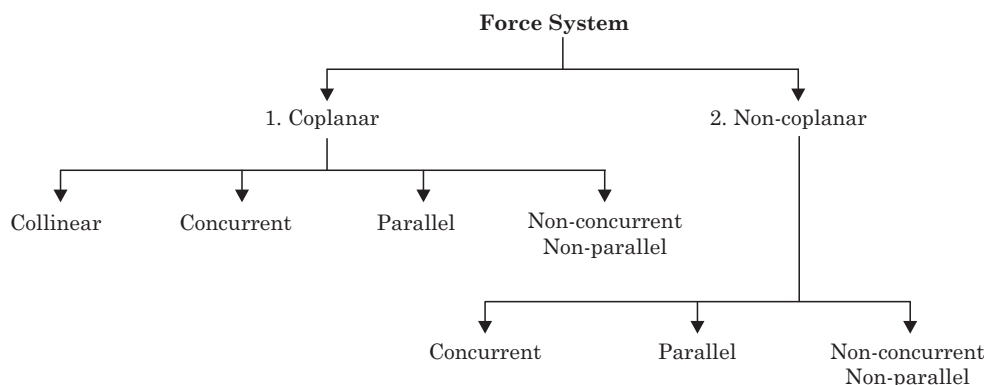


Fig. 2.1

In this chapter, we shall discuss only coplanar force system, in which the forces may be :

- (i) Collinear
- (ii) Concurrent
- (iii) Parallel
- (iv) Non-concurrent, non-parallel (or General system of forces).

2.2.1. Coplanar Collinear. Fig. 2.2 shows three forces F_1 , F_2 and F_3 acting in a plane. These three forces are in the same line, *i.e.*, these three forces are having a common line of action. This system of forces is known as coplanar collinear force system. Hence in coplanar collinear system of forces, all the forces act in the same plane and have a common line of action.

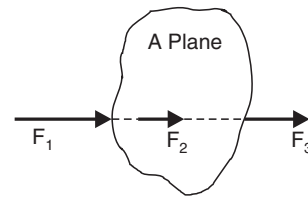


Fig. 2.2. Coplanar Collinear Forces.

2.2.2. Coplanar Concurrent. Fig. 2.3 shows three forces F_1 , F_2 and F_3 acting in a plane and these forces intersect or meet at a common point O . This system of forces is known as coplanar concurrent force system. Hence in coplanar concurrent system of forces, all the forces act in the same plane and they intersect at a common point.

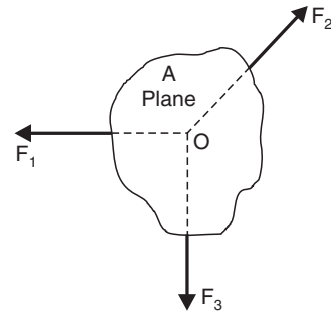


Fig. 2.3. Concurrent Coplanar Forces.

2.2.3. Coplanar Parallel. Fig. 2.4 shows three forces F_1 , F_2 and F_3 acting in a plane and these forces are parallel. This system of forces is known as coplanar parallel force system. Hence in coplanar parallel system of forces, all the forces act in the same plane and are parallel.

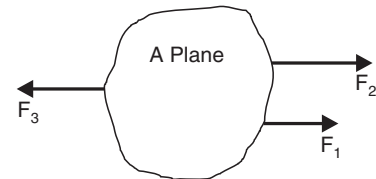


Fig. 2.4. Coplanar Parallel Forces.

2.2.4. Coplanar Non-concurrent Non-parallel. Fig. 2.5 shows four forces F_1 , F_2 , F_3 and F_4 acting in a plane. The lines of action of these forces lie in the same plane but they are neither parallel nor meet or intersect at a common point. This system of forces is known as coplanar non-concurrent non-parallel force system. Hence in coplanar non-concurrent non-parallel system of forces, all the forces act in the same plane but the forces are neither parallel nor meet at a common point. This force system is also known as *general system of forces*.

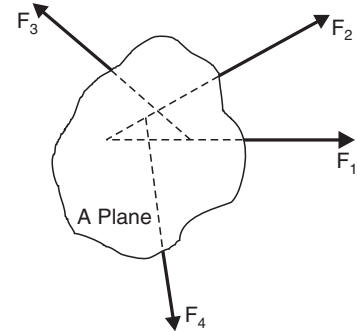


Fig. 2.5. Non-concurrent Non-parallel.

2.3. RESULTANT OF SEVERAL FORCES

When a number of coplanar forces are acting on a rigid* body, then these forces can be replaced by a single force which has the same effect on the rigid body as that of all the forces acting together, then this single force is known as the *resultant* of several forces. Hence a single force which can replace a number of forces acting on a rigid body, without causing any change in the external effects on the body, is known as the *resultant force*.

*Rigid body is a body which does not deform under the action of loads or external forces. In case of rigid body, the distance between any two points of the body remains constant, when this body is subjected to loads. Though all the bodies do deform to some extent under the action of loads, but in many situation, this deformation is negligible small.

2.4. RESULTANT OF COPLANAR FORCES

The resultant of coplanar forces may be determined by following two methods :

1. Graphical method
2. Analytical method.

The resultant of the following coplanar forces will be determined by the above two methods :

- (i) Resultant of collinear coplanar forces
- (ii) Resultant of concurrent coplanar forces.

2.5. RESULTANT OF COLLINEAR COPLANAR FORCES

As defined in Art. 2.2.1, collinear coplanar forces are those forces which act in the same plane and have a common line of action. The resultant of these forces are obtained by analytical method or graphical method.

2.5.1. Analytical Method. The resultant is obtained by adding all the forces if they are acting in the same direction. If any one of the forces is acting in the opposite direction, then resultant is obtained by subtracting that force.

Fig. 2.6 shows three collinear coplanar forces F_1 , F_2 and F_3 acting on a rigid body in the same direction. Their resultant R will be sum of these forces.

$$\therefore R = F_1 + F_2 + F_3 \quad \dots(2.1)$$

If any one of these forces (say force F_2) is acting in the opposite direction, as shown in Fig. 2.7, then their resultant will be given by

$$R = F_1 - F_2 + F_3 \quad \dots(2.2)$$

2.5.2. Graphical Method. Some suitable scale is chosen and vectors are drawn to the chosen scale. These vectors are added/or subtracted to find the resultant. The resultant of the three collinear forces F_1 , F_2 and F_3 acting in the same direction will be obtained by adding all the vectors. In Fig. 2.8, the force $F_1 = ab$ to some scale, force $F_2 = bc$ and force $F_3 = cd$. Then the length ad represents the magnitude of the resultant on the scale chosen.

The resultant of the forces F_1 , F_2 and F_3 acting on a body shown in Fig. 2.7 will be obtained by subtracting the vector F_2 . This resultant is shown in Fig. 2.9, in which the force $F_1 = ab$ to some suitable scale. This force is acting from a to b . The force F_2 is taken equal to bc on the same scale in opposite direction. This force is acting from b to c . The force F_3 is taken equal to cd . This force is acting from c to d . The resultant force is represented in magnitude by ad on the chosen scale.

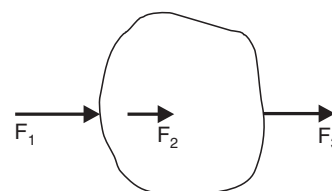


Fig. 2.6

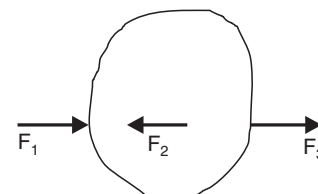


Fig. 2.7

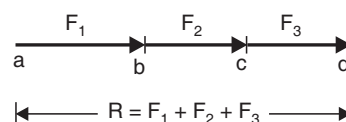


Fig. 2.8

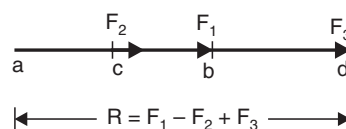


Fig. 2.9

Problem 2.1. Three collinear horizontal forces of magnitude 200 N, 100 N and 300 N are acting on a rigid body. Determine the resultant of the forces analytically and graphically when

- (i) all the forces are acting in the same direction,
- (ii) the force 100 N acts in the opposite direction.

Sol. Given : $F_1 = 200$ N, $F_2 = 100$ N and $F_3 = 300$ N

(a) Analytical method

(i) When all the forces are acting in the same direction, then resultant is given by equation (2.1) as

$$R = F_1 + F_2 + F_3 = 200 + 100 + 300 = \mathbf{600 \text{ N. Ans.}}$$

(ii) When the force 100 N acts in the opposite direction, then resultant is given by equation (2.2) as

$$R = F_1 + F_2 + F_3 = 200 - 100 + 300 = \mathbf{400 \text{ N. Ans.}}$$

(b) Graphical method

Select a suitable scale. Suppose 100 N = 1 cm. Then to this scale, we have

$$F_1 = \frac{200}{100} = 2 \text{ cm,}$$

$$F_2 = \frac{100}{100} = 1 \text{ cm,}$$

and $F_3 = \frac{300}{100} = 3 \text{ cm.}$

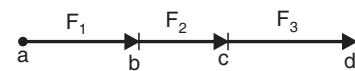


Fig. 2.10

(i) When all the forces act in the same direction.

Draw vectors $ab = 2$ cm to represent F_1 ,

vector $bc = 1$ cm to represent F_2 and

vector $cd = 3$ cm to represent F_3 as shown in Fig. 2.10.

Measure vector ad which represents the resultant.

By measurement, length $ad = 6$ cm

$$\therefore \text{Resultant} = \text{Length } ad \times \text{chosen scale} \quad (\because \text{Chosen scale is } 1 \text{ cm} = 100 \text{ N})$$

$$= 6 \times 100 = \mathbf{600 \text{ N. Ans.}}$$

(ii) When force 100 N = F_2 , acts in the opposite direction

Draw length $ab = 2$ cm to represent force F_1 .

From b , draw $bc = 1$ cm in the opposite direction to represent F_2 . From c , draw $cd = 3$ cm to represent F_3 as shown in Fig. 2.10 (a).



Fig. 2.10 (a)

Measure length ad . This gives the resultant.

By measurement, length $ad = 4$ cm

$$\therefore \text{Resultant} = \text{Length } ad \times \text{chosen scale}$$

$$= 4 \times 100 = \mathbf{400 \text{ N. Ans.}}$$

2.6. RESULTANT OF CONCURRENT COPLANAR FORCES

As defined in Art. 2.2.2, concurrent coplanar forces are those forces which act in the same plane and they intersect or meet at a common point. We will consider the following two cases :

- (i) When two forces act at a point
- (ii) When more than two forces act at a point.

2.6.1. When Two Forces Act at a Point

(a) Analytical method

In Art. 1.4, we have mentioned that when two forces act at a point, their resultant is found by the law of parallelogram of forces. The magnitude of resultant is obtained from equation (1.1) and the direction of resultant with one of the forces is obtained from equation (1.2).

Suppose two forces P and Q act at point O as shown in Fig. 2.11 and α is the angle between them. Let θ is the angle made by the resultant R with the direction of force P .

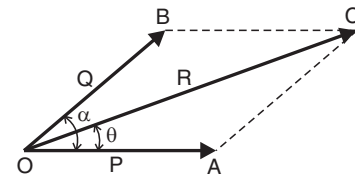


Fig. 2.11

Forces P and Q form two sides of a parallelogram and according to the law, the diagonal through the point O gives the resultant R as shown.

The magnitude* of resultant is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$$

The above method of determining the resultant is also known as the *cosine law method*.

The direction* of the resultant with the force P is given by

$$\theta = \tan^{-1} \left(\frac{Q \sin \alpha}{P + Q \cos \alpha} \right)$$

(b) Graphical method

(i) Choose a convenient scale to represent the forces P and Q .

(ii) From point O , draw a vector $Oa = P$.

(iii) Now from point O , draw another vector $Ob = Q$ and at an angle of α as shown in Fig. 2.12.

(iv) Complete the parallelogram by drawing lines $ac \parallel$ to Ob and $bc \parallel$ to Oa .

(v) Measure the length OC .

Then resultant R will be equal to length $OC \times$ chosen scale.

(vi) Also measure the angle θ , which will give the direction of resultant.

The resultant can also be determine graphically by drawing a triangle oac as explained below and shown in Fig. 2.13.

(i) Draw a line oa parallel to P and equal to P .

(ii) From a , draw a vector ac at an angle α with the horizontal and cut ac equal to Q .

(iii) Join oc . Then oc represents the magnitude and direction of resultant R .

Magnitude of resultant $R = \text{Length } OC \times \text{chosen scale}$. The direction of resultant is given by angle θ . Hence measure the angle θ .

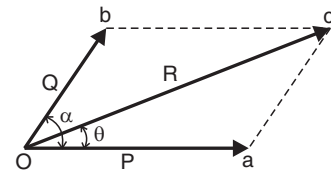


Fig. 2.12

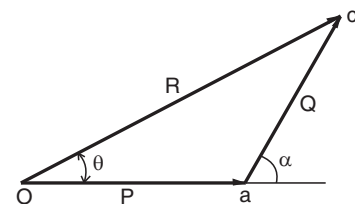


Fig. 2.13

*Refer Art. 1.4, for the derivation of magnitude and direction of resultant on page 4.

2.6.2. When More than Two Forces Act at a Point

(a) Analytical method

The resultant of three or more forces acting at a point is found analytically by a method which is known as rectangular components methods (Refer to Art. 1.7). According to this method all the forces acting at a point are resolved into horizontal and vertical components and then *algebraic summation** of horizontal and vertical components is done separately. The summation of horizontal component is written as ΣH and that of vertical as ΣV . Then resultant R is given by

$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}.$$

The angle made by the resultant with horizontal is given by

$$\tan \theta = \frac{(\Sigma V)}{(\Sigma H)}$$

∴ Let four forces F_1, F_2, F_3 and F_4 act at a point O as shown in Fig. 2.14.

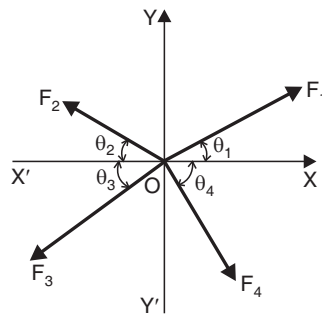


Fig. 2.14

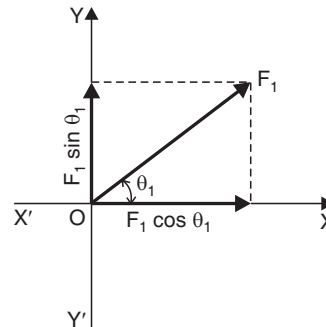


Fig. 2.14 (a)

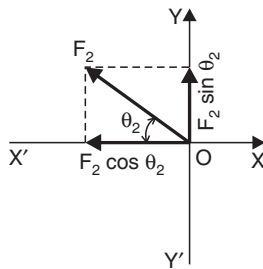


Fig. 2.14 (b)

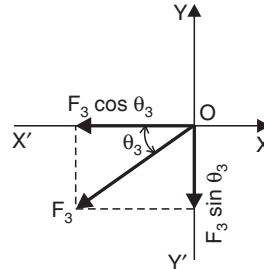


Fig. 2.14 (c)

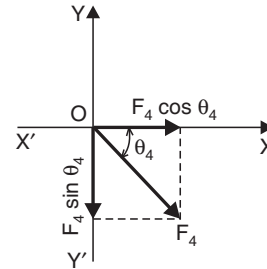


Fig. 2.14 (d)

The inclination of the forces is indicated with respect to horizontal direction. Let

θ_1 = Inclination of force F_1 with OX

θ_2 = Inclination of force F_2 with OX'

θ_3 = Inclination of force F_3 with OX'

θ_4 = Inclination of force F_4 with OX .

*Summation means addition. Algebraic summation of horizontal components means that if all the horizontal components are in the same direction then they are added. But if one horizontal component is in opposite direction then it is subtracted.

The force F_1 is resolved into horizontal and vertical components and these components are shown in Fig. 2.14 (a). Similarly, Figs. 2.14 (b), (c) and (d) shows the horizontal and vertical components of forces F_2 , F_3 and F_4 respectively. The various horizontal components are :

$$F_1 \cos \theta_1 \rightarrow (+)$$

$$F_2 \cos \theta_2 \leftarrow (-)$$

$$F_3 \cos \theta_3 \leftarrow (-)$$

$$F_4 \cos \theta_4 \rightarrow (+)$$

$\therefore$ Summation or algebraic sum of horizontal components :

$$\Sigma H = F_1 \cos \theta_1 - F_2 \cos \theta_2 - F_3 \cos \theta_3 + F_4 \cos \theta_4$$

Similarly, various vertical components of all forces are :

$$F_1 \sin \theta_1 \uparrow (+)$$

$$F_2 \sin \theta_2 \uparrow (+)$$

$$F_3 \sin \theta_3 \downarrow (-)$$

$$F_4 \sin \theta_4 \downarrow (-)$$

$\therefore$ Summation or algebraic sum of vertical components :

$$\Sigma V = F_1 \sin \theta_1 + F_2 \sin \theta_2 - F_3 \sin \theta_3 - F_4 \sin \theta_4$$

Then the resultant will be given by $R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$... (2.3)

And the angle (θ) made by resultant with x-axis is given by $\tan \theta = \frac{(\Sigma V)}{(\Sigma H)}$... (2.4)

(b) *Graphical method*

The resultant of several forces acting at a point is found graphically with the help of the *polygon law of forces*, which may be stated as

“If a number of coplanar forces are acting at a point such that they can be represented in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order.

Let the four forces F_1 , F_2 , F_3 and F_4 act at a point O as shown in Fig. 2.15. The resultant is obtained graphically by drawing polygon of forces as explained below and shown in Fig. 2.15 (a).

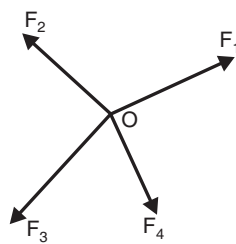


Fig. 2.15

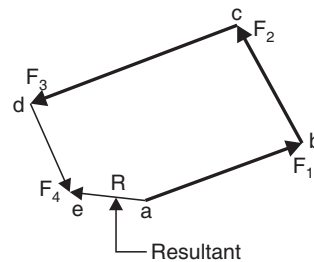


Fig. 2.15 (a)

- (i) Choose a suitable scale to represent the given forces.
- (ii) Take any point a . From a , draw vector ab parallel to OF_1 . Cut ab = force F_1 to the scale.
- (iii) From point b , draw bc parallel to OF_2 . Cut bc = force F_2 .

- (iv) From point C , draw cd parallel to OF_3 . Cut $cd = \text{force } F_3$.
- (v) From point d , draw de parallel to OF_4 . Cut $de = \text{force } F_4$.
- (vi) Join point a to e . This is the closing side of the polygon. Hence ae represents the resultant in magnitude and direction.

Magnitude of resultant $R = \text{Length } ae \times \text{scale}$.

The resultant is acting from a to e .

Problem 2.2. Two forces of magnitude 240 N and 200 N are acting at a point O as shown in Fig. 2.16. If the angle between the forces is 60° , determine the magnitude of the resultant force. Also determine the angle β and γ as shown in the figure.

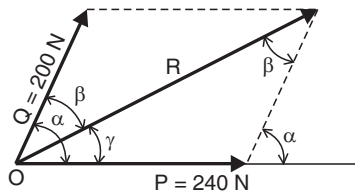


Fig. 2.16

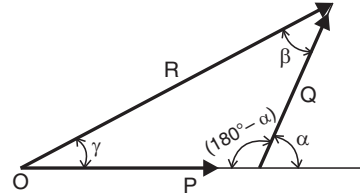


Fig. 2.16(a)

Sol. Given :

Force $P = 240 \text{ N}$, $Q = 200 \text{ N}$

Angle between the forces, $\alpha = 60^\circ$

The magnitude of resultant R is given by,

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} = \sqrt{240^2 + 200^2 + 2 \times 240 \times 200 \times \cos 60^\circ}$$

$$= \sqrt{57600 + 40000 + 48000} = 381.57 \text{ N. Ans.}$$

Now refer to Fig. 2.16 (a). Using sine formula, we get

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin (180^\circ - \alpha)} \quad \dots(i)$$

or

$$\frac{P}{\sin \beta} = \frac{R}{\sin (180^\circ - \alpha)}$$

$$\therefore \sin \beta = \frac{P \sin (180^\circ - \alpha)}{R} = \frac{240 \sin (180 - 60)}{381.57}$$

($\because P = 240 \text{ N}$, $\alpha = 60^\circ$, $R = 381.57 \text{ N}$)

$$= \frac{240 \times \sin 120^\circ}{381.57} = 0.5447$$

$$\therefore \beta = \sin^{-1} 0.5447 = 33^\circ. \text{ Ans.}$$

From equation (i), also we have $\frac{Q}{\sin \gamma} = \frac{R}{\sin (180 - \alpha)}$

$$\therefore \sin \gamma = \frac{Q \sin (180 - \alpha)}{R}$$

$$= \frac{200 \sin (180 - 60)}{381.57} = \frac{200 \sin 120^\circ}{381.57} = 0.4539$$

$$\therefore \gamma = \sin^{-1} 0.4539 = 26.966^\circ. \text{ Ans.}$$

Problem 2.3. Two forces P and Q are acting at a point Q as shown in Fig. 2.17. The resultant force is 400 N and angles β and γ are 35° and 25° respectively. Find the two forces P and Q .

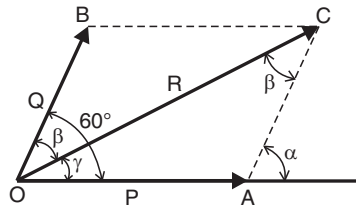


Fig. 2.17

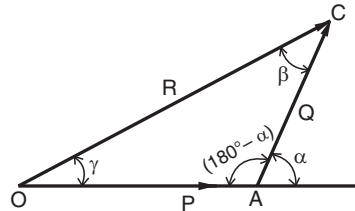


Fig. 2.17(a)

Sol. Given :

Resultant, $R = 400\text{ N}$

Angles, $\beta = 35^\circ$, $\gamma = 25^\circ$

$\therefore$ Angle between the two forces, $\alpha = \beta + \gamma = 35^\circ + 25^\circ = 60^\circ$

Refer to Fig. 2.17 (a). Using sine formula for ΔOAC , we get

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin (180 - \alpha)} \quad \dots(i)$$

$$\frac{P}{\sin \beta} = \frac{R}{\sin (180 - \alpha)}$$

$$\begin{aligned} \therefore P &= \frac{R \sin \beta}{\sin (180 - \alpha)} = \frac{400 \times \sin 35^\circ}{\sin (180 - 60)} \quad (\because R = 400, \beta = 35, \alpha = 60^\circ) \\ &= \frac{400 \times 0.5736}{0.866} = 264.93\text{ N. Ans.} \end{aligned}$$

Also from equation (i), we have $\frac{Q}{\sin \gamma} = \frac{R}{\sin (180 - \alpha)}$

$$\therefore Q = \frac{R \sin \gamma}{\sin (180 - \alpha)} = \frac{400 \times \sin 25^\circ}{\sin (180 - 60)} = \frac{400 \times 0.4226}{0.866} = 195.19\text{ N. Ans.}$$

Problem 2.4. Two forces P and Q are acting at a point O as shown in Fig. 2.18. The force $P = 240\text{ N}$ and force $Q = 200\text{ N}$. If the resultant of the forces is equal to 400 N , then find the values of angles β , γ and α .

Sol. Given :

Forces, $P = 240\text{ N}$, $Q = 200\text{ N}$

Resultant, $R = 400\text{ N}$

Let $\beta =$ Angle between R and Q ,

$\gamma =$ Angle between R and P .

From Fig. 2.18, it is clear that, $\alpha = \beta + \gamma$.

Let us first calculate the angle α (i.e., angle between the two forces).

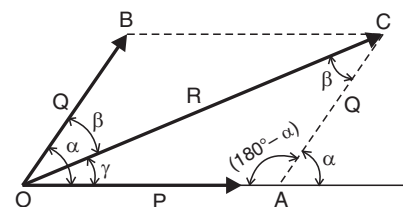


Fig. 2.18

Using the relation,

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha} \quad \text{or} \quad R^2 = P^2 + Q^2 + 2PQ \cos \alpha$$

or $400^2 = 240^2 + 200^2 + 2 \times 240 \times 200 \times \cos \alpha$

or $16000 = 57600 + 40000 + 96000 \times \cos \alpha$

$$\therefore \cos \alpha = \frac{16000 - 57600 - 40000}{96000} = 0.65$$

$$\therefore \alpha = \cos^{-1} 0.65 = 49.458^\circ = 49^\circ (0.458 \times 60') = 49^\circ 27.5'$$

Now using sine formula for $\triangle OAC$ of Fig. 2.18, we get

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin (180 - \alpha)} \quad \dots(i)$$

or $\frac{P}{\sin \beta} = \frac{R}{\sin (180 - \alpha)}$

$$\therefore \sin \beta = \frac{P \sin (180 - \alpha)}{R} = \frac{240 \sin (180 - 49.458)}{400} \quad (\because P = 240, \alpha = 49.458^\circ)$$

$$= \frac{240 \sin (130.542^\circ)}{400} = 0.4559$$

$$\therefore \beta = \sin^{-1} 0.4559 = 27.12^\circ. \quad \text{Ans.}$$

Also from equation (i), we have $\frac{Q}{\sin \gamma} = \frac{R}{\sin (180 - \alpha)}$

$$\therefore \sin \gamma = \frac{Q \sin (180 - \alpha)}{R} = \frac{200 \times \sin (180 - 49.458)}{400}$$

$$= \frac{200 \times \sin (130.542^\circ)}{400} = 0.3799$$

$$\therefore \gamma = \sin^{-1} 0.3799 = 22.33^\circ. \quad \text{Ans.}$$

Problem 2.5. A force of 100 N is acting at a point making an angle of 30° with the horizontal. Determine the components of this force along X and Y directions.

Sol. Given :

Force, $F = 100 \text{ N}$

Angle made by F with horizontal, $\theta = 30^\circ$

Let F_x = Component along x-axis

F_y = Component along y-axis

Then $F_x = F \cos \theta = 100 \cos 30^\circ$

$$= 100 \times 0.866 = 86.6 \text{ N.} \quad \text{Ans.}$$

and $F_y = F \sin \theta = 100 \sin 30^\circ$

$$= 100 \times 0.5 = 50 \text{ N.} \quad \text{Ans.}$$

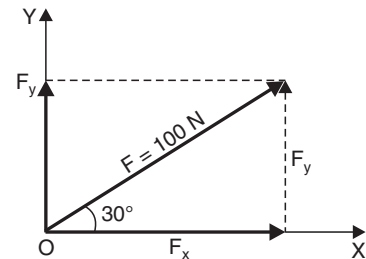


Fig. 2.19

Problem 2.6. A small block of weight 100 N is placed on an inclined plane which makes an angle $\theta = 30^\circ$ with the horizontal. (i) What is the component of this weight ; (ii) parallel to the inclined plane; and (iii) perpendicular to the inclined plane ?

Sol. Given :

Weight of block, $W = 100 \text{ N}$

Inclination of plane, $\theta = 30^\circ$

The weight of block $W = 100 \text{ N}$ is acting vertically downwards through the C.G. of the block. Resolve this weight into two components *i.e.*, one perpendicular to the inclined plane and other parallel to the inclined plane as shown in Fig. 2.20. The perpendicular (normal) component makes an angle of 30° with the direction of W .

Hence component of the weight perpendicular to the inclined plane

$$= W \cos 30^\circ = 100 \times 0.866 = \mathbf{8.66 \text{ N.} \quad \text{Ans.}}$$

Component of the weight (W) parallel to the inclined plane

$$= W \sin 30^\circ = 100 \times 0.5 = \mathbf{50 \text{ N.} \quad \text{Ans.}}$$

Problem 2.7. The four coplanar forces are acting at a point as shown in Fig. 2.21. Determine the resultant in magnitude and direction analytically and graphically.

Sol. Given :

$$\begin{aligned} \text{Forces,} \quad F_1 &= 104 \text{ N,} \\ F_2 &= 156 \text{ N,} \\ F_3 &= 252 \text{ N, and} \\ F_4 &= 228 \text{ N.} \end{aligned}$$

(a) **Analytical method.** Resolve each force along horizontal and vertical axes. The horizontal components along OX will be considered as +ve whereas along OX' as -ve. Similarly, vertical components in upward direction will be +ve whereas in downward direction as -ve.

(i) Consider force $F_1 = 104 \text{ N}$. Horizontal and vertical components are shown in Fig. 2.21 (a).

Horizontal component,

$$\begin{aligned} F_{x_1} &= F_1 \cos 10^\circ = 104 \times 0.9848 \\ &= 102.42 \text{ N} \end{aligned}$$

Vertical component,

$$\begin{aligned} F_{y_1} &= F_1 \sin 10^\circ = 104 \times 0.1736 \\ &= 18.06 \text{ N.} \end{aligned}$$

(ii) Consider force $F_2 = 156 \text{ N}$. Horizontal and vertical components are shown in Fig. 2.21 (b).

Angle made by F_2 with horizontal axis

$$OX' = 90 - 24 = 66^\circ$$

$\therefore$ Horizontal components,

$$\begin{aligned} F_{x_2} &= F_2 \cos 66^\circ = 156 \times 0.4067 \\ &= 63.44 \text{ N.} \end{aligned}$$

It is negative as it is acting along OX' .

Vertical component,

$$\begin{aligned} F_{y_2} &= F_2 \sin 66^\circ = 156 \times 0.9135 \\ &= 142.50 \text{ N.} \end{aligned}$$

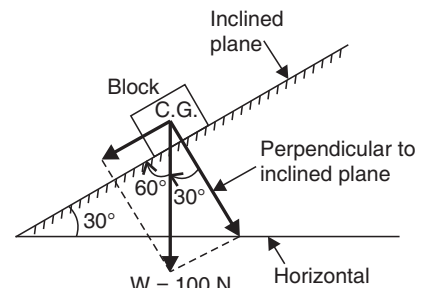


Fig. 2.20

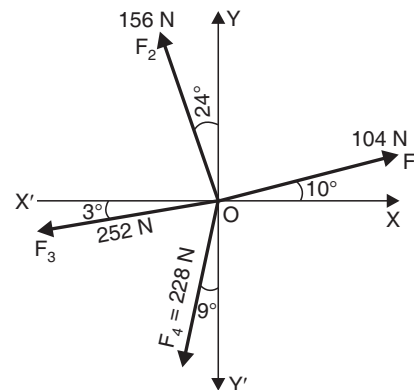


Fig. 2.21

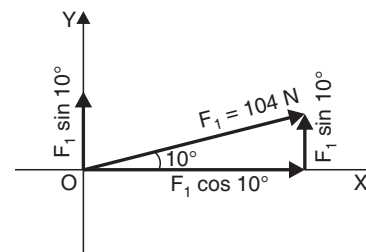


Fig. 2.21 (a)

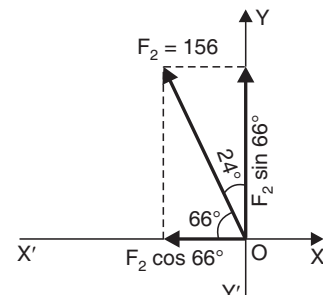


Fig. 2.21 (b)

(+ve)

(iii) Consider force $F_3 = 252 \text{ N}$. Horizontal and vertical components are shown in Fig. 2.21 (c).

Horizontal component,

$$F_{x_3} = F_3 \cos 3^\circ = 252 \times 0.9986 \\ = 251.64 \text{ N.}$$

Vertical component,

$$F_{y_3} = F_3 \sin 3^\circ = 252 \times 0.0523 \\ = 13.18 \text{ N.} \quad (-ve)$$

(iv) Consider force $F_4 = 228 \text{ N}$. Horizontal and vertical components are shown in Fig. 2.21 (d).

Angle made by F_4 with horizontal axis

$$OX' = 90 - 9 = 81^\circ.$$

$\therefore$ Horizontal component,

$$F_{x_4} = F_4 \cos 81^\circ = 228 \times 0.1564 \\ = 35.66 \text{ N} \quad (-ve)$$

Vertical component,

$$F_{y_4} = F_4 \sin 81^\circ = 228 \times 0.9877 \\ = 225.2 \text{ N.} \quad (-ve)$$

Now algebraic sum of horizontal components is given by,

$$\Sigma H = F_{x_1} - F_{x_2} - F_{x_3} - F_{x_4} \\ = 102.4 - 63.44 - 251.64 - 35.66 \\ = -248.32 \text{ N.}$$

-ve sign means that ΣH is acting along OX' as shown in Fig. 2.21 (e).

Similarly, the algebraic sum of vertical components is given

by,

$$\Sigma V = 18.06 = 142.50 + 13.18 - 225.2 \\ = -77.82 \text{ N.}$$

-ve sign means that ΣV is acting along OY' as shown in Fig. 2.21 (e).

The magnitude of resultant (*i.e.*, R) is obtained by using equation (2.1).

$$\therefore R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} \\ = \sqrt{(248.32)^2 + (77.82)^2} = \mathbf{260.2 \text{ N.} \quad \text{Ans.}}$$

The direction of resultant is given by equation (2.2).

$$\therefore \tan \theta = \frac{\Sigma V}{\Sigma H} = \frac{77.82}{248.32} = 0.3134 \\ \therefore \theta = \tan^{-1} 0.3134 = \mathbf{17.4^\circ. \quad \text{Ans.}}$$

(b) **Graphical method.** Fig. 2.22 (a), shows the point at which four forces 104 N, 156 N, 252 N and 228 N are acting. The resultant force is obtained graphically by drawing polygon of forces as explained below and shown in Fig. 2.22 (b) :

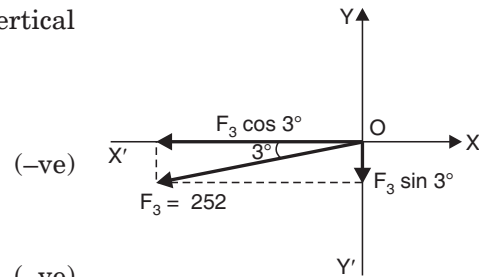


Fig. 2.21 (c)

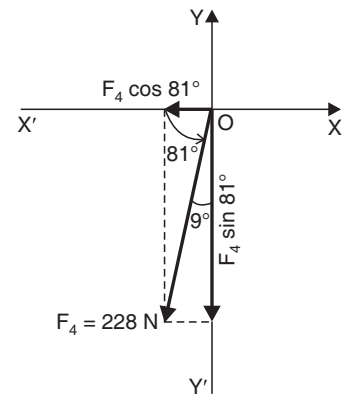


Fig. 2.21 (d)

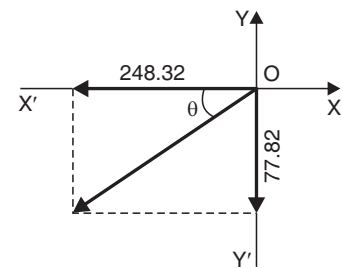


Fig. 2.21 (e)

(i) Choose a suitable scale to represent the given forces. Let the scale is $25 \text{ N} = 1 \text{ cm}$. Hence the force 104 N will be represented by $\frac{104}{25} = 4.16 \text{ cm}$, force 156 N will be represented by $\frac{156}{25} = 6.24 \text{ cm}$ force 252 N will be represented by $\frac{252}{25} = 10.08 \text{ cm}$ and the force 228 N will be represented by $\frac{228}{25} = 9.12 \text{ cm}$.

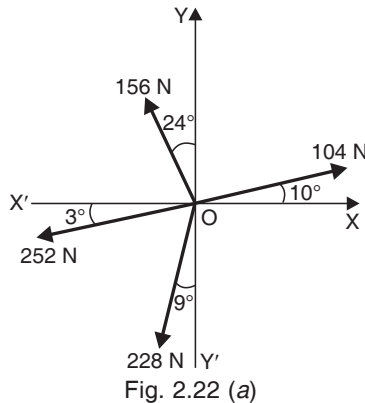


Fig. 2.22 (a)

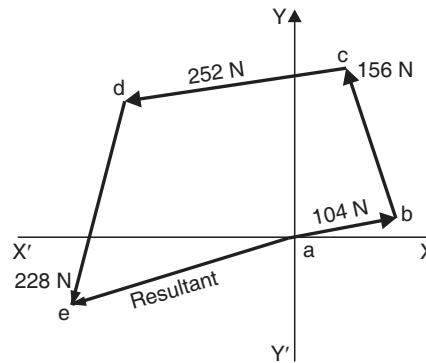


Fig. 2.22 (b)

(ii) Take any point a . From point a , draw vector ab parallel to line of action of force 104 N . Cut $ab = 4.16 \text{ cm}$. Then ab represents the force 104 N in magnitude and direction.

(iii) From point b , draw vector bc parallel to force 156 N and cut $bc = 6.24 \text{ cm}$. Then vector cd represents the force 156 N in magnitude and direction.

(iv) From point c , draw a vector cd parallel 252 N force and cut $cd = 10.08 \text{ cm}$. Then vector cd represents the force 252 N in magnitude and direction.

(v) Now from point d , draw the vector de parallel to 228 N force and cut $de = 9.12 \text{ cm}$. Then vector de represents the force 228 N in magnitude and direction.

(vi) Join point a to e . The line ae is the closing side of the polygon. Hence the side ae represents the resultant in magnitude and direction. Measure the length of ae .

By measurement, length $ae = 10.4 \text{ cm}$

$\therefore$ Resultant, $R = \text{Length } ae \times \text{Scale} = 10.4 \times 25$ ($\because 1 \text{ cm} = 25 \text{ N}$)
 $= 260 \text{ N}$. **Ans.**

Now measure angle made by ae with horizontal. This angle is 17.4° with axis OX' . **Ans.**

Problem 2.8. The resultant of four forces which are acting at a point O as shown in Fig. 2.23, is along Y -axis. The magnitude of forces F_1 , F_3 and F_4 are 10 kN , 20 kN and 40 kN respectively. The angles made by 10 kN , 20 kN and 40 kN with X -axis are 30° , 90° and 120° respectively. Find the magnitude and direction of force F_2 if resultant is 72 kN .

Sol. Given :

$$F_1 = 10 \text{ kN}, \theta_1 = 30^\circ$$

$$F_2 = ?, \theta_2 = \theta$$

$$F_3 = 20 \text{ kN}, \theta_3 = 90^\circ$$

$$F_4 = 40 \text{ kN}, \theta_4 = 120^\circ$$

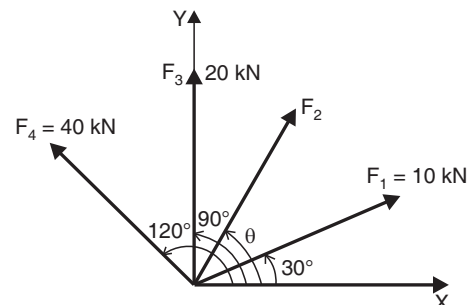


Fig. 2.23

Resultant, $R = 72 \text{ kN}$

Resultant is along Y-axis.

Hence the algebraic sum of horizontal component should be zero and algebraic sum of vertical components should be equal to the resultant.

$$\therefore \Sigma H = 0 \quad \text{and} \quad \Sigma V = R = 72 \text{ kN}$$

$$\begin{aligned} \text{But} \quad \Sigma H &= F_1 \cos 30^\circ + F_2 \cos \theta + F_3 \cos 90^\circ + F_4 \cos 120^\circ \\ &= 10 \times 0.866 + F_2 \cos \theta + 20 \times 0 + 40 \times \left(-\frac{1}{2}\right) \\ &= 8.66 + F_2 \cos \theta + 0 - 20 \\ &= F_2 \cos \theta - 11.34 \end{aligned}$$

$$\therefore \Sigma H = 0 \quad \text{or} \quad F_2 \cos \theta - 11.34 = 0$$

$$\text{or} \quad F_2 \cos \theta = 11.34 \quad \dots(i)$$

$$\begin{aligned} \text{Now} \quad \Sigma V &= F_1 \sin 30^\circ + F_2 \sin \theta + F_3 \sin 90^\circ + F_4 \sin 120^\circ \\ &= 10 \times \frac{1}{2} + F_2 \sin \theta + 20 \times 1 + 40 \times 0.866 \\ &= 5 + F_2 \sin \theta + 20 + 34.64 \\ &= F_2 \sin \theta + 59.64 \end{aligned}$$

$$\text{But} \quad \Sigma V = R$$

$$\therefore F_2 \sin \theta + 59.64 = 72$$

$$\therefore F_2 \sin \theta = 72 - 59.64 = 12.36 \quad \dots(ii)$$

Dividing equation (ii) and (i),

$$\frac{F_2 \sin \theta}{F_2 \cos \theta} = \frac{12.36}{11.34} \quad \text{or} \quad \tan \theta = 1.0899$$

$$\therefore \theta = \tan^{-1} 1.0899 = 47.46^\circ. \quad \text{Ans.}$$

Substituting the value of θ in equation (ii), we get $F_2 \sin (47.46^\circ) = 12.36$

$$\text{or} \quad F_2 = \frac{12.36}{\sin (47.46^\circ)} = \frac{12.36}{0.7368} = 16.77 \text{ kN.} \quad \text{Ans.}$$

Problem 2.9. Find the magnitude and direction of the resultant of the system of coplanar forces shown in Fig. 2.23 (a) given below. (U.P. Technical University, 2000–2001)

Sol. To find the resultant force in magnitude and direction, we must calculate the resultant force in horizontal as well as in vertical direction.

Resultant force in horizontal direction i.e.,

$$\begin{aligned} \Sigma H &= 10 + (10\sqrt{2}) \times \cos 45^\circ - 20 \\ &= 10 + (10\sqrt{2}) \times \frac{1}{\sqrt{2}} - 20 \\ &= 10 + 10 - 20 = 0 \end{aligned}$$

Resultant force in vertical direction i.e.,

$$\begin{aligned} \Sigma V &= -10 + (10\sqrt{2}) \times \sin 45^\circ \\ &= -10 + 10\sqrt{2} \times \frac{1}{\sqrt{2}} = -10 + 10 = 0 \end{aligned}$$

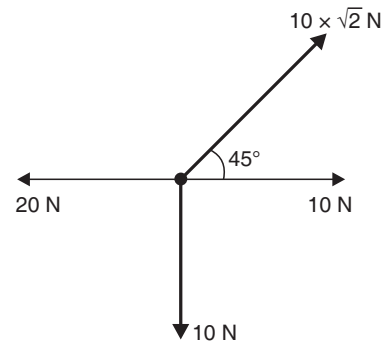


Fig. 2.23 (a)

∴ Resultant force,

$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{0^2 + 0^2} = 0. \text{ Ans.}$$

2.7. RESULTANT OF NON-CONCURRENT COPLANAR FORCES

Fig. 2.24 shows four forces F_1, F_2, F_3 and F_4 acting in a plane. The lines of action of these forces lie in the same plane but they are neither parallel nor meet or intersect at a common point. This system of forces is known as coplanar non-concurrent non-parallel force system. Hence in coplanar non-concurrent non-parallel system of forces, all the forces act in the same plane but the forces are neither parallel nor meet at a common point. This force system is also known as *general system of forces*.

Resultant of non-concurrent coplanar force system is obtained as explained below.

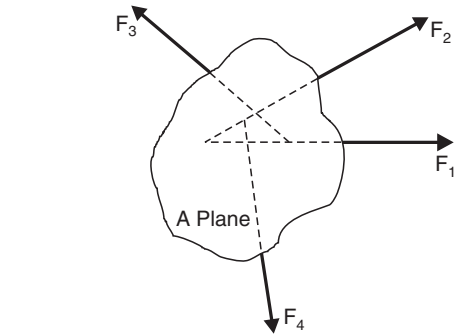


Fig. 2.24. Non-concurrent Non-parallel.

Forces F_1 and F_2 form a concurrent system and their lines of action intersect at O as shown in Fig. 2.24 (a). The resultant R_1 of these forces is obtained from law of parallelogram of forces. Now the forces R_1 and F_3 form a concurrent system and their lines of action intersect at O_1 . The resultant R_2 of these forces (R_1 and F_3) can be obtained from law of parallelogram of forces. The forces R_2 and F_4 may be :

1. **Parallel.** Then the resultant is obtained as described for parallel forces.
2. **Concurrent.** Then the resultant R is obtained by law of parallelogram of forces.

The force R is the resultant of forces F_1, F_2, F_3 and F_4 .

Here forces R_2 and F_4 are concurrent and their lines of action intersect at O_2 . The resultant R of forces R_2 and F_4 can be obtained from law of parallelogram of forces.

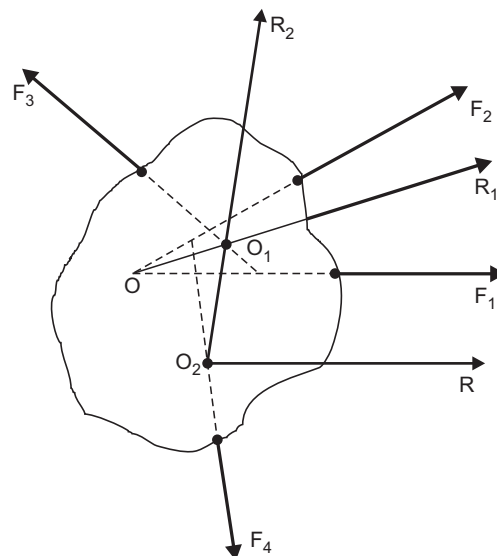


Fig. 2.24. (a) Non-concurrent non-parallel coplanar forces.

Problem 2.10. Determine the magnitude, direction and position of a single force P , which keeps in equilibrium the system of forces acting on the corners of a rectangular block as shown in Fig. 2.25. The position of force P may be stated by reference to axes with origin O and coinciding with the edges of the block.

Sol. Given :

Length $OC = 4$ m, Length $BC = 3$ m

Force at $O = 20$ N ($\leftarrow$), Force at $C = 35$ N ($\downarrow$)

Force at $B = 25$ N ($\rightarrow$), Force at $A = 50$ N ($\downarrow$)

Let O be the origin and OX and OY be the reference axes as shown in Fig. 2.26.

All these forces are neither concurrent nor-parallel. Some forces are parallel while others are concurrent.

Forces 50 N and 20 N form a concurrent system and their line of action intersect at O . These forces are at right angles.

The resultant of these forces

$$R_1 = \sqrt{50^2 + 20^2} = \sqrt{2900} = 53.85 \text{ N}$$

and $\theta_1 = \tan^{-1} \left(\frac{20}{50} \right) = 21.8^\circ$ with vertical axis.

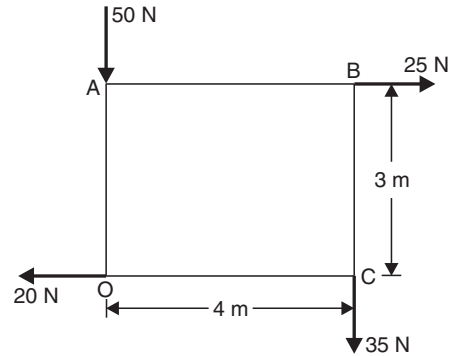


Fig. 2.25

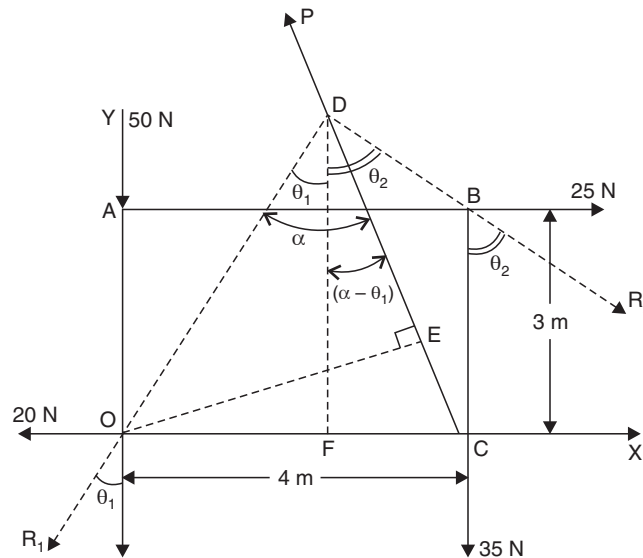


Fig. 2.26

Similarly the forces 35 N and 25 N form a concurrent system and their line of action intersect at B . These forces are also at right angles.

The resultant of these forces

$$R_2 = \sqrt{25^2 + 35^2} = \sqrt{1850} = 43.01 \text{ N}$$

and

$$\theta_2 = \tan^{-1} \left(\frac{25}{35} \right) = 35.53^\circ \text{ with } BC \text{ i.e., with vertical line.}$$

These two forces R_1 and R_2 intersect at D . The angle between these forces is $\theta_1 + \theta_2$ i.e., angle $R_1DR_2 = \theta_1 + \theta_2 = 21.8^\circ + 35.53^\circ = 57.33^\circ$. The resultant of R_1 and R_2 can be obtained by using equations (1.1) and (1.2).

Let P be the resultant of the forces R_1 and R_2 .

$$\begin{aligned} \therefore P &= \sqrt{R_1^2 + R_2^2 + 2R_1 \times R_2 \times \cos(57.33^\circ)} \\ &= \sqrt{53.85^2 + 43.01^2 + 2 \times 53.85 \times 43.01 \times \cos 57.33^\circ} \\ &= \sqrt{2900 + 1850 + 4632.17 \times 0.5398} = \mathbf{85.15 \text{ N. Ans.}} \end{aligned}$$

The angle made by the resultant P with R_1 is given by [See equation (1.2)]

$$\begin{aligned} \tan \alpha &= \frac{R_2 \sin 57.33^\circ}{R_1 + R_2 \cos 57.33^\circ} = \frac{43.01 \sin 57.33^\circ}{53.85 + 43.01 \times \cos 57.33^\circ} \\ &= \frac{43.01 \times 0.8418}{53.85 + 23.21} = \frac{36.2058}{77.06} = 0.4698 \end{aligned}$$

$$\therefore \alpha = \tan^{-1} 0.4698 = 25.16^\circ$$

Hence the resultant P makes $(\alpha - \theta_1)$ angle with vertical in anti-clockwise direction i.e., P makes $(25.16 - 21.8 = 3.36^\circ)$. **Ans.**

Position of the Force P

The position of the force P is obtained by equating the clockwise moments and anti-clockwise moments about O (Refer Fig. 2.26).

Let OE = Perpendicular distance between O and line of action of the force P .

Taking moments of all forces about O ,

$$20 \times 0 + 50 \times 0 + 35 \times 4 + 25 \times 3 = P \times OE$$

$$\text{or} \quad 0 + 0 + 140 + 75 = 85.147 \times OE \quad \text{or} \quad OE = \frac{215}{85.147} = 2.525 \text{ m}$$

$$\text{From right angled triangle } OED, \sin \alpha = \frac{OE}{OD}$$

$$\therefore OD = \frac{OE}{\sin \alpha} = \frac{OE}{\sin 25.16^\circ} = \frac{2.525}{0.4241} = 5.939$$

Let x and y are the co-ordinates of the force P with reference to the axes with origin O .

Then $x = OF$ and $y = DF$

In right angled triangle OFD ,

$$\begin{aligned} OF &= OD \times \sin \theta_1 = 5.939 \times \sin 21.8^\circ & (\because \theta_1 = 21.8^\circ) \\ &= 2.20 \text{ m} \end{aligned}$$

$$\text{Also} \quad FD = OD \times \cos \theta_1 = 5.939 \times \cos 21.8^\circ = 5.514 \text{ m}$$

$$\therefore x = OF = \mathbf{2.20 \text{ m. Ans.}}$$

and

$$y = FD = \mathbf{5.514 \text{ m. Ans.}}$$

Graphical method [Refer to Fig. 2.27(a)]

(i) To a suitable scale, take $OG = 50$ N and $GH = 20$ N. Join OH . Then OH represents the resultant R_1 in magnitude and direction. Produce the line HO backward.

(ii) From point B , take $BJ = 35$ N and $JK = 25$ N. Join BK , which represents the resultant R_2 in magnitude and direction. Produce KB in the backward direction to intersect the line of action of R_1 at point D .

(iii) To find the resultant of R_1 and R_2 (i.e., force P) refer to Fig. 2.27(b).

(iv) Take any point 'a'. From this point draw line ab parallel to R_1 and equal to R_1 . From point 'b', draw line bc parallel to R_2 and equal to R_2 . Join the point c to a .

(v) Then ca represents in magnitude and direction the force P . Hence measure ca . Then

$$P = ca = 85.15 \text{ N. Ans.}$$

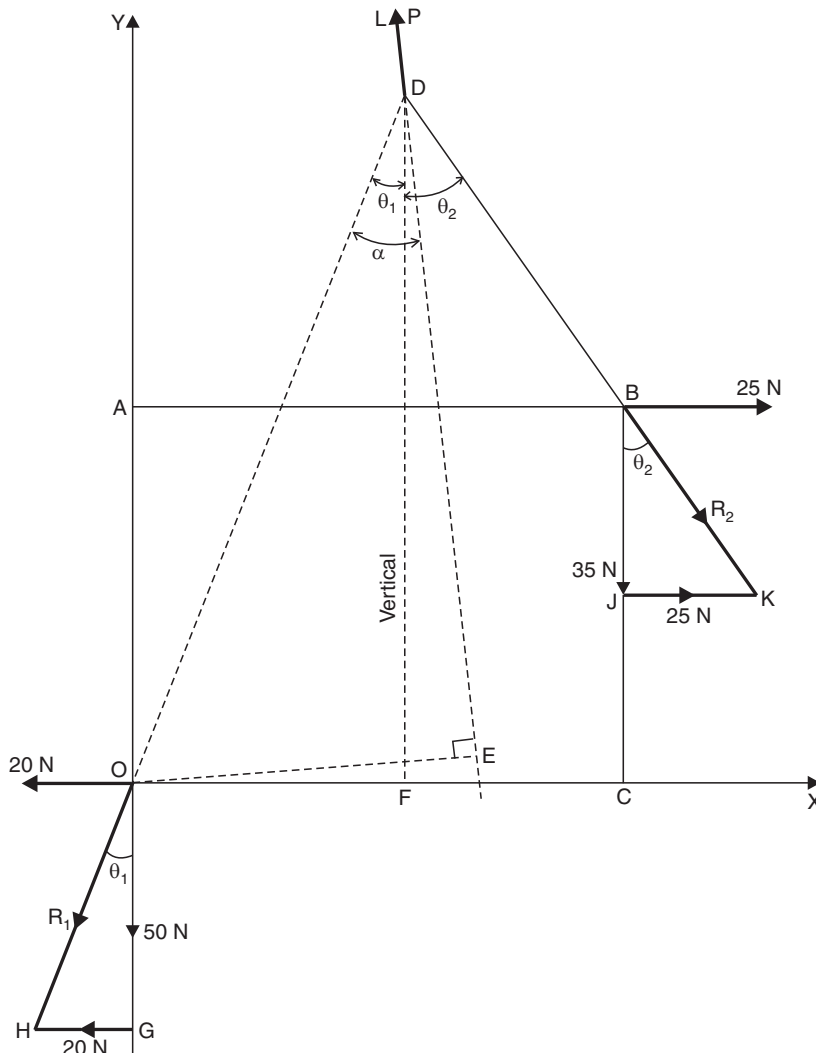


Fig. 2.27(a)

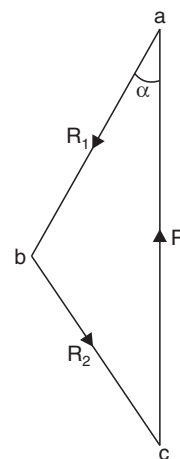


Fig. 2.27(b)

(vi) From point D , draw the line DL parallel to ca . Hence DL represents the direction of the force P .

(vii) To find the position of the force P which is acting at point D , draw DF parallel to axes OY . Then OF represents the x -coordinate and FD represents the y -coordinate of the force P . Measure OF and FD . Then by measurement,

$$OF = x = 2.20 \text{ m. Ans.}$$

and

$$FD = y = 5.514. \text{ Ans.}$$

HIGHLIGHTS

1. Coplanar forces means the forces are acting in one plane.
2. Concurrent forces means the forces are intersecting at a common point.
3. Collinear forces means the forces are having same line of action.
4. The resultant of coplanar forces are determined by analytical and graphical methods.
5. The resultant (R) of three collinear forces F_1 , F_2 and F_3 acting in the same direction, is given by $R = F_1 + F_2 + F_3$. If the force F_2 is acting in opposite direction then their resultant will be, $R = F_1 - F_2 + F_3$.
6. The resultant of the two forces P and Q having an angle α between them and acting at a point, is given by cosine law method as $R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$. And the direction of the resultant with the force P is given by, $\tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}$.
7. The resultant of three or more forces acting at a point is given by, $R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$, where ΣH = Algebraic sum of horizontal components of all forces, ΣV = Algebraic sum of vertical components of all forces. The angle made by the resultant with horizontal is given by, $\tan \theta = \frac{(\Sigma V)}{(\Sigma H)}$.
8. The resultant of several forces acting at a point is found graphically by using polygon law of forces.
9. Polygon law of forces states that if a number of coplanar forces are acting at a point such that they can be represented in magnitude and direction by the sides of a polygon taken in the same order, then their resultant is represented in magnitude and direction by the closing side of the polygon taken in the opposite order.

EXERCISE 2

(A) Theoretical Questions

1. Define and explain the following terms :
 (i) Coplanar and non-coplanar forces (ii) Collinear and concurrent forces
 (iii) Parallel and non-parallel forces.
2. What is the difference between collinear and concurrent forces ?
3. State and explain the following laws of forces :
 (i) Law of parallelogram of forces (ii) Law of triangle of forces
 (iii) Law of polygon of forces.
4. Derive an expression for the resultant in magnitude and direction of two coplanar concurrent forces using cosine law method.
5. Explain in detail the method of finding resultant in magnitude and direction of three or more forces acting at a point by analytical and graphical method.

6. Explain the procedure of resolving a given force into two components at right angles to each other.
7. Three collinear forces F_1 , F_2 and F_3 are acting on a body. What will be the resultant of these forces, if
 - (a) all are acting in the same direction
 - (b) force F_3 is acting in opposite direction?
8. State the law of parallelogram of forces and show that the resultant $R = \sqrt{P^2 + Q^2}$ when the two forces P and Q are acting at right angles to each other. Find the value of R if the angle between the forces is zero.
9. Match the following columns :

<i>Column-I</i>	<i>Column-II</i>
(i) Coplanar forces	(P) Line of action of all forces lie in the same plane and passes through a common point
(ii) Concurrent forces	(Q) Line of action of all forces lie in the same plane
(iii) Concurrent coplanar forces	(R) Line of action of all forces lie along the same line
(iv) Collinear forces	(S) Line of action of all forces pass through a common point

(U.P. Tech. University, May 2009)

[Ans. (i) and (Q), (ii) and (S), (iii) and (P), (iv) and (R)]

(B) Numerical Problems

1. Three collinear horizontal forces of magnitude 300 N, 100 N and 250 N are acting on rigid body. Determine the resultant of the forces analytically and graphically when : (i) all the forces are acting in the same direction ; (ii) the force 100 N acts in the opposite direction.
 [Ans. (i) 650 N, (ii) 450 N]
2. Two forces of magnitude 15 N and 12 N are acting at a point. The angle between the forces is 60° . Find the resultant is magnitude.
 [Ans. 20.43 N]
3. A force of 1000 N is acting at a point, making an angle of 60° with the horizontal. Determine the components of this force along horizontal and vertical directions.
 [Ans. 500 N, 866 N]
4. A small block of weight 100 N is placed on an inclined plane which makes an angle of 60° with the horizontal. Find the components of this weight (i) perpendicular to the inclined plane and (ii) parallel to the inclined plane.
 [Ans. (i) 50 N, (ii) 86.6 N]
5. Two forces P and Q are acting at a point O as shown in Fig. 2.28. The force $P = 264.9$ N and force $Q = 195.2$ N. If the resultant of the forces is equal to 400 N then find the values of angles β , γ , α .
 [Ans. $\beta = 35^\circ$, $\gamma = 25^\circ$, $\alpha = 60^\circ$]
6. A small block of unknown weight is placed on an inclined plane which makes an angle of 30° with horizontal plane. The component of this weight parallel to the inclined plane is 100 N. Find the weight of the block.
 [Ans. 200 N]
7. In question 6, find the component of the weight perpendicular to the inclined plane.
 [Ans. 173.2 N]

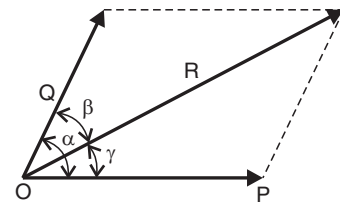


Fig. 2.28

8. The four coplanar forces are acting at a point as shown in Fig. 2.29. Determine the resultant in magnitude and direction analytically and graphically.

[Ans. 1000 N, $\theta = 60^\circ$ with OX]

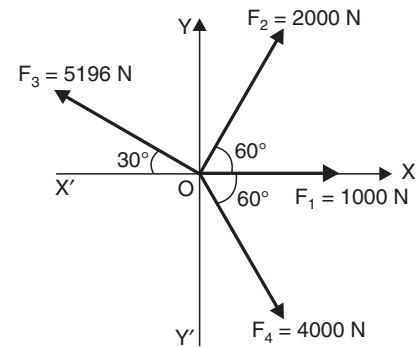


Fig. 2.29

9. The four coplanar forces are acting at a point as shown in Fig. 2.30. One of the forces is unknown and its magnitude is shown by P . The resultant is having a magnitude 500 N and is acting along x -axis. Determine the unknown force P and its inclination with x -axis.

[Ans. $P = 286.5$ N and $\theta = 53^\circ 15'$]

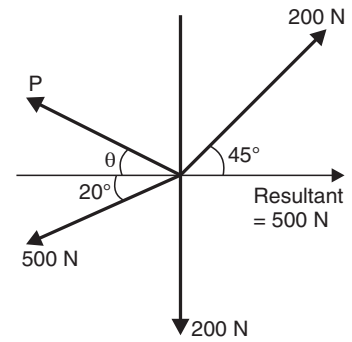


Fig. 2.30

Moment of a Force and Varignon's Theorem

3.1. PARALLEL FORCES

The forces, which are having their line of actions parallel to each other, are known as parallel forces. The two parallel forces will not intersect at a point. The resultant of two coplanar concurrent forces (*i.e.*, forces intersecting at the same point) can be directly determined by the method of parallelogram of forces. This method along with other methods for finding resultant of collinear and concurrent coplanar forces, were discussed in earlier chapters.

The parallel forces are having their lines of action parallel to each other. Hence, for finding the resultant of two parallel forces, (two parallel forces do not intersect at a point) the parallelogram cannot be drawn. The resultant of such forces can be determined by applying the *principle of moments*. Hence in this chapter first the concepts of moment and principle of moments will be dealt with. Thereafter the methods of finding resultant of parallel and even non-parallel forces will be explained.

3.2. MOMENT OF A FORCE

The product of a force and the perpendicular distance of the line of action of the force from a point is known as moment of the force about that point.

Let F = A force acting on a body as shown in Fig. 3.1.

r = Perpendicular distance from the point O on the line of action of force F .

Then moment (M) of the force F about O is given by,

$$M = F \times r$$

The tendency of this moment is to rotate the body in the clockwise direction about O . Hence this moment is called *clockwise moment*. If the tendency of a moment is to rotate the body in anti-clockwise direction, then that moment is known as *anti-clockwise moment*. If clockwise moment is taken $-ve$ then anti-clockwise moment will be $+ve$.

In S.I. system, moment is expressed in Nm (Newton metre).

Fig. 3.2 shows a body on which three forces F_1 , F_2 and F_3 are acting. Suppose it is required to find the resultant moments if these forces about point O .

Let r_1 = Perpendicular distance from O on the line of action of force F_1 .

r_2 and r_3 = Perpendicular distances from O on the lines of action of force F_2 and F_3 respectively.

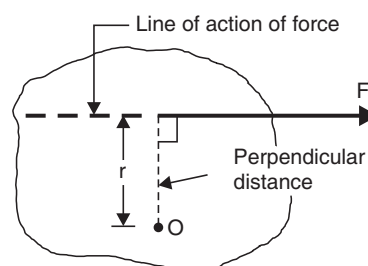


Fig. 3.1

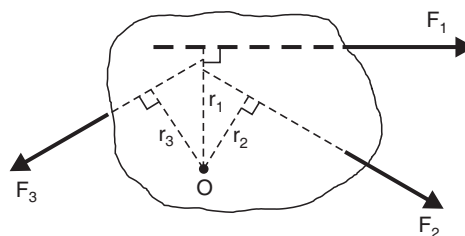


Fig. 3.2

Moment of F_1 about $O = F_1 \times r_1$ (clockwise) $(-)$

Moment of F_2 about $O = F_2 \times r_2$ (clockwise) $(-)$

Moment of F_3 about $O = F_3 \times r_3$ (anti-clockwise) $(+)$

The resultant moment will be *algebraic sum* of all the moments.

$\therefore$ The resultant moment of F_1 , F_2 and F_3 about O

$$= -F_1 \times r_1 - F_2 \times r_2 + F_3 \times r_3.$$

Problem 3.1. Four forces of magnitude 10 N, 20 N, 30 N and 40 N are acting respectively along the four sides of a square ABCD as shown in Fig. 3.3. Determine the resultant moment about the point A. Each side of the square is given 2 m.

Sol. Given :

Length $AB = BC = CD$
 $= DA = 2 \text{ m}$

Force at $B = 10 \text{ N}$,

Force at $C = 20 \text{ N}$,

Force at $D = 30 \text{ N}$,

Force at $A = 40 \text{ N}$,

The resultant moment about point A is to be determined.

The forces at A and B passes through point A. Hence perpendicular distance from A on the lines of action of these forces will be zero.

Hence their moments about A will be zero. The moment of the force at C about point A.

$$\begin{aligned} &= \text{Force at } C \times \perp \text{ distance from A on the line of action of force at } C. \\ &= (20 \text{ N}) \times (\text{Length } AB). \\ &= 20 \times 2 \text{ Nm} = 40 \text{ Nm (anti-clockwise)}. \end{aligned}$$

The moment of force at D about point A.

$$\begin{aligned} &= \text{Force at } D \times \perp \text{ distance from A on the line of action of force at } D. \\ &= (30 \text{ N}) \times (\text{Length } AD). \\ &= 30 \times 2 \text{ Nm} = 60 \text{ Nm (anti-clockwise)}. \end{aligned}$$

$\therefore$ Resultant moment of all forces about A.

$$= 40 + 60 = 100 \text{ Nm (anti-clockwise).} \quad \text{Ans.}$$

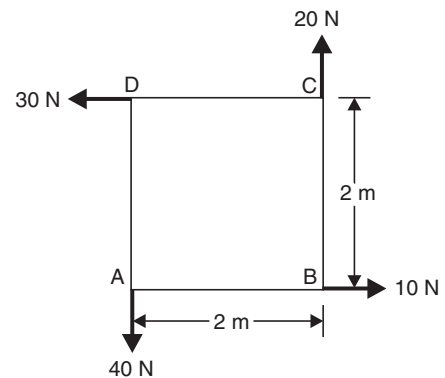


Fig. 3.3

3.3. VARIGNON'S THEOREM (OR PRINCIPLE OF MOMENTS)

Varignon's Theorem states that the moment of a force about any point is equal to the *algebraic sum* of the moments of its components about *that point*.

Principle of moments states that the moment of the resultant of a number of forces about any point is equal to the *algebraic sum* of the moments of all the forces of the system about the same point.

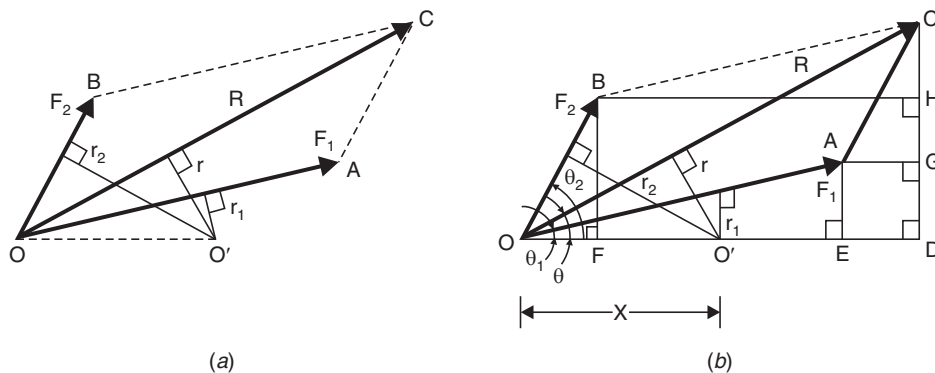
Proof of Varignon's Theorem

Fig. 3.4

Fig. 3.4 (a) shows two forces F_1 and F_2 acting at point O . These forces are represented in magnitude and direction by OA and OB . Their resultant R is represented in magnitude and direction by OC which is the diagonal of parallelogram $OACB$. Let O' is the point in the plane about which moments of F_1 , F_2 and R are to be determined. From point O' , draw perpendiculars on OA , OC and OB .

Let r_1 = Perpendicular distance between F_1 and O' .

r = Perpendicular distance between R and O' .

r_2 = Perpendicular distance between F_2 and O' .

Then according to Varignon's principle ;

Moment of R about O' must be equal to algebraic sum of moments of F_1 and F_2 about O' .

or $R \times r = F_1 \times r_1 + F_2 \times r_2$

Now refer to Fig. 3.4 (b). Join OO' and produce it to D . From points C , A and B draw perpendiculars on OD meeting at D , E and F respectively. From A and B also draw perpendiculars on CD meeting the line CD at G and H respectively.

Let θ_1 = Angle made by F_1 with OD ,
 θ = Angle made by R with OD , and
 θ_2 = Angle made by F_2 with OD .

In Fig. 3.4 (b), $OA = BC$ and also OA parallel to BC , hence the projection of OA and BC on the same vertical line CD will be equal i.e., $GD = CH$ as GD is the projection of OA on CD and CH is the projection of BC on CD .

Then from Fig. 3.4 (b), we have

$$F_1 \sin \theta_1 = AE = GD = CH$$

$$F_1 \cos \theta_1 = OE$$

$$F_2 \sin \theta_2 = BF = HD$$

$$F_2 \cos \theta_2 = OF = ED$$

($\because OB = AC$ and also $OB \parallel AC$. Hence projections of OB and AC on the same horizontal line OD will be equal i.e., $OF = ED$)

$$R \sin \theta = CD$$

$$R \cos \theta = OD$$

Let the length $OO' = x$.

Then $x \sin \theta_1 = r_1$, $x \sin \theta = r$ and $x \sin \theta_2 = r_2$

Now moment of R about O'

$$\begin{aligned}
 &= R \times (\perp \text{ distance between } O' \text{ and } R) = R \times r \\
 &= R \times x \sin \theta \quad (\because r = x \sin \theta) \\
 &= (R \sin \theta) \times x \\
 &= CD \times x \quad (\because R \sin \theta = CD) \\
 &= (CH + HD) \times x \\
 &= (F_1 \sin \theta_1 + F_2 \sin \theta_2) \times x \quad (\because CH = F_1 \sin \theta_1 \text{ and } HD = F_2 \sin \theta_2) \\
 &= F_1 \times x \sin \theta_1 + F_2 \times x \sin \theta_2 \\
 &= F_1 \times r_1 + F_2 \times r_2 \quad (\because x \sin \theta_1 = r_1 \text{ and } x \sin \theta_2 = r_2) \\
 &= \text{Moment of } F_1 \text{ about } O' + \text{Moment of } F_2 \text{ about } O'.
 \end{aligned}$$

Hence moment of R about any point is the algebraic sum of moments of its components (*i.e.*, F_1 and F_2) about the same point. Hence Varignon's principle is proved.

The principle of moments (or Varignon's principle) is not restricted to only two concurrent forces but is also applicable to any coplanar force system, *i.e.*, concurrent or non-concurrent or parallel force system.

Problem 3.2. A force of 100 N is acting at a point A as shown in Fig. 3.5. Determine the moments of this force about O.

Sol. Given :

Force at A = 100 N

Draw a perpendicular from O on the line of action of force 100 N. Hence OB is the perpendicular on the line of action of 100 N as shown in Fig. 3.5.

1st Method

Triangle OBC is a right-angled triangle. And angle $OCB = 60^\circ$.

$$\therefore \sin 60^\circ = \frac{OB}{OC}$$

$$\therefore OB = OC \sin 60^\circ = 3 \times 0.866 = 2.598 \text{ m}$$

Moment of the force 100 N about O

$$\begin{aligned}
 &= 100 \times OB = 100 \times 2.598 \\
 &= \mathbf{259.8 \text{ Nm (clockwise). Ans.}
 \end{aligned}$$

2nd Method

The moment of force 100 N about O, can also be determined by using Varignon's principle. The force 100 N is replaced by its two rectangular components at any convenient point. Here the convenient point is chosen as C. The horizontal and vertical components of force 100 N acting at C are shown in Fig. 3.6.

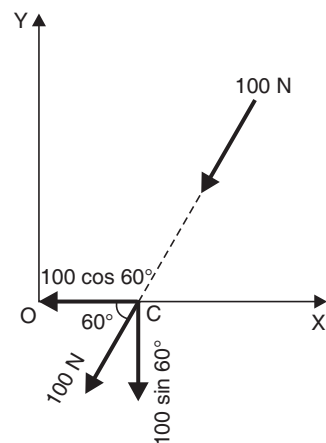
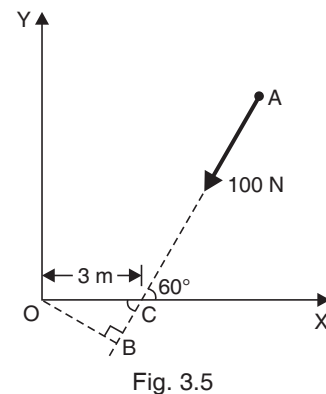
The horizontal component

$$= 100 \times \cos 60^\circ = 50 \text{ N}$$

But this force is passing through O and hence has no moment about O.

The vertical component

$$= 100 \times \sin 60^\circ = 100 \times 0.866 = 86.6 \text{ N}$$



This force is acting vertical downwards at C . Moment of this force about O .

$$= 86.6 \times OC = 86.6 \times 3$$

$$(\because OC = 3 \text{ m})$$

$$= 259.8 \text{ N (clockwise). Ans.}$$

3.4. TYPES OF PARALLEL FORCES

The following are the important types of parallel forces :

1. Like parallel forces,
2. Unlike parallel forces.

3.4.1. Like Parallel Forces. The parallel forces which are acting in the same direction, are known as like parallel forces. In Fig. 3.7, two parallel forces F_1 and F_2 are shown. They are acting in the same direction. Hence they are called as like parallel forces. These forces may be equal or unequal in magnitude.

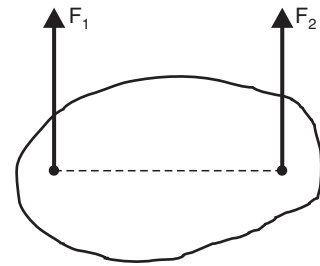


Fig. 3.7

3.4.2. Unlike Parallel Forces. The parallel forces which are acting in the opposite direction, are known as unlike parallel forces. In Fig. 3.8, two parallel forces F_1 , F_2 are acting in opposite direction. Hence they are called as unlike parallel forces. These forces may be equal or unequal in magnitude.

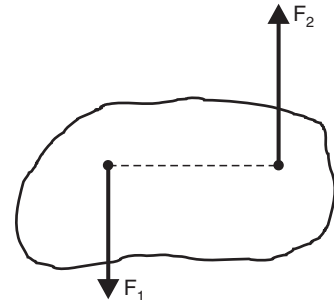


Fig. 3.8

The unlike parallel forces may be divided into : (i) unlike equal parallel forces, and (ii) unlike unequal parallel forces.

Unlike equal parallel forces are those which are acting in opposite direction and are equal in magnitude.

Unlike unequal parallel forces are those which are acting in opposite direction and are unequal in magnitude.

3.5. RESULTANT OF TWO PARALLEL FORCES

The resultant of following two parallel forces will be considered :

1. Two parallel forces are like.
2. Two parallel forces are unlike and are unequal in magnitude.
3. Two parallel forces are unlike but equal in magnitude.

3.5.1. Resultant of Two like Parallel Forces. Fig. 3.9 shows a body on which two like parallel forces F_1 and F_2 are acting. It is required to determine the resultant (R) and also the point at which the resultant R is acting. For the two parallel forces which are acting in the same direction, obviously the resultant R is given by,

$$R = F_1 + F_2$$

In order to find the point at which the resultant is acting, Varignon's principle (or method of moments) is used. According to this, the algebraic sum of moments of F_1 and F_2 about any point should be equal to the moment of the resultant (R) about that point. Now arbitrarily choose any point O along line AB and take moments of all forces about this point.

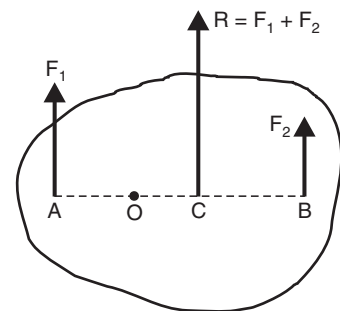


Fig. 3.9

Moment of F_1 about $O = F_1 \times AO$ (clockwise) (–)

Moment of F_2 about $O = F_2 \times BO$ (anti-clockwise) (+ve)

∴ Algebraic sum of moments of F_1 and F_2 about O

$$= -F_1 \times AO + F_2 \times BO$$

Moment of resultant about $O = R \times OC$ (anti-clockwise)(+)

But according to principle of moments the algebraic sum of moments of F_1 and F_2 about O should be equal to the moment of resultant about the same point O .

$$\therefore -F_1 \times AO + F_2 \times BO = +R \times CO = (F_1 + F_2) \times CO \quad (\because R = F_1 + F_2)$$

or $F_1(AO + CO) = F_2(BO + CO)$

or $F_1 \times AC = F_2 \times BC \quad (\because AO + CO = AC \text{ and } BO + CO = BC)$

or $\frac{F_1}{F_2} = \frac{BC}{AC}$

The above relation shows that the resultant R acts at the point C , parallel to the lines of action of the given forces F_1 and F_2 in such a way that the resultant divides the distance AB in the ratio inversely proportional to the magnitudes of F_1 and F_2 . Also the point C lies in line AB i.e., point C is not outside AB .

The location of the point C , at which the resultant R is acting, can also be determined by taking moments about points A of Fig. 3.9. As the force F_1 is passing through A , the moment of F_1 about A will be zero.

The moment of F_2 about $A = F_2 \times AB$ (anti-clockwise) (+)

Algebraic sum of moments of F_1 and F_2 about O

$$= 0 + F_2 \times AB = F_2 \times AB \text{ (anti-clockwise) (+)} \quad \dots(i)$$

The moment of resultant R about A

$$= R \times AC \text{ (anti-clockwise)(+)} \quad \dots(ii)$$

But according to the principle of moments, the algebraic sum of moments of F_1 and F_2 about A should be equal to the moment of resultant about the same point A . Hence equating equations (i) and (ii),

$$F_2 \times AB = R \times AC$$

But $R = (F_1 + F_2)$ hence the distance AC should be less than AB . Or in other words, the point C will lie inside AB .

3.5.2. Resultant of Two Unlike Parallel Forces (Unequal in Magnitude).

Fig. 3.10 shows a body on which two unlike parallel forces F_1 and F_2 are acting which are unequal in magnitude. Let us assume that force F_1 is more than F_2 . It is required to determine the resultant R and also the point at which the resultant R is acting. For the two parallel forces, which are acting in opposite direction, obviously the resultant is given by,

$$R = F_1 - F_2$$

Let the resultant R is acting at C as shown in Fig. 3.10.

In order to find the point C , at which the resultant is acting, principle of moments is used.

Choose arbitrarily any point O in line AB . Take the moments of all forces (i.e., F_1 , F_2 and R) about this point.

Moment of F_1 about $O = F_1 \times AO$ (clockwise)

Moment of F_2 about $O = F_2 \times BO$ (clockwise)

Algebraic sum of moments of F_1 and F_2 about O

$$= F_1 \times AO + F_2 \times BO \quad \dots(i)$$

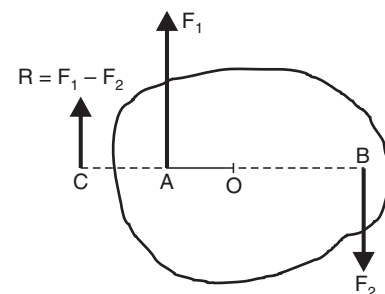


Fig. 3.10

Moment of resultant force R about O

$$\begin{aligned} &= R \times CO \text{ (clockwise)} \\ &= (F_1 - F_2) \times CO \quad (\because R = F_1 - F_2) \\ &= F_1 \times CO - F_2 \times CO \quad \dots(ii) \end{aligned}$$

But according to the principle of moments, the algebraic sum of moments of all forces about any point should be equal to the moment of resultant about that point. Hence equating equations (i) and (ii), we get

$$\begin{aligned} F_1 \times AO + F_2 \times BO &= F_1 \times CO - F_2 \times CO \\ \text{or } F_2(BO + CO) &= F_1(CO - AO) \\ F_2 \times BC &= F_1 \times AC \quad (\because BO + CO = BC \text{ and } CO - AO = AC) \\ \text{or } \frac{BC}{AC} &= \frac{F_1}{F_2} \quad \text{or} \quad \frac{F_1}{F_2} = \frac{BC}{AC} \end{aligned}$$

But $F_1 > F_2$, hence BC will be more than AC . Hence point O lies outside of AB and on the same side as the larger force F_1 . Thus in case of two unlike parallel forces the resultant lies outside the line joining the points of action of the two forces and on the same side as the larger force.

The location of the point C , at which the resultant R is acting, can also be determined by taking moments about point A , of Fig. 3.10. As the force F_1 is passing through A , the moment of F_1 about A will be zero.

The moment of F_2 about $A = F_2 \times AB$ (clockwise) (–)

$$\begin{aligned} \text{Algebraic sum of moments of } F_1 \text{ and } F_2 \text{ about } A \\ &= 0 + F_2 \times AB = F_2 \times AB \text{ (clockwise) (–)} \quad \dots(i) \end{aligned}$$

The moment of resultant R about A should be equal to the algebraic sum of moments of F_1 and F_2 (i.e., $= F_2 \times AB$) according to the principle of moments. Also the moment of resultant R about A should be clockwise. As R is acting upwards [$\because F_1 > F_2$ and $R = (F_1 - F_2)$ so R is acting in the direction of F_1], the moment of resultant R about A would be clockwise only if the point C is towards the left of point A . Hence the point C will be outside the line AB and on the side of F_1 (i.e., larger force).

Now the moment of resultant R about A

$$= R \times AC \text{ (clockwise) (–)} \quad \dots(ii)$$

Equating equations (i) and (ii),

$$\begin{aligned} F_2 \times AB &= R \times AC \\ &= (F_1 - F_2) \times AC \quad (\because R = F_1 - F_2) \end{aligned}$$

As F_1 , F_2 and AB are known, hence AC can be calculated. Or in other words, the location of point C is known.

3.5.3. Resultant of Two Unlike Parallel Forces which are Equal in Magnitude. When two equal and opposite parallel forces act on a body, at some distance apart, the two forces form a couple which has a tendency to rotate the body. The perpendicular distance between the parallel forces is known as *arm of the couple*.

Fig. 3.11 shows a body on which two parallel forces, which are acting in opposite direction but equal in magnitude are acting. These two forces will form a couple which will have a tendency to rotate the body in clockwise direction. The moment of the couple is the product of either one of the forces and perpendicular distance between the forces.

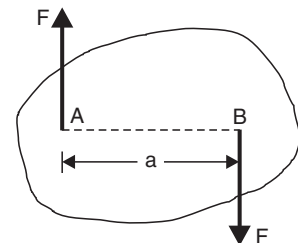


Fig. 3.11

Let F = Force at A or at B

a = Perpendicular distance (or arm of the couple)

The moment (M) of the couple is given by, $M = F \times a$.

The units of moment will be Nm.

Problem 3.3. Three like parallel forces 100 N, 200 N and 300 N are acting at points A, B and C respectively on a straight line ABC as shown in Fig. 3.12. The distances are AB = 30 cm and BC = 40 cm. Find the resultant and also the distance of the resultant from point A on line ABC.

Sol. Given :

Force at A = 100 N

Force at B = 200 N

Force at C = 300 N

Distance AB = 30 cm, BC = 40 cm. As all the forces are parallel and acting in the same direction, their resultant R is given by

$$R = 100 + 200 + 300 = 600 \text{ N}$$

Let the resultant is acting at a distance of x cm from the point A as shown in Fig. 3.12.

Now take the moments of all forces about point A. The force 100 N is passing A, hence its moment about A will be zero.

$\therefore$ Moment of 100 N force about A = 0

Moment of 200 N force about A = $200 \times 30 = 6000 \text{ N cm}$ (anti-clockwise)

Moment of 300 N force about A = $300 \times AC$
 $= 300 \times 70 = 21000 \text{ N cm}$ (anti-clockwise)

Algebraic sum of moments of all forces about A

$$= 0 + 6000 + 21000 = 27000 \text{ N cm (anti-clockwise)}$$

Moment of resultant R about A = $R \times x$

$$= 600 \times x \text{ N cm} \quad (\because R = 600)$$

But algebraic sum of moments of all forces about A

= Moment of resultant about A

$$\text{or} \quad 27000 = 600 \times x \quad \text{or} \quad x = \frac{27000}{600} = 45 \text{ cm. Ans.}$$

Problem 3.4. The three like parallel forces of magnitude 50 N, F and 100 N are shown in Fig. 3.13. If the resultant $R = 250 \text{ N}$ and is acting at a distance of 4 m from A, then find

(i) Magnitude of force F .

(ii) Distance of F from A.

Sol. Given :

Forces at A = 50 N, at B = F and D = 100 N

$R = 250 \text{ N}$, Distance AC = 4 m, CD = 3 m.

(i) Magnitude of force F

The resultant R of three like forces is given by,

$$R = 50 + F + 100$$

$$\text{or} \quad 250 = 50 + F + 100 \quad (\because R = 250)$$

$$\therefore F = 250 - 50 - 100 = 100 \text{ N. Ans.}$$

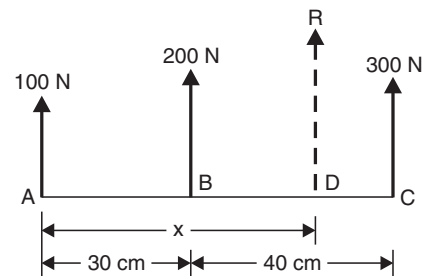


Fig. 3.12

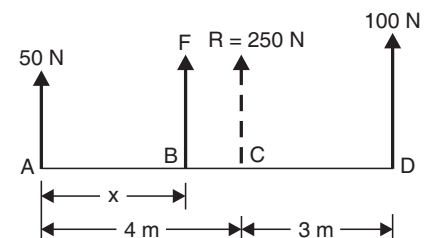


Fig. 3.13

(ii) *Distance of F from A*

Take the moments of all forces about point A .

Moment of force 50 N about $A = 0$ ($\because$ Force 50 N is passing through)

Moment of force F about $A = F \times x$ (anti-clockwise)

Moment of force 100 N about $A = 100 \times AD = 100 \times 7 = 700$ Nm (anti-clockwise)

$\therefore$ Algebraic sum of moments of all forces about A

$$= 0 + F \times x + 700 \text{ Nm}$$

$$= F \times x + 700 \text{ Nm} \quad (\text{anti-clockwise})$$

Moment of resultant R about $A = R \times 4 = 250 \times 4 = 1000$ Nm (anti-clockwise)

But algebraic sum of moments of all forces about A must be equal to the moment of resultant R about A .

$$\therefore F \times x + 700 = 1000 \quad \text{or} \quad F \times x = 1000 - 700 = 300$$

$$\text{or} \quad x = \frac{300}{F} = \frac{300}{100} \quad (\because F = 100 \text{ N})$$

$$= 3 \text{ m. Ans.}$$

Problem 3.5. Four parallel forces of magnitudes 100 N, 150 N, 25 N and 200 N are shown in Fig. 3.14. Determine the magnitude of the resultant and also the distance of the resultant from point A .

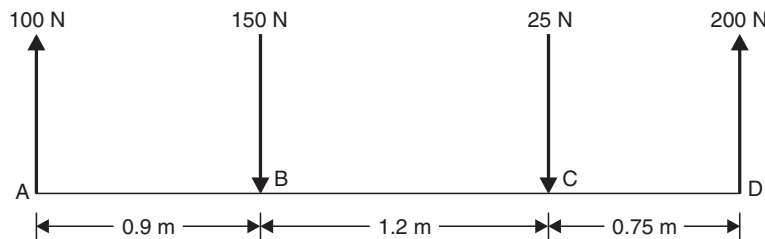


Fig. 3.14

Sol. Given :

Forces are 100 N, 150 N, 25 N and 200 N.

Distances $AB = 0.9$ m, $BC = 1.2$ m, $CD = 0.75$ m.

As all the forces are acting vertically, hence their resultant R is given by

$$R = 100 - 150 - 25 + 200$$

(Taking upward force +ve and downward as -ve)

$$= 300 - 175 = 125 \text{ N}$$

+ve sign shows that R is acting vertically upwards. To find the distance of R from point A , take the moments of all forces about point A .

Let x = Distance of R from A in metre.

As the force 100 N is passing through A , its moment about A will be zero.

Moment of 150 N force about $A = 150 \times AB$

$$= 150 \times 0.9 \text{ (clockwise) } (-) = -135 \text{ Nm}$$

Moment of 25 N force about $A = 25 \times AC = 25 \times (0.9 + 1.2)$

$$= 25 \times 2.1 \text{ (clockwise) } (-) = -52.5 \text{ Nm.}$$

Moment of 200 N force about A = $200 \times AD$

$$= 200 \times (0.9 + 1.2 + 0.75)$$

$$= 200 \times 2.85 \text{ (anti-clockwise) (+)} = 570 \text{ Nm}$$

Algebraic sum of moments of all forces about A

$$= -135 - 52.5 + 570 = 382.5 \text{ Nm} \quad \dots(i)$$

+ve sign shows that this moment is anti-clockwise. Hence the moment of resultant R about A must be 382.5 Nm, i.e., moment of R should be anti-clockwise about A. The moment of R about A will be anti-clockwise if R is acting upwards and towards the right of A.

Now moment of R about A = $R \times x$. But $R = 125$

$$= 125 \times x \quad \text{(anti-clockwise) (+)}$$

$$= +125 \times x \quad \dots(ii)$$

Equating (i) and (ii),

$$382.5 = 125 \times x$$

or $x = \frac{382.5}{125} = 3.06 \text{ m. Ans.}$

$\therefore$ Resultant ($R = 125 \text{ N}$) will be 125 N upwards and is acting at a distance of 3.06 m to the right of point A as shown in Fig. 3.14 (a).

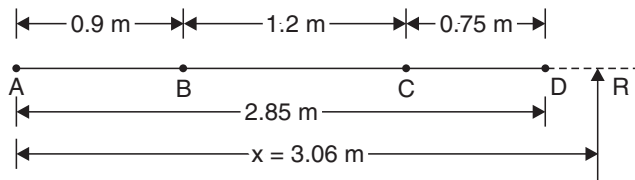


Fig. 3.14 (a)

3.6. RESOLUTION OF A FORCE INTO A FORCE AND A COUPLE

A given force F applied to a body at any point A can always be replaced by an equal and parallel force applied at another point B together with a couple which will be equivalent to the original force. This is proved as given below :

Let the given force F is acting at point A as shown in Fig. 3.15 (a).

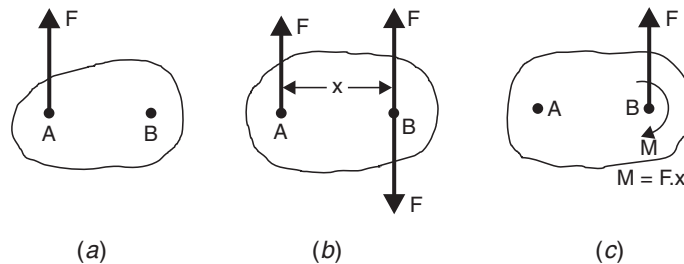


Fig. 3.15

This force is to be replaced at the point B. Introduce two equal and opposite forces at B, each of magnitude F and acting parallel to the force at A as shown in Fig. 3.15 (b). The force system of Fig. 3.15 (b) is equivalent to the single force acting at A of Fig. 3.15 (a). In Fig. 3.15 (b) three equal forces are acting. The two forces i.e., force F at A and the oppositely directed force F at B (i.e., vertically downward force at B) form a couple. The moment of this couple is $F \times x$ clockwise where x is the perpendicular distance between the lines of action of forces at A and B. The third force is acting at B in the same direction in which the force at A is acting. In Fig. 3.15 (c), the couple is shown by curved arrow with symbol M . The force system of Fig. 3.15 (c) is equivalent to Fig. 3.15 (b). Or in other words the Fig. 3.15 (c) is equivalent to Fig. 3.15 (a). Hence the given force F acting at A has been replaced by an equal and parallel force applied at point B in the same direction together with a couple of moment $F \times x$.

Thus a force acting at a point in a rigid body can be transferred to an equal and parallel force at any other point in the body, and a couple.

Problem 3.6. A system of parallel forces are acting on a rigid bar as shown in Fig. 3.16. Reduce this system to :

- (i) a single force
- (ii) a single force and a couple at A
- (iii) a single force and a couple at B.

Sol. Given :

Forces at A, C, D and B are 32.5 N, 150 N, 67.5 N and 10 N respectively.

Distances AC = 1 m, CD = 1 m and BD = 1.5 m.

(i) *Single force system.* The single force system will consist only resultant force in magnitude and location. All the forces are acting in the vertical direction and hence their resultant (R) in magnitude is given by

$$R = 32.5 - 150 + 67.5 - 10 = -60 \text{ N. Ans.}$$

Negative sign shows that resultant is acting vertically downwards.

Let x = Distance of resultant from A towards right. To find the location of the resultant take the moments of all forces about A, we get moment of resultant about A.

= Algebraic sum of moments of all forces about A

$$\text{or } R \times x = -150 \times AC + 67.5 \times AD - 10 \times AB$$

(Taking clockwise moment -ve and anti-clockwise moment +ve)

$$\text{or } (-60)x = -150 \times 1 + 67.5 \times 2 - 10 \times 3.5$$

(The moment due to R at A is clockwise and hence it is -ve)

$$\text{or } -60x = -150 + 135 - 35 = -50$$

$$\therefore x = \frac{-50}{-60} = 0.833 \text{ m. Ans.}$$

Hence the given system of parallel forces is equivalent to a single force 60 N acting vertically downwards at point E at a distance of 0.833 m from A shown in Fig. 3.16 (a).

(ii) *A single force and a couple at A.* The resultant force R acting at point E as shown in Fig. 3.16 (a) can be replaced by an equal force applied at point A in the same direction together with a couple. This is shown in Fig. 3.16 (c).

The moment of the couple = $60 \times 0.833 \text{ Nm}$ (clockwise)

$$= -49.98 \text{ Nm. Ans.} \quad (-\text{ve sign is due to clockwise})$$

(iii) *A single force and a couple at B.* First find distance BE. But from Fig. 3.16 (b), the distance

$$BE = AB - AE = 3.5 - 0.833 = 2.667 \text{ m.}$$

Now if the force $R = 60 \text{ N}$ is moved to the point B, it will be accompanied by a couple of moment $60 \times BE$ or $60 \times 2.667 \text{ Nm}$. This is shown in Fig. 3.16 (e).

$$\begin{aligned} \text{The moment of the couple} &= 60 \times 2.667 \text{ Nm (anti-clockwise)} \\ &= 160 \text{ Nm. Ans.} \end{aligned}$$

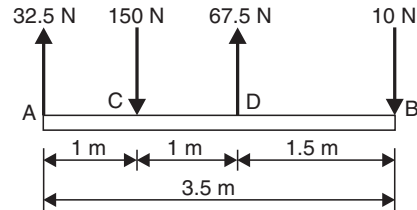


Fig. 3.16

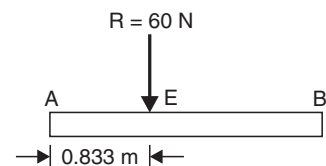


Fig. 3.16 (a)

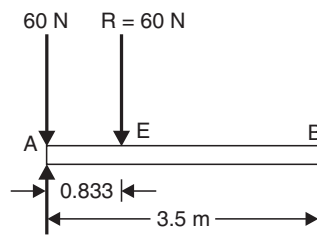


Fig. 3.16 (b)

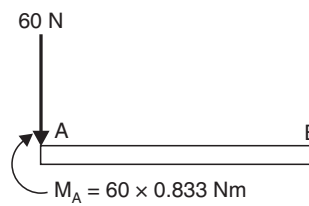


Fig. 3.16 (c)

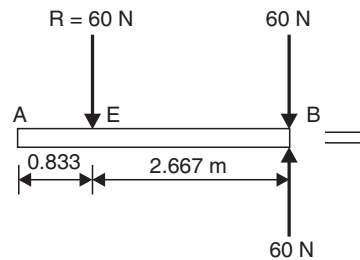


Fig. 3.16 (d)



Fig. 3.16 (e)

Problem 3.6 (A). A rigid bar is subjected to a system of parallel forces as shown in Fig. 3.16 (f). Reduce this system to :

- (i) a single force system,
- (ii) a single force moment system at B.

(U.P. Tech. University 2001–2002)

Sol. (i) A Single Force System

A single force will consist only a resultant force in magnitude and location. All the forces are acting in the vertical direction and hence their resultant (R) in magnitude is given by

$$R = -15 + 60 - 10 + 25 = 60 \text{ N. Ans.}$$

(Here upward direction is considered –ve whereas downward direction +ve).

As R is +ve, hence it acts downward as shown in Fig. 3.16 (g).

To find the distance of the resultant from end A, take the moments of all forces about A, then

Moment of R about A = Algebraic sum of moments of all forces about A

$$-R \times x = -60 \times 0.4 + 10(0.4 + .3) - 25(0.4 + .3 + .7)$$

$$-R \times x = -24 + 7 - 35$$

$$\text{or} \quad -60x = -52$$

$$\text{or} \quad x = \frac{-52}{-60} = 0.867 \text{ m. Ans.}$$

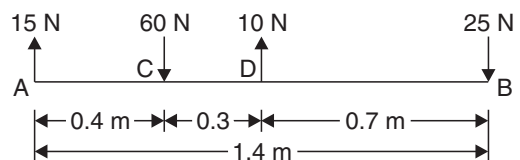


Fig. 3.16 (f)

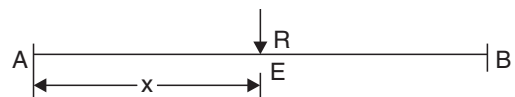


Fig. 3.16 (g)

Hence the given system of parallel forces is equivalent to a single force 60 N acting vertically downward at point E at a distance of 0.867 m from A as shown in Fig. 3.16 (g).

(ii) A Single Force Moment System at B

$$\begin{aligned}\text{Distance } BE &= AB - AE = AB - x \\ &= 1.4 - 0.867 = 0.533 \text{ m}\end{aligned}$$

If the force $R = 60 \text{ N}$ is moved to the point B , it will be accompanied by a couple of moment $60 \times 0.533 \text{ Nm}$ as shown in Fig. 3.16 (h).

$$\begin{aligned}\therefore \text{Moment of couple} &= 60 \times 0.533 \text{ Nm (anti-clockwise)} \\ &= \mathbf{31.98 \text{ Nm. Ans.}}\end{aligned}$$

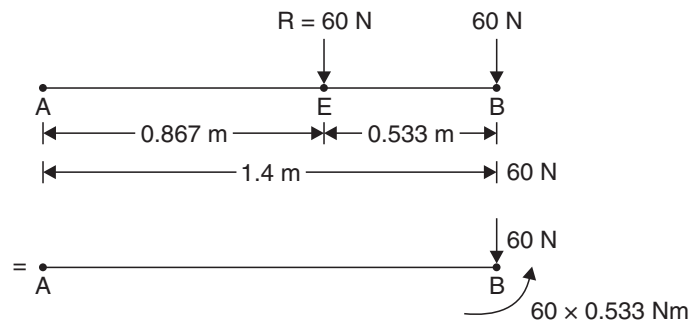


Fig. 3.16 (h)

3.7. GENERAL CASE OF PARALLEL FORCES IN A PLANE

Fig. 3.17 shows a number of parallel forces acting on a body in one plane. The forces F_1 , F_2 and F_4 are acting in one direction, whereas the forces F_3 and F_5 are acting in the opposite direction. Let R_1 = Resultant of forces F_1 , F_2 and F_4 and R_2 = Resultant of forces F_3 and F_5 . The resultants R_1 and R_2 are acting in opposite direction and are parallel to each other. Now three important cases are possible.

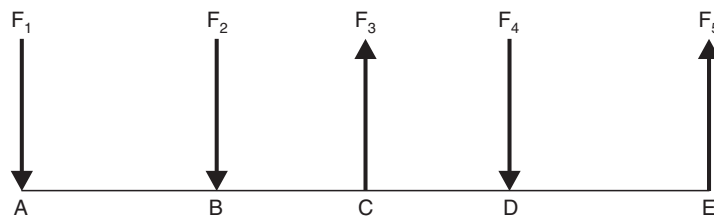


Fig. 3.17

1. R_1 may not be equal to R_2 . Then we shall have two unequal parallel forces (R_1 and R_2) acting in the opposite direction. The resultant R of these two forces (R_1 and R_2) can be easily obtained. The point of application of resultant R can be obtained by equating the moment of R about any point to the algebraic sum of the moments of individual forces about the same point.

2. R_1 is equal to R_2 . Then we shall have two equal parallel forces (R_1 and R_2) acting in the opposite direction. The resultant R of these two forces will be zero. Now the system may reduce to a couple or the system is in equilibrium. To distinguish between these two cases, the algebraic sum of moments of all forces ($F_1, F_2, \dots, F_5$) about any point is taken. If the sum of moments is not zero, the system reduces a resultant couple. The calculated moment gives the moment of this couple.

3. R_1 is equal to R_2 and sum of moments of all forces ($F_1, F_2, F_3, F_4, F_5, \dots$) about any point is zero, then the system will not be subjected to any resultant couple but the system will be in equilibrium.

Problem 3.7. Determine the resultant of the parallel force system shown in Fig. 3.18.

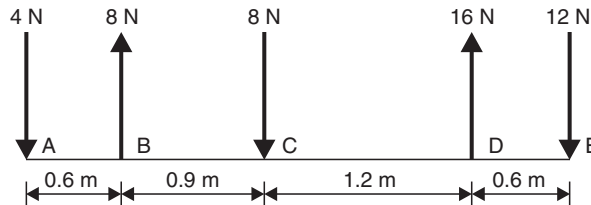


Fig. 3.18

Sol. Given :

Forces at A, B, C, D and E and 4 N, 8 N, 8 N, 16 N and 12 N respectively.

Distances $AB = 0.6$ m, $BC = 0.9$ m,
 $CD = 1.2$ m, and $DE = 0.6$ m.

Since all the forces are vertical and parallel, hence their resultant is given by

$$R = -4 + 8 - 8 + 16 - 12 = 0$$

As the resultant force on the system is zero, there will be two possibilities. The system has a resultant couple or the system is in equilibrium. To distinguish between these two possibilities, take the sum of moments of all forces about any point. Let us take the moments about point A.

$$\begin{aligned} \therefore \text{Algebraic sum of moments of all forces about A} \\ &= 4 \times 0 + 8 \times AB - 8 \times AC + 16 \times AD - 12 \times AE \\ &= 0 + 8 \times 0.6 - 8 \times (0.6 + 0.9) + 16 \times (0.6 + 0.9 + 1.2) - 12 \times (0.6 + 0.9 + 1.2 + 0.6) \\ &= 0 + 4.8 - 12 + 16 \times 2.7 - 12 \times 3.3 \text{ Nm} \\ &= 4.8 - 12 + 43.2 - 39.6 = 48 - 51.6 \\ &= -3.6 \text{ Nm} \end{aligned}$$

As the algebraic sum of moments of all forces about any point is not zero, the system will have a resultant couple of magnitude -3.6 Nm i.e., a clockwise couple. **Ans.**

Problem 3.8. Determine the resultant of the parallel forces acting on a body as shown in Fig. 3.19.

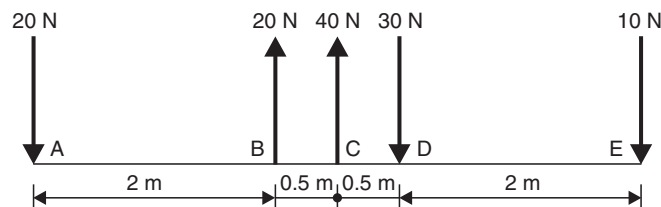


Fig. 3.19

Sol. Since all the forces are vertical and parallel, hence their resultant is given by

$$R = -20 + 20 + 40 - 30 - 10 = 0$$

Taking moments of all forces about the point A, we get

$$\begin{aligned}\text{Resultant moment} &= 20 \times 0 + 20 \times 2 + 40 \times 2.5 - 30 \times 3 - 10 \times 5 \\ &= 0 + 40 + 100 - 90 - 50 = 140 - 140 = 0\end{aligned}$$

As the resultant moment is zero and also the resultant force on the body is zero, the body will be in equilibrium. **Ans.**

HIGHLIGHTS

1. Parallel forces are having their lines of action parallel to each other.
2. The moment of a force about any point is the product of force and perpendicular distance between the point and line of action of force.
3. Anti-clockwise moment is taken +ve whereas clockwise moment is taken -ve.
4. Varignon's principle states that the moment of a force about any point is equal to the algebraic sum of moments of its components about that point.
5. Like parallel forces are parallel to each other and are acting in the same direction, whereas the unlike parallel forces are acting in opposite direction.
6. The resultant of two like parallel forces is the sum of the two forces and acts at a point between the line in such a way that the resultant divides the distance in the ratio inversely proportional to the magnitudes of the forces.
7. When two equal and opposite parallel forces act on a body at some distance apart, the two forces form a couple which has a tendency to rotate the body. The moment of this couple is the product of either one of the forces and perpendicular distance between the forces.
8. A given force F applied to a body at any point A can always be replaced by an equal force applied at another point B in the same direction together with a couple.
9. If the resultant of a number of parallel forces is not zero, the system can be reduced to a single force, whose magnitude is equal to the algebraic sum of all forces. The point of application of this single force is obtained by equating the moment of this single force about any point to the algebraic sum of moments of all forces acting on the system about the same point.
10. If the resultant of a number of parallel forces is zero, then the system may have a resultant couple or may be in equilibrium. If the algebraic sum of moments of all forces about any point is not zero, then system will have a resultant couple. But if the algebraic sum of moments of all forces about any point is zero, the system will be in equilibrium.

EXERCISE 3

(A) Theoretical Questions

1. Define the terms : Coplanar parallel forces, like parallel forces and unlike parallel forces.
2. Define and explain the moment of a force. Differentiate between clockwise moment and anti-clockwise moment.
3. (a) State the Varignon's principle. Also give the proof of Varignon's principle.
(U.P. Tech. University, 2002-2003)
(b) Differentiate between :
 - (i) Concurrent and non-concurrent forces,
 - (ii) Coplanar and non-coplanar forces,
 - (iii) Moment of a force and couple.
4. Define moment of a force about a point and show that the algebraic sum of the moments of two coplanar forces about a point is equal to the moment of their resultant about that point.

5. What are the different types of parallel forces ? Distinguish between like and unlike parallel forces.
6. Prove that the resultant of two like parallel forces F_1 and F_2 is $F_1 + F_2$. Also prove that the resultant divides the line of joining the points of action of F_1 and F_2 internally in the inverse ratio of the forces.
7. Prove that in case of two unlike parallel forces the resultant lies outside the line joining the points of action of the two forces and on the same side as the larger force.
8. Describe the method of finding the line of action of the resultant of a system of parallel forces.
9. The resultant of a system of parallel forces is zero, what does it signify ?
10. Describe the method of finding the resultant of two unlike parallel forces which are equal in magnitude.
11. Prove that a given force F applied to a body at any point A can always be replaced by an equal force applied at another point B together with a couple.
12. State the principle of moment.
13. Indicate whether the following statements are True or False.
 - (i) Force is an agency which tends to cause motion.
 - (ii) The tension member of a framework is called a struts.
 - (iii) The value of g reduces slightly as we move from poles towards the equator.
 - (iv) Coplanar forces are those which have the same magnitude and direction.
 - (v) A couple consists of two unequal and parallel forces acting on a body, having the same line of action.
 - (vi) A vector diagram of a force represents its magnitude, direction, sense and point of application.
 - (vii) The force of gravitation on a body is called its weight.
 - (viii) The centre of gravity of a body is the point, through which the resultant of parallel forces passes in whatever position may the body be placed.

[Ans. (i) True (ii) False (iii) True (iv) False (v) False (vi) False (vii) True (viii) True.]

(B) Numerical Problems

1. Four forces of magnitudes 20 N, 40 N, 60 N and 80 N are acting respectively along the four sides of a square $ABCD$ as shown in Fig. 3.20. Determine the resultant moment about point A .
Each side of square is 2 m. [Ans. 200 Nm anti-clockwise]

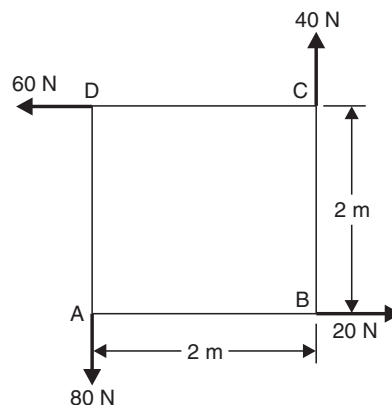


Fig. 3.20

2. A force of 50 N is acting at a point A as shown in Fig. 3.21. Determine the moment of this force about O. [Ans. 100 Nm clockwise]

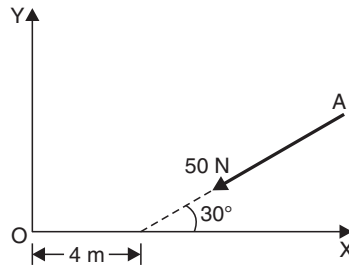


Fig. 3.21

3. Three like parallel forces 20 N, 40 N and 60 N are acting at points A, B and C respectively on a straight line ABC. The distances are $AB = 3$ m and $BC = 4$ m. Find the resultant and also the distance of the resultant from point A on line ABC. [Ans. 120 N, 4.5 m]
4. The three like parallel forces 101 N, F and 300 N are acting as shown in Fig. 3.22. If the resultant $R = 600$ N and is acting at a distance of 45 cm from A, then find the magnitude of force F and distance of F and A. [Ans. 200 N, 30 cm]

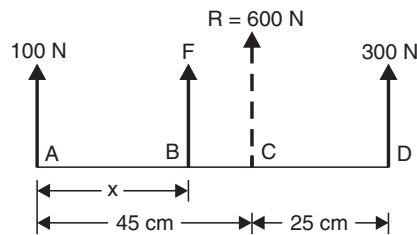


Fig. 3.22

5. Four parallel forces of magnitudes 100 N, 200 N, 50 N and 400 N are shown in Fig. 3.23. Determine the magnitude of the resultant and also the distance of the resultant from point A. [Ans. $R = 350$ N, 3.07 m]

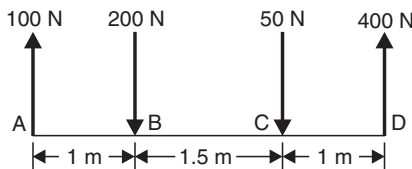


Fig. 3.23

6. A system of parallel forces are acting on a rigid bar as shown in Fig. 3.24. Reduce this system to :
 (i) a single force, [Ans. (i) $R = 120$ N at 2.83 m from A
 (ii) a single force and a couple at A (ii) $R = 120$ N and $M_A = -340$ Nm
 (iii) a single force and a couple at B. (iii) $R = 120$ N and $M_B = 120$ Nm]

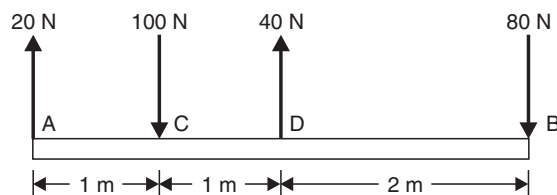


Fig. 3.24

7. Five forces are acting on a body as shown in Fig. 3.25. Determine the resultant.

[Ans. $R = 0$, Resultant couple = 10 Nm]

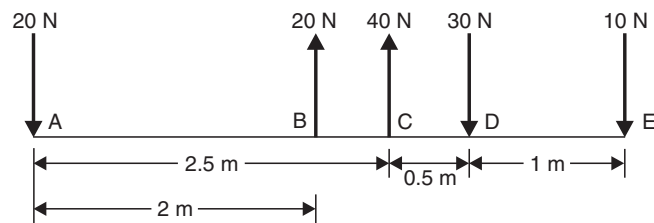


Fig. 3.25

8. Determine the resultant of the parallel forces shown in Fig. 3.26. [Ans. Body is in equilibrium]

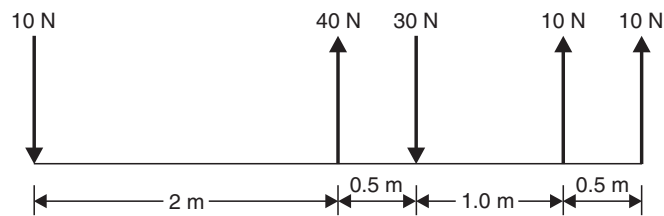


Fig. 3.26

4

Equilibrium of Coplanar Force System and Free Body Diagram

4.1. EQUILIBRIUM OF COPLANAR FORCE SYSTEM

When some external forces (which may be concurrent or parallel) are acting on a stationary body, the body may start moving or may start rotating about any point. But if the body does not start moving *and also* does not start rotating about any point, then the body* is said to be in equilibrium. In this chapter, the conditions of equilibrium for coplanar concurrent forces (*i.e.*, forces meeting at a point) and for coplanar parallel forces will be described. Also the concept of free body diagram, different types of support reactions and determination of reactions will be explained.

4.1.1. Principle of Equilibrium. The principle of equilibrium states that, a stationary body which is subjected to coplanar forces (concurrent or parallel) will be in equilibrium if the algebraic sum of all the external forces is zero *and also* the algebraic sum of moments of all the external forces about any point in their plane is zero. Mathematically, it is expressed by the equations :

$$\Sigma F = 0 \quad \dots(4.1)$$

$$\Sigma M = 0 \quad \dots(4.2)$$

The sign Σ is known as sigma which is a Greek letter. This sign represents the *algebraic sum* of forces or moments.

The equation (4.1) is also known as force law of equilibrium whereas the equation (4.2) is known as moment law of equilibrium.

The forces are generally resolved into horizontal and vertical components. Hence equation (4.1) is written as

$$\Sigma F_x = 0 \quad \dots(4.3)$$

and $\Sigma F_y = 0 \quad \dots(4.4)$

where ΣF_x = Algebraic sum of all horizontal components

and ΣF_y = Algebraic sum of all vertical components.

1. Equations of Equilibrium for Coplanar Non-concurrent Force Systems. A non-concurrent force systems will be in equilibrium if the resultant of all forces and moment is zero.

Hence the equations of equilibrium are

$$\Sigma F_x = 0, \Sigma F_y = 0 \quad \text{and} \quad \Sigma M = 0.$$

*A body will be in equilibrium if both the resultant of the forces and resultant moment are zero, whereas a particle will be in equilibrium if the resultant force acting on it is zero.

2. Equations of Equilibrium for Coplanar Concurrent Force System. For the concurrent forces, the lines of action of all forces meet at a point, and hence the moment of those force about that very point will be zero or $\Sigma M = 0$ automatically.

Thus for concurrent force system, the condition $\Sigma M = 0$ becomes redundant and only two conditions, i.e., $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are required.

4.1.2. Force Law of Equilibrium. Force law of equilibrium is given by equation (4.1) or by equations (4.3) and (4.4). Let us apply this law to the following important force system :

- (i) Two force system
- (ii) Three force system
- (iii) Four force system.

1. Two Force System. When a body is subjected to two forces, then the body will be in equilibrium if the two forces are *collinear, equal and opposite* as shown in Fig. 4.1.

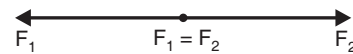


Fig. 4.1

If the two forces acting on a body are equal and opposite but are parallel, as shown in Fig. 4.1 (a), then the body will not be in equilibrium. This is due to the fact that the three conditions of equilibrium will not be satisfied. This is proved as given below :

(i) Here $\Sigma F_x = 0$ as there is no horizontal force acting on the body. Hence 1st condition of equilibrium is satisfied.

(ii) Also here $\Sigma F_y = 0$ as $F_1 = F_2$.

Hence second condition of equilibrium is also satisfied.

(iii) ΣM about any point should be zero. The resultant moment about point A is given by

$$M_A = -F_2 \times AB$$

(-ve sign is due to clockwise moment)

But M_A is not equal to zero. Hence the third condition is not satisfied.

Hence a body will not be in equilibrium under the action of two equal and opposite parallel forces.

Two equal and opposite parallel forces produce a couple and moment of the couple is $-F_1 \times AB$ [see Fig. 4.1 (a)].

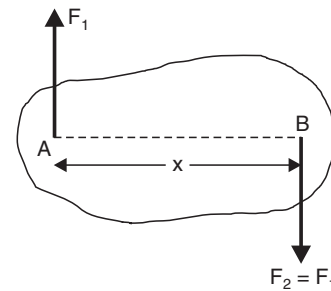


Fig. 4.1 (a)

2. Three Force System. The three forces acting on a body which is in equilibrium may be either concurrent or parallel. Let us first consider that the body is in equilibrium when three forces, acting on the body, are concurrent. This is shown in Fig. 4.2.

(a) *When three forces are concurrent.* The three concurrent forces F_1 , F_2 and F_3 are acting on a body at point O and the body is in equilibrium. The resultant of F_1 and F_2 is given by R . If the force F_3 is collinear, equal and opposite to the resultant R , then the body will be in equilibrium. The force F_3 which is equal and opposite to the resultant R is known as *equilibrant*. Hence for three concurrent forces acting on a body when the body is in equilibrium, the resultant of the two forces should be equal and opposite to the third force.

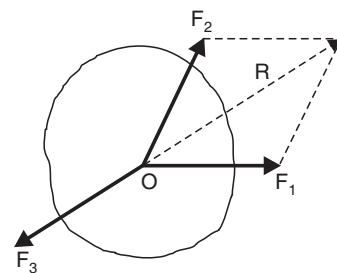


Fig. 4.2

(b) *When three forces are parallel.* Fig. 4.3 shows a body on which three parallel forces F_1 , F_2 and F_3 are acting and the body is in equilibrium. If three forces F_1 , F_2 and F_3 are acting in

the same direction, then there will be a resultant $R = F_1 + F_2 + F_3$ and body will not be in equilibrium. The three forces are acting in opposite direction and their magnitude is so adjusted that there is no resultant force and body is in equilibrium. Let us suppose that F_2 is acting in opposite direction as shown in Fig. 4.3.

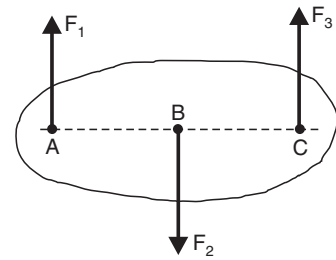


Fig. 4.3

Now let us apply the three conditions of equilibrium :

(i) $\Sigma F_x = 0$ as there is no horizontal force acting on the body

(ii) $\Sigma F_y = 0$ i.e., $F_1 + F_3 = F_2$

(iii) $\Sigma M = 0$ about any point.

Taking the moments of F_1 , F_2 and F_3 about point A,

$$\Sigma M_A = -F_2 \times AB + F_3 \times AC$$

(Moment of F_3 is anti-clockwise whereas moment of F_2 is clockwise)

For equilibrium, ΣM_A should be zero

$$\text{i.e.,} \quad -F_2 \times AB + F_3 \times AC = 0$$

If the distances AB and AC are such that the above equation is satisfied, then the body will be in equilibrium under the action of three parallel forces.

3. Four Force System. The body will be in equilibrium if the resultant force in horizontal direction is zero (i.e., $\Sigma F_x = 0$), resultant force in vertical direction is zero (i.e., $\Sigma F_y = 0$) and moment of all forces about any point in the plane of forces is zero (i.e., $\Sigma M = 0$).

Problem 4.1. Two forces F_1 and F_2 are acting on a body and the body is in equilibrium. If the magnitude of the force F_1 is 100 N and its acting at O along x-axis as shown in Fig. 4.4, then determine the magnitude and direction of force F_2 .

Sol. Given :

Force, $F_1 = 100$ N

The body is in equilibrium under the action of two forces F_1 and F_2 .

When two forces are acting on a body and the body is in equilibrium, then the two forces should be collinear, equal and opposite.

$$\therefore F_2 = F_1 = 100 \text{ N}$$

The force F_2 should pass through O, and would be acting in the opposite direction of F_1 .

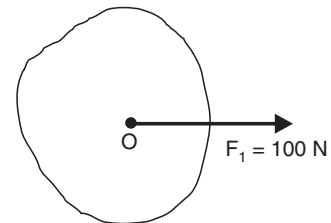


Fig. 4.4

Problem 4.2. Three forces F_1 , F_2 and F_3 are acting on a body as shown in Fig. 4.5 and the body is in equilibrium. If the magnitude of force F_3 is 400 N, find the magnitudes of force F_1 and F_2 .

Sol. Given :

Force, $F_3 = 400$ N.

As the body is in equilibrium, the resultant force in x-direction should be zero and also the resultant force in y-direction should be zero.

1st Method

(i) For $\Sigma F_x = 0$, we get

$$F_1 \cos 30^\circ - F_2 \cos 30^\circ = 0$$

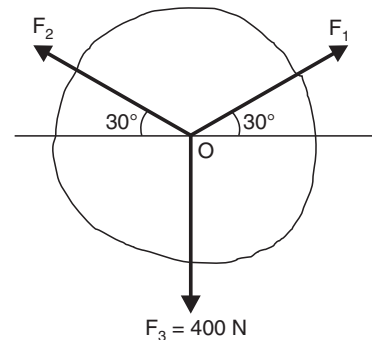


Fig. 4.5

or $F_1 - F_2 = 0$

or $F_1 = F_2$... (i)

(ii) For $\Sigma F_y = 0$, we get

$$F_1 \sin 30^\circ + F_2 \sin 30^\circ - 400 = 0$$

or $F_1 \times 0.5 + F_2 \times 0.5 = 400$

or $F_1 \times 0.5 + F_1 \times 0.5 = 400$ ($\because F_2 = F_1$)

or $F_1 = 400 \text{ N. Ans.}$

Also

$$F_2 = F_1 = 400 \text{ N. Ans.}$$

2nd Method

If three forces are acting on a body at a point and the body is in equilibrium, Lami's Theorem can be applied.

Using Lami's theorem,

$$\frac{F_1}{\sin 120^\circ} = \frac{F_2}{\sin 120^\circ} = \frac{400}{\sin 120^\circ}$$

or $F_1 = F_2 = 400 \text{ N. Ans.}$

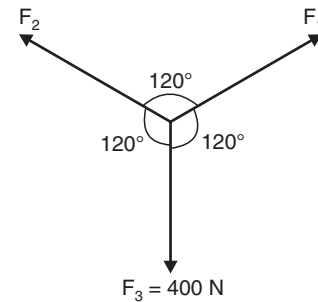


Fig. 4.5 (a)

Problem 4.3. Three parallel forces F_1 , F_2 and F_3 are acting on a body as shown in Fig. 4.6 and the body is in equilibrium. If force $F_1 = 250 \text{ N}$ and $F_3 = 1000 \text{ N}$ and the distance between F_1 and F_2 is 1.0 m , then determine the magnitude of force F_2 and the distance of F_2 from force F_3 .

Sol. Given :

Force, $F_1 = 250 \text{ N}$

Force, $F_3 = 1000 \text{ N}$

Distance, $AB = 1.0 \text{ m}$

The body is in equilibrium.

Find F_2 and distance BC .

For the equilibrium of the body, the resultant force in the vertical direction should be zero (here there is no force in horizontal direction).

$\therefore$ For $\Sigma F_y = 0$, we get

$$F_1 + F_3 - F_2 = 0$$

or $250 + 1000 - F_2 = 0$

or $F_2 = 250 + 1000 = 1250 \text{ N. Ans.}$

For the equilibrium of the body, the moment of all forces about any point must be zero.

Taking moments of all forces about point A and considering distance $BC = x$, we get

$$F_2 \times AB - AC \times F_3 = 0$$

or $1250 \times 1.0 - (1 + x) \times 1000 = 0$

($\because AC = AB + BC = 1 + x$)

or $1250 - 1000 - 1000x = 0$

or $250 = 1000x$

or $x = \frac{250}{1000} = 0.25 \text{ m. Ans.}$

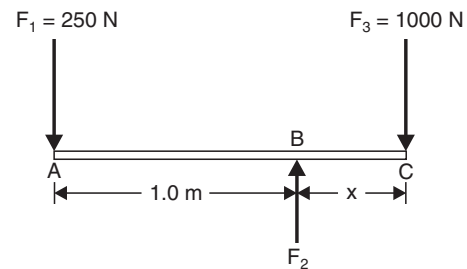


Fig. 4.6

Problem 4.3 (A). The frictionless pulley A shown in Fig. 4.6 (a) is supported by two bars AB and AC which are hinged at B and C to a vertical wall. The flexible cable DG hinged at D

goes over the pulley and supports a load of 20 kN at G. The angles between various members are shown in figure. Determine the forces in AB and AC. Neglect the size of pulley.

(U.P. Tech. University, 2001–2002)

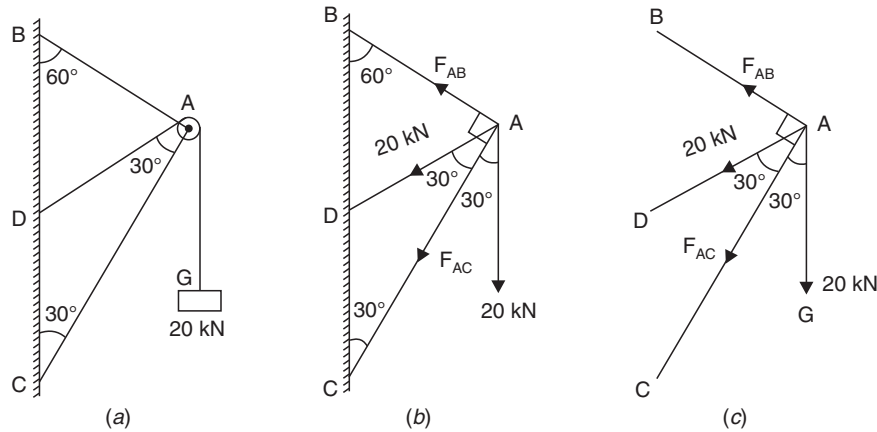


Fig. 4.6

Sol. As the pulley A is frictionless, the tension in the flexible cable DA will be equal to load 20 kN. Also if the size of the pulley is neglected, then AB, AD, AC and AG will meet at one point i.e. will meet at A as shown in Fig. 4.6 (b) and 4.6 (c).

The angle $BAC = 90^\circ$, $\angle CDA = 120^\circ$, $\angle ADB = 180 - 120 = 60^\circ$; $\angle ABD = 60^\circ$.

Let F_{AB} = Force in bar AB and
 F_{AC} = Force in bar AC.

Four forces are meeting at point A. The two forces F_{AB} and F_{AC} are at right angles. The remaining two forces i.e. $F_{AD} = 20$ kN and $F_{AG} = 20$ kN can be resolved along AB and AC. The components of F_{AD} along AB and AC are $20 \sin 30^\circ$ and $20 \cos 30^\circ$ respectively. Similarly the components of force AG along AC and AB are $20 \cos 30^\circ$ and $20 \sin 30^\circ$ respectively. The component along AB is in opposite direction.

For equilibrium of point A, the resultant force in any direction should be zero. Resolving all forces along AB, we get

$$F_{AB} + 20 \sin 30^\circ - 20 \sin 30^\circ = 0$$

or $F_{AB} = 0$. **Ans.**

Now resolving all forces along AC, we get

$$F_{AC} + 20 \cos 30^\circ + 20 \cos 30^\circ = 0$$

$$\therefore F_{AC} = -20 \cos 30^\circ - 20 \cos 30^\circ = -40 \cos 30^\circ$$

$$= -40 \times 0.866 = -34.64 \text{ N. } \mathbf{Ans.}$$

Negative sign shows that force in F_{AC} will be in opposite direction. This means force F_{AC} will be acting towards A.

Problem 4.4. The five forces F_1 , F_2 , F_3 , F_4 and F_5 are acting at a point on a body as shown in Fig. 4.7 and the body is in equilibrium. If $F_1 = 18$ N, $F_2 = 22.5$ N, $F_3 = 15$ N and $F_4 = 30$ N, find the force F_5 in magnitude and direction.

Sol. Given :

Forces, $F_1 = 18$ N, $F_2 = 22.5$ N,
 $F_3 = 15$ N and $F_4 = 30$ N.

The body is in equilibrium. Find force F_5 in magnitude and direction. This problem can be solved analytically and graphically.

1. Analytical Method

Let θ = Angle made by force F_5 with horizontal axis $O-X'$.

As the body is in equilibrium, the resultant force in x -direction and y -direction should be zero.

(i) For $\Sigma F_x = 0$, we get

$$\begin{aligned} F_1 + F_2 \cos 45^\circ - F_4 \cos 30^\circ - F_5 \cos \theta &= 0 \\ \text{or } 18 + 22.5 \times 0.707 - 30 \times 0.866 - F_5 \cos \theta &= 0 \\ \text{or } 18 + 15.9 - 25.98 - F_5 \cos \theta &= 0 \\ \text{or } F_5 \cos \theta &= 18 + 15.9 - 25.98 \\ \text{or } F_5 \cos \theta &= 7.92 \quad \dots(i) \end{aligned}$$

(ii) For $\Sigma F_y = 0$, we get

$$\begin{aligned} F_2 \sin 45^\circ + F_3 - F_4 \sin 30^\circ - F_5 \sin \theta &= 0 \\ \text{or } 22.5 \times 0.707 + 15 - 30 \times 0.5 - F_5 \sin \theta &= 0 \\ \text{or } 15.9 + 15 - 15 - F_5 \sin \theta &= 0 \\ \text{or } F_5 \sin \theta &= 15.9 \quad \dots(ii) \end{aligned}$$

Dividing equation (ii) by equation (i), we get $\frac{F_5 \sin \theta}{F_5 \cos \theta} = \frac{15.9}{7.92}$ or $\tan \theta = 2.0075$

$$\therefore \theta = \tan^{-1} 2.0075 = 63.52^\circ. \quad \text{Ans.}$$

Substituting the value of θ in equation (i), we get

$$F_5 \cos 63.52^\circ = 7.92$$

$$\therefore F_5 = \frac{7.92}{\cos 63.52^\circ} = 17.76 \text{ N.} \quad \text{Ans.}$$

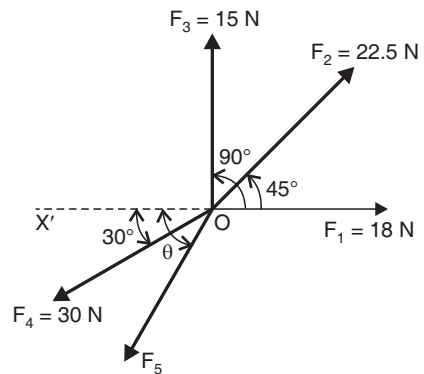


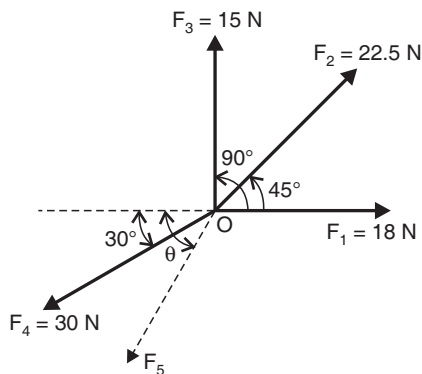
Fig. 4.7

2. Graphical Method

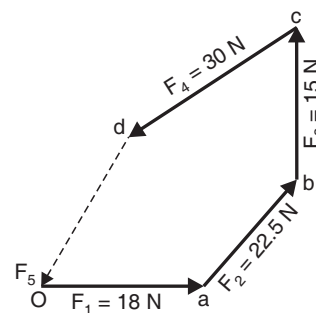
(i) First draw a space diagram with given four forces F_1, F_2, F_3 and F_4 at correct angles as shown in Fig. 4.8 (a).

(ii) Now choose a suitable scale, say 1 cm = 5 N for drawing a force diagram. Take any point O in the force diagram as shown in Fig. 4.8 (b).

(iii) Draw line Oa parallel to force F_1 and cut $Oa = F_1 = 18$ N to the same scale.



(a) Space diagram



(b) Force diagram

Fig. 4.8

- (iv) From a , draw the line ab parallel to F_2 and cut $ab = F_2 = 22.5$ N
- (v) From b , draw the line bc parallel to F_3 and cut $bc = F_3 = 15$ N
- (vi) From c , draw the line cd parallel to F_4 and cut $cd = F_4 = 30$ N
- (vii) Now join d to O . Then the closing side dO represents the force F_5 in magnitude and direction. Now measure the length dO .

By measurement, length $dO = 3.55$ cm.

$\therefore$ Force $F_5 = \text{Length } dO \times \text{Scale} = 3.55 \times 5 = \mathbf{17.75 \text{ N. Ans.}}$

The direction is obtained in the space diagram by drawing the force F_5 parallel to line dO .

Measure the angle θ , which is equal to 63.5° . Or the force F_5 is making an angle of $180^\circ + 63.5^\circ = 243.5^\circ$ with the force F_1 .

Problem 4.5. Fig. 4.8 (c) shows the coplanar system of forces acting on a flat plate. Determine : (i) the resultant and (ii) x and y intercepts of the resultant.

Sol. Given :

Force at $A = 2240$ N.

Angle with x -axis = 63.43°

Force at $B = 1805$ N.

Angle with x -axis = 33.67°

Force at $C = 1500$ N.

Angle with x -axis = 60°

Lengths $OA = 4$ m,

$DB = 3$ m,

$DC = 2$ m

and $OD = 3$ m.

Each force is resolved into X and Y components as shown in Fig. 4.8 (d).

(i) Force at $A = 2240$ N.

Its X -component = $2240 \times \cos 63.43^\circ = 1001.9$ N,

Its Y -component = $2240 \times \sin 63.43^\circ = 2003.4$ N

(ii) Force at $B = 1805$ N.

X -component = $1805 \times \cos 33.67^\circ = 1502.2$ N

Y -component = $1805 \times \sin 33.67^\circ = 1000.7$ N

(iii) Force at $C = 1500$ N.

X -component = $1500 \times \cos 60^\circ = 750$ N

Y -component = $1500 \times \sin 60^\circ = 1299$ N

The net force along X -axis,

$$R_x = \Sigma F_x = 1001.9 - 1502.2 - 750 = -1250.3 \text{ N}$$

The net force along Y -axis,

$$R_y = \Sigma F_y = -2003.4 - 1000.7 + 1299 = -1705.1 \text{ N}$$

(i) The resultant force is given by,

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2} = \sqrt{(-1250.3)^2 + (-1705.1)^2} \\ &= \sqrt{1563250 + 2907366} = \mathbf{2114.4 \text{ N. Ans.}} \end{aligned}$$

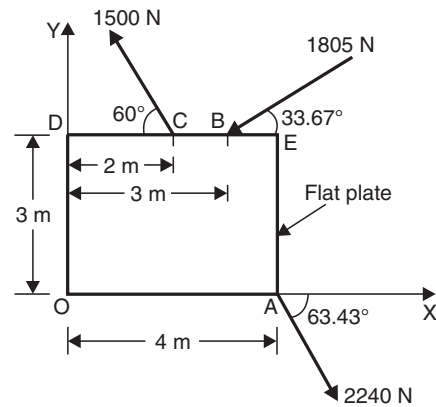


Fig. 4.8 (c)

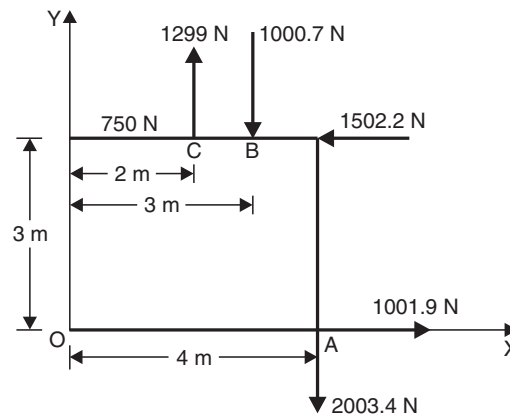


Fig. 4.8 (d)

The angle made by the resultant with x -axis is given by

$$\tan \theta = \frac{R_y}{R_x} = \frac{-1705.1}{-1250.3} = 1.363$$

$$\therefore \theta = \tan^{-1} 1.363 = 53.70$$

The net moment* about point O ,

$$\begin{aligned} M_0 &= 2003.4 \times 4 + 1000.7 \times 3 - 1299 \times 2 - 1502.2 \times 3 - 750 \times 3 \\ &= 8012.16 + 3002.1 - 2598 - 4506.6 - 2250 \\ &= 11014.26 - 9354.6 = 1659.55 \text{ Nm} \end{aligned}$$

As the net moment about O is clockwise, hence the resultant must act towards right of origin O , making an angle $= 53.7^\circ$ with x -axis as shown in Fig. 4.8 (e). The components R_x and R_y are also negative. Hence this condition is also satisfied.

(ii) *Intercepts of resultant on x -axis and y -axis*
[Refer to Fig. 4.8 (e)].

Let x = Intercept of resultant along x -axis.

y = Intercept of resultant along y -axis.

The moment of a force about a point is equal to the sum of the moments of the components of the force about the same point. Resolving the resultant (R) into its component R_x and R_y at F .

Moment of R about O = Sum of moments of R_x and R_y at O

But moment of R about O = 1659.66

$$(M_0 = 1659.66)$$

$$\therefore 1659.66 = R_x \times 0 + R_y \times x$$

(as R_x at F passes through O hence it has no moment)

$$\therefore 1659.66 = 1705.1 \times x$$

$$(\because R_y = 1705.1)$$

$$\therefore x = \frac{1659.66}{1705.1} = 0.97 \text{ m right of } O. \quad \text{Ans.}$$

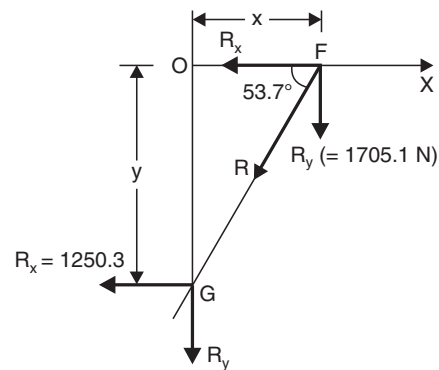


Fig. 4.8 (e)

*Considering clockwise moment positive and anti-clockwise moment as negative. At A , the X component of 1001.9 N passes through O and hence has no moment.

To find y -intercept, resolve the resultant R at G into its component R_x and R_y .

$\therefore$ Moment of R about O = Sum of moments of R_x and R_y at O

or $1659.66 = R_x \times y + R_y \times O.$

(At G , R_y passes through O and hence has no moment)

$\therefore 1659.66 = 1250.3 \times y$

$\therefore y = \frac{1659.66}{1250.30} = 1.32 \text{ m below } O. \quad \text{Ans.}$

Problem 4.6. A lamp weighing 5 N is suspended from the ceiling by a chain. It is pulled aside by a horizontal cord until the chain makes an angle of 60° with the ceiling as shown in Fig. 4.9. Find the tensions in the chain and the cord by applying Lami's theorem and also by graphical method.

Sol. Given :

Weight of lamp = 5 N

Angle made by chain with ceiling = 60°

Cord is horizontal as shown in Fig. 4.9.

(i) By Lami's theorem

Let T_1 = Tension (or pull) in the cord

T_2 = Tension (or pull) in the chain.

Now from the geometry, it is obvious that angles between T_1 and lamp will be 90° , between lamp and T_2 is 150° and between T_2 and T_1 is 120° . [Refer to Fig. 4.9 (b)].

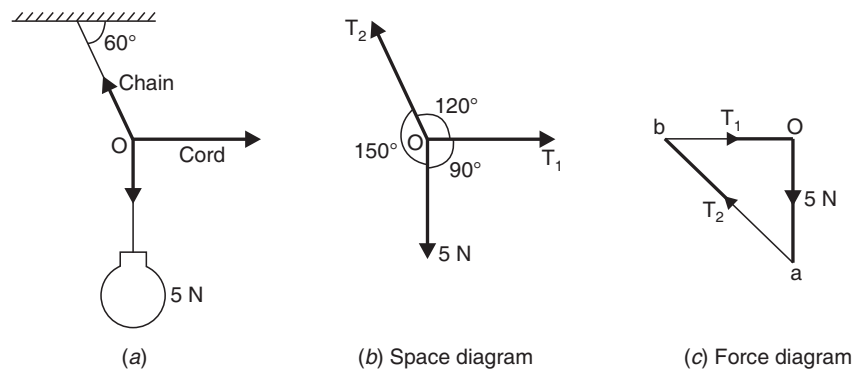


Fig. 4.9

Applying Lami's theorem, we get

$$\frac{T_1}{\sin 150^\circ} = \frac{T_2}{\sin 90^\circ} = \frac{5}{\sin 120^\circ}$$

$\therefore T_1 = 5 \times \frac{\sin 150^\circ}{\sin 120^\circ} = 2.887 \text{ N.} \quad \text{Ans.}$

and $T_2 = 5 \times \frac{\sin 90^\circ}{\sin 120^\circ} = 5.774 \text{ N.} \quad \text{Ans.}$

(ii) By Graphical method

(1) First draw the space diagram at correct angles as shown in Fig. 4.9 (b). Now choose a suitable scale say $1 \text{ cm} = 1 \text{ N}$ for drawing a force diagram as shown in Fig. 4.9 (c). Take any point O in the force diagram.

(2) From O , draw the line Oa vertically downward to represent the weight of the lamp. Cut $Oa = 5$ N.

(3) From a , draw the line ab parallel to T_2 . The magnitude of T_2 is unknown. Now from O , draw the line Ob horizontally (*i.e.*, parallel to T_1) cutting the line ab at point b .

(4) Now measure the lengths ab and bO .

Then ab represents T_2 and bO represents T_1 . By measurements, $ab = 5.77$ cm and $bO = 2.9$ cm.

$\therefore$ Pull in the cord $= bO = 2.9 \text{ cm} \times \text{scale} = 2.9 \times 1$
 $= 2.9 \text{ N. Ans.}$

Pull in the chain $= ab = 5.77 \text{ cm} \times \text{scale} = 5.77 \times 1$
 $= 5.77 \text{ N. Ans.}$

Problem 4.7. On a horizontal line $PQRS$ 12 cm long, where $PQ = QR = RS = 4$ cm, forces of 1000, 1500, 1000 and 500 N are acting at P , Q , R and S respectively, all downwards, their lines of action making angles of 90, 60, 45 and 30 degrees respectively with PS . Obtain the resultant of the system completely in magnitude, direction and position graphically and check the answer analytically.

Sol. Given :

$$PQ = QR = RS = 4 \text{ cm}$$

Force at $P = 1000$ N. Angle with $PS = 90^\circ$

Force at $Q = 1500$ N. Angle with $QS = 60^\circ$

Force at $R = 1000$ N. Angle with $RS = 45^\circ$

Force at $S = 500$ N. Angle with $PS = 30^\circ$

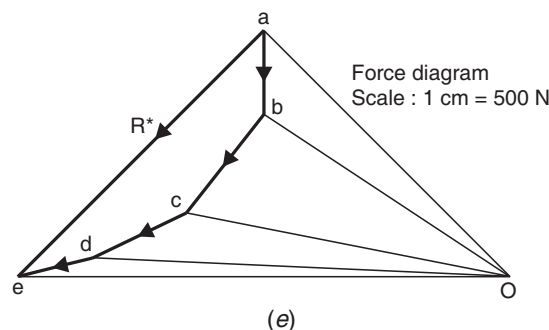
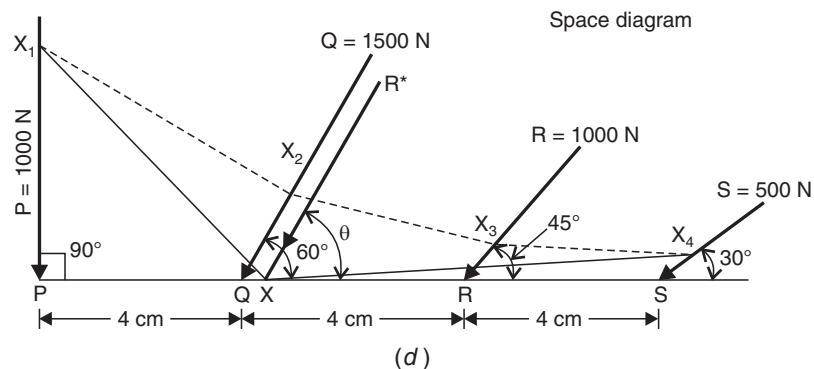


Fig. 4.9

Graphical Method

Draw the space diagram of the forces as shown in Fig. 4.9 (d). The procedure is as follows :

- (i) Draw a horizontal line $PQRS = 12$ cm in which $PQ = QR = RS = 4$ cm.
- (ii) Draw the line of action of forces P, Q, R, S of magnitude 1000 N, 1500 N, 1000 N and 500 N respectively at an angle of $90^\circ, 60^\circ, 45^\circ$ and 30° respectively with line PS as shown in Fig. 4.9 (d).

Magnitude and Direction of Resultant Force (R^*)

To find the magnitude and direction of the resultant force, the force diagram is drawn as shown in Fig. 4.9 (e) as given above :

- (i) Draw the vector ab to represent the force 1000 N to a scale of $1 \text{ cm} = 500 \text{ N}$. The vector ab is parallel to the line of action of force P .
 - (ii) From point b , draw vector, $bc = 1500 \text{ N}$ and parallel to the line of action of force Q . Similarly the vectors, $cd = 1000 \text{ N}$ and parallel to line of action of force R and $de = 500 \text{ N}$ and parallel to the line of action of force S , are drawn.
 - (iii) Join ae which gives the magnitude of the resultant. Measuring ae , the resultant force is equal to 3770 N.
 - (iv) To get the line of action of the resultant, choose any point O on force diagram (called the pole) and join Oa, Ob, Oc, Od and Oe .
 - (v) Now choose any point X_1 on the line of action of force P and draw a line parallel to Oa .
 - (vi) Also from the point X_1 , draw another line parallel to Ob , which cuts the line of action of force Q at X_2 . Similarly from point X_2 , draw a line parallel to Oc to cut the line of action of force R at X_3 . From point X_3 , draw a line parallel to Od to cut the line of action of force S at X_4 .
 - (vii) From point X_4 , draw a line parallel to Oe .
 - (viii) Produce the first line (*i.e.*, the line from X_1 and parallel to Oa) and the last line (*i.e.*, the line from X_4 and parallel to Oe) to intersect at X . Then the resultant must pass through this point.
 - (ix) From point X , draw a line parallel to ae which determines the line of action of resultant force. Measure PX . By measurements :
- Resultant force, $R^* = 3770 \text{ N}$
 Point of action, $PX = 4.20 \text{ cm}$
 Direction, $\theta = 60^\circ 30'$ with PS .

Analytical Method

In analytical method, all the forces acting can be resolved horizontally and vertically. Resultant of all vertical and horizontal forces can be calculated separately and then the final resultant can be obtained.

Resolving all forces and considering the system for vertical forces only.

Vertical force at $P = 1000 \text{ N}$

Vertical force at $Q = 1500 \sin 60^\circ = 1299 \text{ N}$

Vertical force at $R = 1000 \sin 45^\circ = 707 \text{ N}$

Vertical force at $S = 500 \sin 30^\circ = 250 \text{ N}$

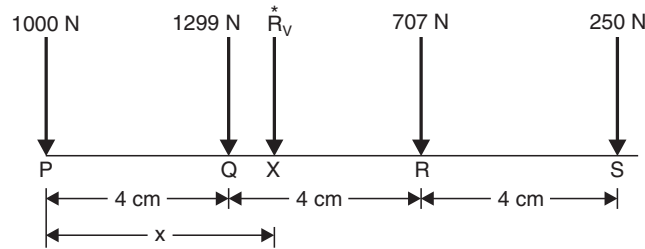


Fig. 4.9 (f)

Let R_V^* = the resultant of all vertical forces and acting at a distance x cm from P .
 $= 1000 + 1299 + 707 + 250 = 3256$ N

Taking moments of all vertical forces about point P ,

$$R_V^* \times x = 1299 \times 4 + 707 \times 8 + 250 \times 12 = 13852$$

$$\therefore x = \frac{13852}{R_V^*} = \frac{13852}{3256} = 4.25 \text{ cm}$$

Now consider the system for horizontal forces only

Horizontal force at $P = 0$

Horizontal force at $Q = 1500 \times \cos 60^\circ = 750$ N

Horizontal force at $R = 1000 \times \cos 45^\circ = 707$ N

Horizontal force at $S = 500 \times \cos 30^\circ = 433$ N

Resultant of all horizontal forces will be,

$$R_H^* = 0 + 750 + 707 + 433 = 1890 \text{ N}$$

The resultant R^* of R_V^* and R_H^* will also pass through point X which is at a distance of 4.25 cm from P .

$$\therefore R^* = \sqrt{R_V^{*2} + R_H^{*2}} = \sqrt{3256^2 + 1890^2} = 3764 \text{ N. Ans.}$$

The resultant will make an angle θ with PS and is given by

$$\tan \theta = \frac{R_V^*}{R_H^*} = \frac{3256}{1890} = 1.723$$

$$\therefore \theta = \tan^{-1} 1.723 = 59.9^\circ$$

Thus the resultant of 3764 N makes an angle 59.9° with PS and passing through point X which is at a distance of 4.25 cm from point P .

This result confirms closely with the values obtained by graphical method.

4.2. ACTION AND REACTION

From the Newton's third law of motion, we know that to every action there is equal and opposite reaction. Hence reaction is always equal and opposite to the action.

Fig. 4.10 (a) shows a ball placed on a horizontal surface (or horizontal plane) such that it is free to move along the plane but cannot move vertically downward. Hence the ball will exert a force vertically downwards at the support as shown in Fig. 4.10 (b). This force is known as *action*. The support will exert an equal force vertically upwards on the ball at the point of contact as shown in Fig. 4.10 (c).

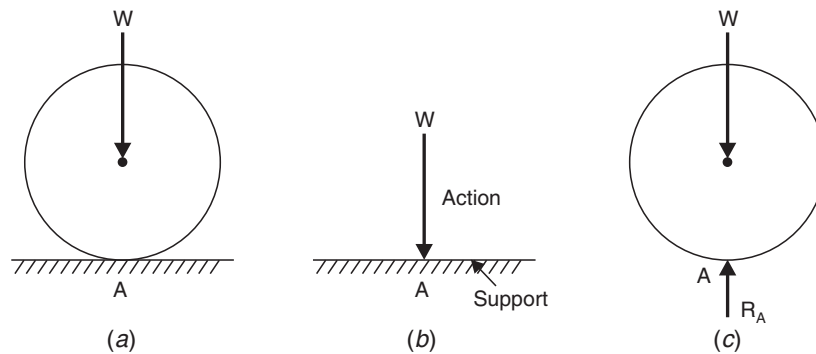


Fig. 4.10

The force, exerted by the support on the ball, is known as *reaction*. Hence ‘any force on a support causes an equal and opposite force from the support so that action and reaction are two equal and opposite forces’.

4.3. FREE BODY DIAGRAMS

The equilibrium of the bodies which are placed on the supports can be considered if we remove the supports and replace them by the reactions which they exert on the body. In Fig. 4.10 (a), if we remove the supporting surface and replace it by the reaction R_A that the surface exerts on the balls as shown in Fig. 4.10 (c), we shall get *free-body diagram* which is written as *FBD*.

The point of application of the reaction R_A will be the point of contact A, and from the law of equilibrium of two forces, we conclude that the reaction R_A must be vertical and equal to the weight W .

Hence Fig. 4.10 (c), in which the ball is *completely isolated* from its support and in which all forces acting on the ball are shown by vectors, is known a free-body diagram. Hence to draw the free-body diagram of a body we remove all the supports (like wall, floor, hinge or any other body) and replace them by the reactions which these support exert on the body. Also the body should be completely isolated.

Problem 4.8. Draw the free body diagram of ball of weight W supported by a string AB and resting against a smooth vertical wall at C as shown in Fig. 4.11 (a).

Sol. Given :

Weight of ball = W

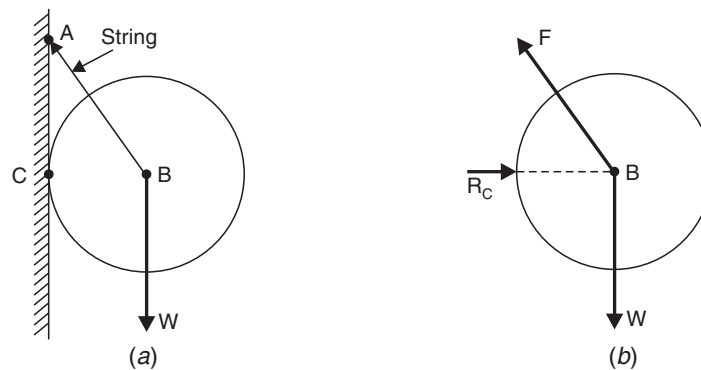


Fig. 4.11

The ball is supported by a string AB and is resting against a vertical wall at C .

To draw the free-body diagram of the ball, isolate the ball completely (*i.e.*, isolate the ball from the support and string). Then besides the weight W acting at B , we have two reactive forces to apply one replacing the string AB and another replacing the vertical wall AC . Since the string is attached to the ball at B and since a string can pull only along its length, we have the reactive force F applied at B and parallel to BA . The magnitude of F is unknown.

The reaction R_C will be acting at the point of contact of the ball with vertical wall *i.e.*, at point C . As the surface of the wall is *perfectly smooth**, the reaction R_C will be normal to the vertical wall (*i.e.*, reaction R_C will be horizontal in this case) and will pass through the point B . The magnitude of R_C is also unknown. The complete free-body diagram is shown in Fig. 4.11 (*b*).

Problem 4.9. A circular roller of weight 100 N and radius 10 cm hangs by a tie rod $AB = 20\text{ cm}$ and rests against a smooth vertical wall at C as shown in Fig. 4.12 (*a*). Determine : (i) the force F in the tie rod, and (ii) the reaction R_C at point C .

Sol. Given :

Weight of roller, $W = 100\text{ N}$

Radius of roller, $BC = 10\text{ cm}$

Length of tie rod, $AB = 20\text{ cm}$

From $\triangle ABC$, we get $\sin \theta = \frac{BC}{AB} = \frac{10}{20} = 0.5$

$\therefore \theta = \sin^{-1} 0.5 = 30^\circ$

The free-body diagram of the roller is shown in Fig. 4.12 (*b*) in which

R_C = Reaction at C

F = Force in the tie rod AB

Free-body diagram shows the equilibrium of the roller. Hence the resultant force in x -direction and y -direction should be zero.

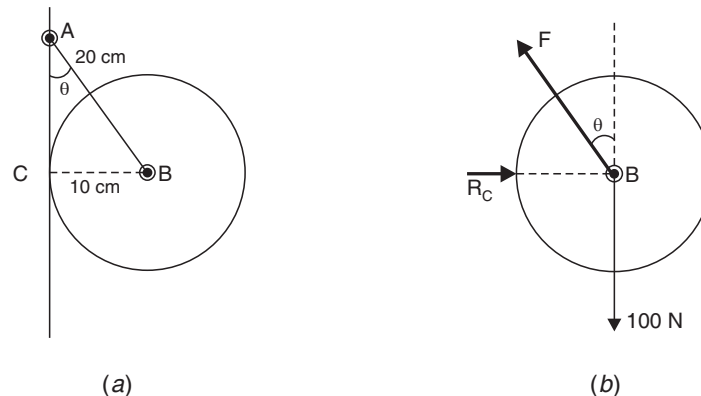


Fig. 4.12

For	$\Sigma F_x = 0$, we get $R_C - F \sin \theta = 0$	
or	$R_C = F \sin \theta$	...(i)
For	$\Sigma F_y = 0$, we get $100 - F \cos \theta = 0$	
or	$100 = F \cos \theta$	

* The reaction at a perfectly smooth surface is always normal to the surface.

or
$$F = \frac{100}{\cos \theta} = \frac{100}{\cos 30^\circ} \quad (\because \theta = 30^\circ)$$

$$= 115.47 \text{ N. Ans.}$$

Substituting the value of F in equation (i),

$$R_C = 115.47 \times \sin 30^\circ = 57.73 \text{ N. Ans.}$$

Problem 4.10. Draw the free-body diagram of a ball of weight W , supported by a string AB and resting against a smooth vertical wall at C and also resting against a smooth horizontal floor at D as shown in Fig. 4.13 (a).

Sol. Given :

To draw the free-body diagram of the ball, the ball should be isolated completely from the vertical support, horizontal support and string AB . Then the forces acting on the isolated ball as shown in Fig. 4.13 (b), will be :

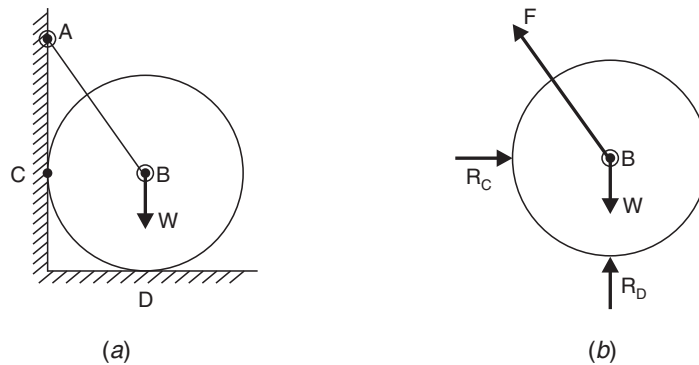


Fig. 4.13

- (i) Reaction R_C at point C , normal to AC .
- (ii) Force F in the direction of string.
- (iii) Weight W of the ball.
- (iv) Reaction R_D at point D , normal to horizontal surface.

The reactions R_C and R_D will pass through the centre of the ball i.e., through point B .

Problem 4.11. A ball of weight 120 N rests in a right-angled groove, as shown in Fig. 4.14 (a). The sides of the groove are inclined to an angle of 30° and 60° to the horizontal. If all the surfaces are smooth, then determine the reactions R_A and R_C at the points of contact.

Sol. Given :

Weight of ball, $W = 120 \text{ N}$

Angle of groove $= 90^\circ$

Angle made by side FD with horizontal $= 30^\circ$

Angle made by side ED with horizontal $= 60^\circ$

$\therefore$ Angle $FDH = 30^\circ$ and angle $EDG = 60^\circ$

Consider the equilibrium of the ball. For this draw the free body diagram of the ball as shown in Fig. 4.14(b).

The forces acting on the isolated ball will be :

- (i) Weight of the ball $= 120 \text{ N}$ and acting vertically downwards.

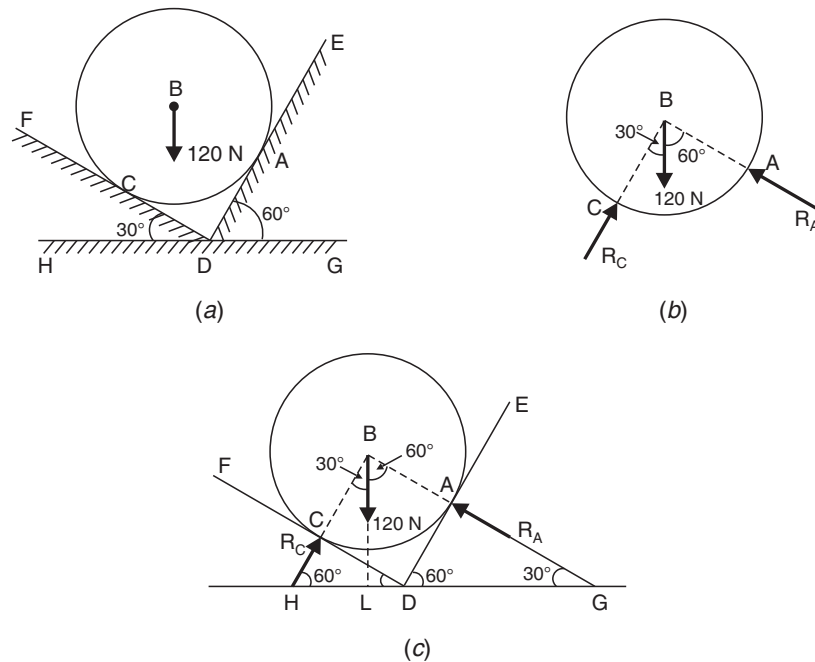


Fig. 4.14

(ii) Reaction R_C acting at C and normal to FD .

(iii) Reaction R_A acting at A and normal to DE .

The reactions R_A and R_C will pass through B , i.e., centre of the ball. The angles made by R_A and R_C at point B will be obtained as shown in Fig. 4.14 (c).

In $\triangle HDC$, $\angle CDH = 30^\circ$ and $\angle DCH = 90^\circ$. Hence $\angle DHC$ will be 60° . Now in $\triangle HBL$, $\angle BLH = 90^\circ$ and angle $LHB = 60^\circ$. Hence $\angle HBL$ will be 30° .

Similarly, $\angle GBL$ may be calculated. This will be equal to 60° .

For the equilibrium of the ball,

$$\Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

$$\text{For } \Sigma F_x = 0, \text{ we have } R_C \sin 30^\circ - R_A \sin 60^\circ = 0$$

$$\text{or } R_C \sin 30^\circ = R_A \sin 60^\circ$$

$$\text{or } R_C = R_A \times \frac{0.866}{\sin 30^\circ} = 1.732 R_A \quad \dots(i)$$

$$\text{For } \Sigma F_y = 0, \text{ we have } 120 - R_A \cos 60^\circ - R_C \cos 30^\circ = 0$$

$$\text{or } 120 = R_A \cos 60^\circ + R_C \cos 30^\circ$$

$$= R_A \times 0.5 + (1.732 R_A) \times 0.866$$

$$= 0.5 R_A + 1.5 R_A = 2 R_A$$

$$\therefore R_A = \frac{120}{2} = 60 \text{ N. Ans.}$$

Substituting this value in equation (i), we get

$$R_C = 1.732 \times 60 = 103.92 \text{ N. Ans.}$$

Problem 4.11 (A). Fig. 4.14 (d) shows a sphere resting in a smooth V shaped groove and subjected to a spring force. The spring is compressed to a length of 100 mm from the free length of 150 mm. If the stiffness of the spring is 2 N/mm, determine the contact reaction at A and B.

(U.P. Tech. University 2002–2003)

Sol. Given :

Weight of sphere, $W = 40 \text{ N}$

Deflection of spring

$$= \text{Free length} - \text{Compressed length}$$

$$= 150 - 100 = 50 \text{ mm}$$

Stiffness of spring

$$= 2 \text{ N/mm}$$

$\therefore$ Spring force

$$= (\text{Stiffness of spring}) \times \text{Deflection}$$

$$= 2 \text{ N/mm} \times 50 \text{ mm} = 100 \text{ N}$$

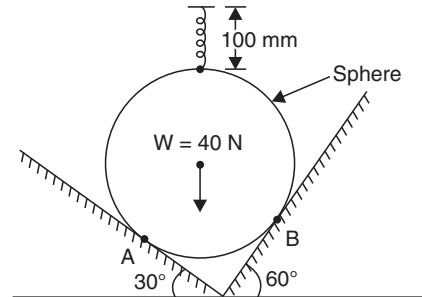


Fig. 4.14 (d)

As the spring is compressed, hence the above force (100 N) will also be acting on the sphere vertically downward direction. This force will be passing through the centre of the sphere.

$\therefore$ Total vertical load acting through the centre of sphere in the downward direction is $100 + 40 = 140 \text{ N}$.

Let $R_A = \text{Reaction at A}$

$R_B = \text{Reaction at B}$

As the groove is smooth the reactions R_A and R_B will be normal to inclined surfaces and will pass through the centre of the sphere as shown in Fig. 4.14 (e).

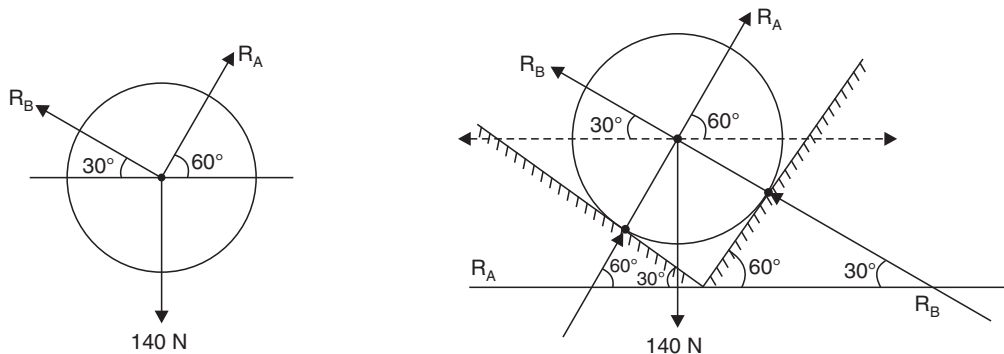


Fig. 4.14 (e)

For equilibrium, $\Sigma F_x = 0$ and $\Sigma F_y = 0$

$$\text{For } \Sigma F_x = 0, \quad R_A \cos 60^\circ - R_B \cos 30^\circ = 0$$

$$\text{or} \quad R_A = \frac{R_B \cos 30^\circ}{\cos 60^\circ} \quad \dots(i)$$

$$\text{For } \Sigma F_y = 0, \quad R_A \sin 60^\circ + R_B \sin 30^\circ - 140 = 0$$

Substituting the value of R_A from (i), in the above equation, we get

$$\left(R_B \times \frac{\cos 30^\circ}{\cos 60^\circ} \right) \times \sin 60^\circ + R_B \sin 30^\circ - 140 = 0$$

$$\text{or } R_B \times \cos 30^\circ \times \sin 60^\circ + R_B \cos 60^\circ \sin 30^\circ - 140 \times \cos 60^\circ = 0$$

$$\text{or } R_B (\cos 30^\circ \sin 60^\circ + \cos 60^\circ \sin 30^\circ) = 140 \times \frac{1}{2} = 70$$

$$\text{or } R_B \sin 90^\circ = 70$$

$$(\because \cos 30^\circ \sin 60^\circ + \cos 60^\circ \sin 30^\circ = \sin (60^\circ + 30^\circ) = \sin 90^\circ)$$

$$\therefore R_B = 70 \text{ N. Ans.} \quad (\because \sin 90^\circ = 1)$$

Substituting this value of R_B in equation (i), we get

$$R_A = 70 \times \frac{\cos 30^\circ}{\cos 60^\circ} = \frac{70 \times 0.866}{0.5} = 121.24 \text{ N. Ans.}$$

Problem 4.12. A circular roller of radius 5 cm and of weight 100 N rests on a smooth horizontal surface and is held in position by an inclined bar AB of length 10 cm as shown in Fig. 4.15. A horizontal force of 200 N is acting at B. Find the tension (or Force) in the bar AB and the vertical reaction at C.

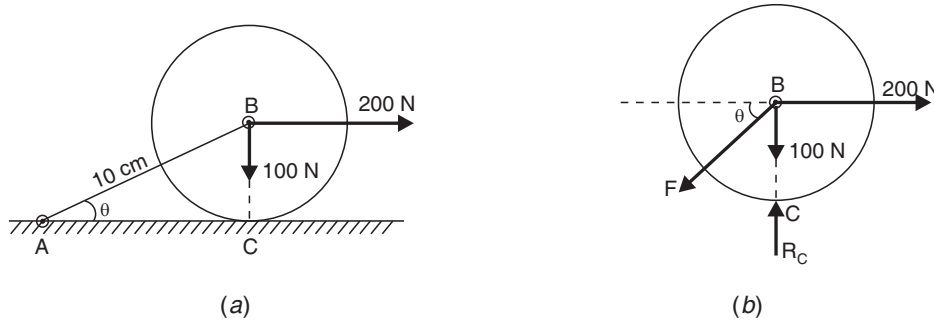


Fig. 4.15

Sol. Given :

Weight, $W = 100 \text{ N}$

Radius i.e., $BC = 5 \text{ cm}$

Length of bar, $AB = 10 \text{ cm}$

Horizontal force at $B = 200 \text{ N}$

$$\text{In } \triangle ABC, \quad \sin \theta = \frac{BC}{AB} = \frac{5}{10} = 0.5$$

$$\therefore \theta = \sin^{-1} 0.5 = 30^\circ$$

Let $F =$ Tension in the string AB.

Consider the equilibrium of the roller. For this draw the free body diagram of the roller as shown in Fig. 4.15 (b).

The reaction R_C at point C will pass through point B.

The tension (or force F) will be acting along the length of the string.

As the roller is in equilibrium in Fig. 4.15 (b), the resultant force in x -direction and y -direction should be zero.

$$\text{For } \Sigma F_x = 0, \text{ we have } F \cos \theta - 200 = 0$$

$$\therefore F = \frac{200}{\cos \theta} = \frac{200}{\cos 30^\circ} \quad (\because \theta = 30^\circ)$$

$$= 230.94 \text{ N. Ans.}$$

$$\text{For } \Sigma F_y = 0, \text{ we have } R_C - W - F \sin \theta = 0$$

or

$$R_C = W + F \sin \theta = 100 + 230.94 \times \sin 30^\circ \\ = 215.47 \text{ N. Ans.}$$

Problem 4.12 (A). A right circular roller of weight 5000 N rest on a smooth inclined plane and is held in position by a cord AC as shown in Fig. 4.15 (c). Find the tension in the cord if there is a horizontal force of magnitude 1000 N acting at C.

(U.P. Tech. University 2002–2003)

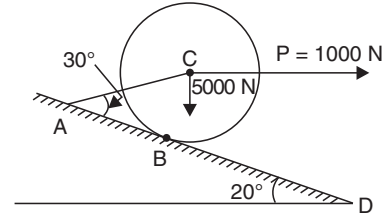


Fig. 4.15 (c)

Sol. Given : Weight, $W = 5000 \text{ N}$,

Angle of inclined plane = 20°

Roller is resting on smooth surface. Hence reaction R_B will be normal to inclined surface AD. Horizontal force, $P = 1000 \text{ N}$.

Angle of cord AC with inclined plane = 30°

The free-body diagram of the roller is shown in Fig. 4.15 (d) and 4.15 (e).

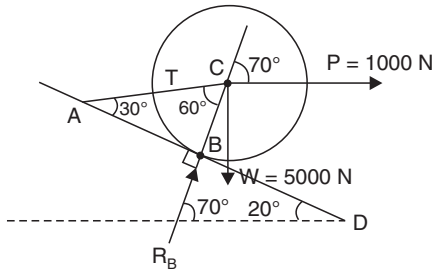


Fig. 4.15 (d)

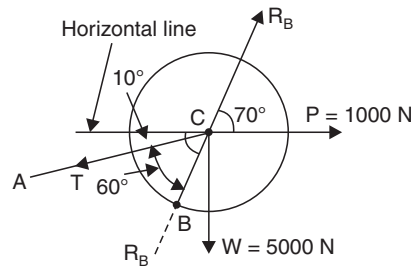


Fig. 4.15 (e)

The reaction R_B at point B is normal to inclined plane and will pass through point C.

Let T = Tension in string AC.

The forces acting on the roller are :

- (i) Weight of roller acting vertically downward through C.
- (ii) Horizontal force P acting through C.
- (iii) Reaction R_B , passing through C and making an angle of 70° with horizontal force P as shown in Fig. 4.15 (d) and 4.15 (e).
- (iv) The tension (T) in string AC, passing through C and making an angle of 10° with horizontal direction.

For the equilibrium of roller, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, we have

$$P + R_B \cos 70^\circ - T \cos 10^\circ = 0, \text{ But } P = 1000 \text{ N}$$

$$\therefore 1000 + R_B \times 0.342 - T \times 0.9848 = 0 \quad \dots(i)$$

For $\Sigma F_y = 0$, we have

$$R_B \sin 70^\circ - W - T \sin 10^\circ = 0, \quad \text{But } W = 5000 \text{ N}$$

$$\therefore R_B \times 0.9396 - 5000 - T \times 0.1736 = 0 \quad \dots(ii)$$

From equations (i) and (ii), the values of T and R_B can be obtained.

From equation (i), we have

$$R_B = \frac{T \times 0.9848 - 1000}{0.342} = \frac{0.9848}{0.342} T - \frac{1000}{0.342} = 2.88T - 2923.9$$

Substituting this value of R_B in equation (ii), we get

$$(2.88T - 2939.9) \times 0.9396 - 5000 - T \times 0.1736 = 0$$

or $2.88 \times 0.9396T - 2939.9 \times 0.9396 - 5000 - 0.1736T = 0$

or $2.706T - 2762.3 - 5000 - 0.1736T = 0$

or $(2.706 - 0.1736)T = 5000 + 2762.3$

or $2.5324T = 7762.3$

or $T = \frac{7762.3}{2.5324} = 3065.2 \text{ N. Ans.}$

Problem 4.13. Two identical rollers P and Q , each of weight W , are supported by an inclined plane and a vertical wall as shown in Fig. 4.16 (a). Assume all the surfaces to be smooth. Draw the free body diagrams of :

(i) roller Q , (ii) roller P and (iii) rollers P and Q taken together.

Sol. Given :

Weight of each roller = W

Radius of each roller = R

Identical rollers means the radius of each roller is same.

Hence the line EF in Fig. 4.16 (a) will be parallel to surface AB .

Each surface is smooth, hence reaction at the point of contact will be normal to the surface.

Let R_A = Reaction at point A

R_B = Reaction at point B

R_C = Reaction at point C

The two rollers are also in contact at point D . Hence there will be a reaction R_D at the point D .

(i) **Free-body diagram of roller Q .** To draw the free-body diagram of roller Q , isolate the roller Q completely and find the forces acting on the roller Q . The roller Q has points of contact at B , C and D . The forces acting on the roller Q will be :

(a) Weight of roller W .

(b) Reaction R_B at point B .

This will be normal to the surface BA at point B .

(c) Reaction R_C at point C . This will be normal to the vertical surface at point C .

(d) Reaction R_D at point D . This will be normal to the tangent at point D .

The reactions R_B , R_C and R_D will pass through the centre E of the roller Q . These three reactions are unknown.

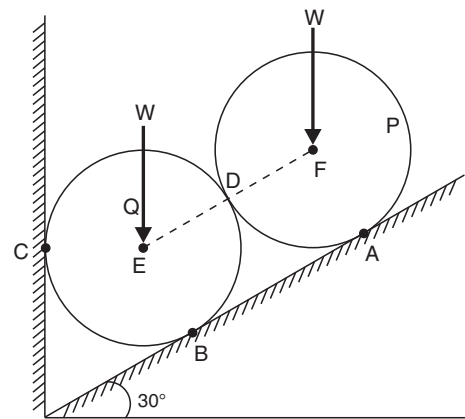


Fig. 4.16 (a)

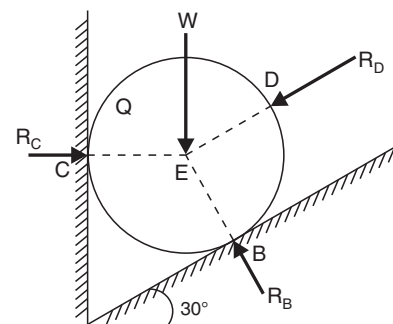


Fig. 4.16 (b)

(ii) **Free-body diagram of roller P.** Free-body diagram of roller P is shown in Fig. 4.16 (c). The roller P has points of contact at A and D . The forces acting on the roller P are :

- (a) Weight W
- (b) Reaction R_A at point A
- (c) Reaction R_D at point D .

The reactions R_A and R_D will pass through point F , i.e., centre of roller P . These two reactions are unknown. If W is given, then these reactions can be calculated.

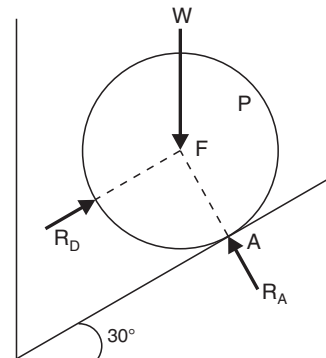


Fig. 4.16 (c)

(iii) **Free-body diagram of rollers P and Q taken together.** When the rollers P and Q are taken together, then points of contacts are A , B and C . The free-body diagram of this case is shown in Fig. 4.16 (d). The forces acting are :

- (a) Weight W on each roller
- (b) Reaction R_A at point A
- (c) Reaction R_B at point B
- (d) Reaction R_C at point C .

In this case there will be no reaction at point D .

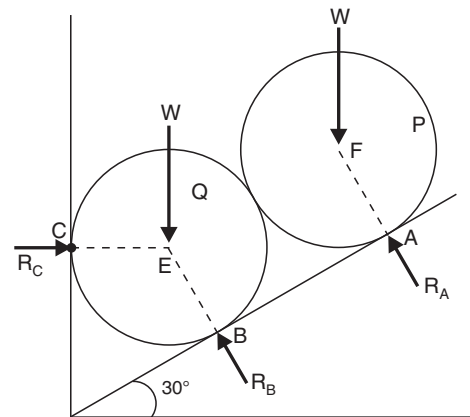


Fig. 4.16 (d)

Problem 4.14. Two identical rollers, each of weight $W = 1000 \text{ N}$, are supported by an inclined plane and a vertical wall as shown in Fig. 4.17 (a). Find the reactions at the points of supports A , B and C . Assume all the surfaces to be smooth.

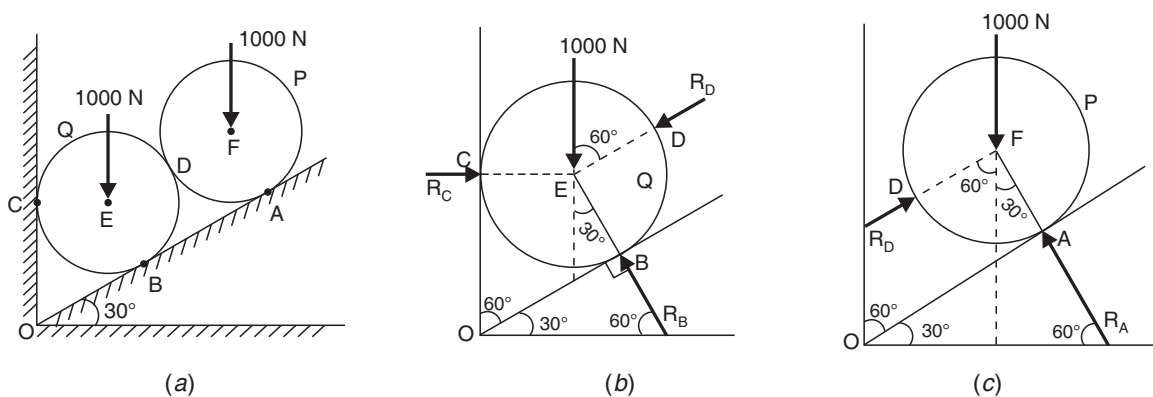


Fig. 4.17

Sol. Given :

Weight of each roller = 1000 N

Radius of each roller is same. Hence line EF will be parallel to AB .

Equilibrium of Roller P

First draw the free-body diagram of roller P as shown in Fig. 4.17 (c). The roller P has points of contact at A and D . Hence the forces acting on the roller P are :

- (i) Weight 1000 N acting vertically downward.
- (ii) Reaction R_A at point A . This is normal to OA .
- (iii) Reaction R_D at point D . This is parallel to line OA .

The resultant force in x and y directions on roller P should be zero.

For $\Sigma F_x = 0$, we have

$$R_D \sin 60^\circ - R_A \sin 30^\circ = 0 \quad \text{or} \quad R_D \sin 60^\circ = R_A \sin 30^\circ$$

$$\therefore R_D = R_A \frac{\sin 30^\circ}{\sin 60^\circ} = 0.577 R_A \quad \dots(i)$$

For $\Sigma F_y = 0$, we have

$$R_D \cos 60^\circ + R_A \cos 30^\circ - 1000 = 0$$

$$(0.577 R_A) \cos 60^\circ + R_A \cos 30^\circ = 1000 \quad (\because R_D = 0.577 R_A)$$

$$\text{or} \quad 0.577 \times 0.5 R_A + R_A \times 0.866 = 1000$$

$$1.1545 R_A = 1000 \quad \text{or} \quad R_A = \frac{1000}{1.1545} = 866.17 \text{ N. Ans.}$$

Substituting this value in equation (i), we get

$$R_D = 0.577 \times 866.17 = 499.78$$

Equilibrium of Roller Q

The free-body diagram of roller Q is shown in Fig. 4.17 (b). The roller Q has points of contact at B , C and D .

The forces acting on the roller Q are :

- (i) Weight $W = 1000$ N ;
- (ii) Reaction R_B at point B and normal to BO ;
- (iii) Reaction R_C at point C and normal to CO ; and
- (iv) Reaction R_D at point D and parallel to BO .

For $\Sigma F_x = 0$, we have

$$R_B \sin 30^\circ + R_D \sin 60^\circ - R_C = 0$$

$$\text{or} \quad R_B \times 0.5 + 499.78 \times 0.866 - R_C = 0$$

$$\text{or} \quad R_C = 0.5 R_B + 432.8 \quad \dots(ii)$$

For $\Sigma F_y = 0$, we have

$$R_B \times \cos 30^\circ - 1000 - R_D \times \cos 60^\circ = 0$$

$$\text{or} \quad R_B \times 0.866 - 1000 - 499.78 \times 0.5 = 0 \quad (\because R_D = 499.78)$$

$$\text{or} \quad 0.866 R_B - 1249.89 = 0 \quad \text{or} \quad R_B = \frac{1249.89}{0.866} = 1443.3 \text{ N. Ans.}$$

Substituting this value in equation (ii), we get

$$R_C = 0.5 \times 1443.3 + 432.8 = 1154.45 \text{ N. Ans.}$$

Problem 4.15. Two spheres, each of weight 1000 N and of radius 25 cm rest in a horizontal channel of width 90 cm as shown in Fig. 4.18. Find the reactions on the points of contact A, B and C.

Sol. Given :

Weight of each sphere, $W = 1000 \text{ N}$

Radius of each sphere, $R = 25 \text{ cm}$

$\therefore AF = BF = FD = DE = CE = 25 \text{ cm}$

Width of horizontal channel = 90 cm

Join the centre E to centre F as shown in Fig. 4.18(b).

Now $EF = 25 + 25 = 50 \text{ cm}$, $FG = 40 \text{ cm}$

In $\triangle EFG$, $EG = \sqrt{EF^2 - FG^2} = \sqrt{50^2 - 40^2} = \sqrt{2500 - 1600} = 30$

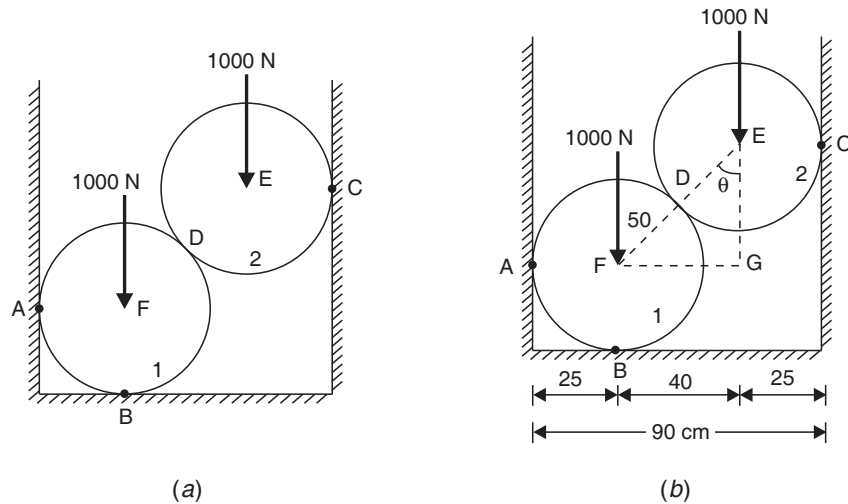


Fig. 4.18

$$\therefore \cos \theta = \frac{EG}{EF} = \frac{30}{50} = \frac{3}{5} \quad \text{and} \quad \sin \theta = \frac{FG}{EF} = \frac{40}{50} = \frac{4}{5}$$

Equilibrium of Sphere No. 2

The sphere 2 has points of contact at C and D.

Let $R_C = \text{Reaction at C}$

and $R_D = \text{Reaction at D}$

The free-body diagram of sphere No. 2 is shown in Fig. 4.18 (c).

The reaction R_D at point D, will pass through the centre E of the sphere No. 2, as any line normal to any point on the circumference of the circle will pass through the centre of circle. For the equilibrium of the sphere No. 2, the resultant force in x and y directions should be zero.

For $\Sigma F_x = 0$, we have $R_D \sin \theta = R_C$... (i)

For $\Sigma F_y = 0$, we have $R_D \cos \theta = 1000$

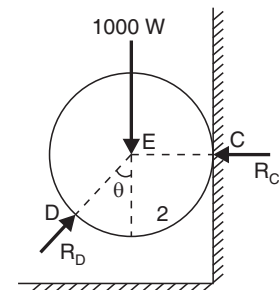


Fig. 4.18 (c)

$$\begin{aligned}
 \text{or} \quad R_D &= \frac{1000}{\cos \theta} = \frac{1000}{\left(\frac{3}{5}\right)} & \left(\because \cos \theta = \frac{3}{5} \right) \\
 &= 1000 \times \frac{5}{3} = \frac{5000}{3} \text{ N} & \dots(ii)
 \end{aligned}$$

Substituting the value of R_D in equation (i),

$$\begin{aligned}
 \frac{5000}{3} \times \sin \theta &= R_C \\
 \text{or} \quad \frac{5000}{3} \times \frac{4}{5} &= R_C & \left(\because \sin \theta = \frac{4}{5} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{or} \quad 1333.33 &= R_C \\
 \therefore R_C &= 1333.33 \text{ N. Ans.}
 \end{aligned}$$

Equilibrium of Sphere No. 1. The sphere 1 has points of contact at A, B and D.

Let R_A = Reaction at point A

R_B = Reaction at point B

The free-body diagram of sphere No. 1 is shown in Fig. 4.18 (d). The reactions R_A , R_B and R_D will pass through the centre F of the sphere No. 1.

For $\Sigma F_x = 0$, we have

$$\begin{aligned}
 R_A - R_D \sin \theta &= 0 \\
 \text{or} \quad R_A &= R_D \sin \theta \\
 &= \frac{5000}{3} \times \frac{4}{5} \\
 &\left(\because R_D = \frac{5000}{3} \text{ and } \sin \theta = \frac{4}{5} \right) \\
 &= 1333.33 \text{ N. Ans.}
 \end{aligned}$$

For $\Sigma F_y = 0$, we have

$$\begin{aligned}
 R_B - 1000 - R_D \cos \theta &= 0 \\
 \therefore R_B &= 1000 + R_D \cos \theta \\
 &= 1000 + \frac{5000}{3} \times \frac{3}{5} & \left(\because \cos \theta = \frac{3}{5} \right) \\
 &= 2000 \text{ N. Ans.}
 \end{aligned}$$

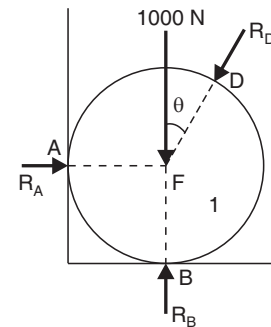


Fig. 4.18 (d)

Problem 4.16. Two smooth circular cylinders, each of weight $W = 1000 \text{ N}$ and radius 15 cm , are connected at their centres by a string AB of length $= 40 \text{ cm}$ and rest upon a horizontal plane, supporting above them a third cylinder of weight $= 2000 \text{ N}$ and radius 15 cm as shown in Fig. 4.19. Find the force S in the string AB and the pressure produced on the floor at the points of contact D and E.

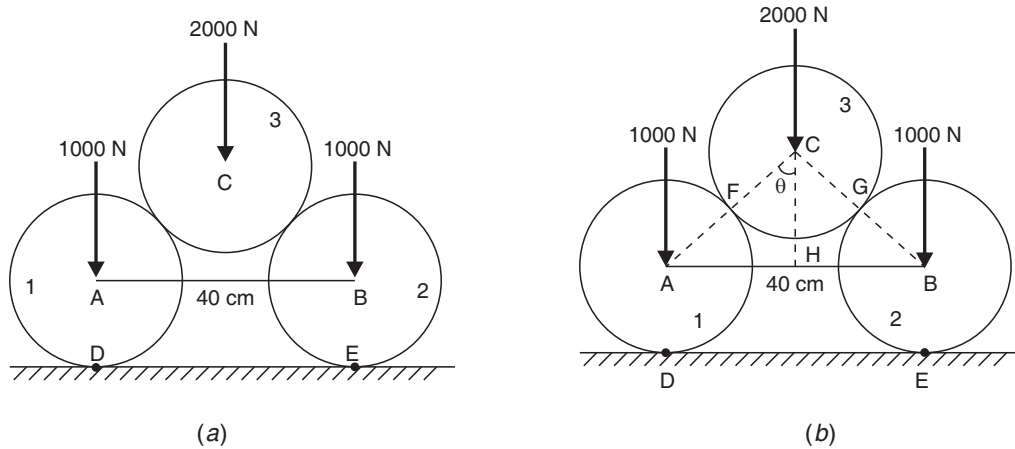


Fig. 4.19

Sol. Given :

Weight of cylinders 1 and 2 = 1000 N

Weight of cylinder 3 = 2000 N

Radius of each cylinder = 15 cm

Length of string $AB = 40$ cm

From Fig. 4.19 (b), $AC = AF + FC = 15 + 15 = 30$ cm

$$AH = \frac{1}{2} \times AB = \frac{1}{2} \times 40 = 20 \text{ cm}$$

$$\text{From } \triangle ACH, \quad \sin \theta = \frac{AH}{AC} = \frac{20}{30} = 0.667$$

$$\therefore \theta = \sin^{-1} 0.667 = 41.836^\circ.$$

Equilibrium of cylinder 3. The cylinder 3 has points of contact at F and G . The reactions R_F and R_G will pass through the centre of sphere 3. The free-body diagram is shown in Fig. 4.19 (c). Resolving forces horizontally,

$$R_F \sin \theta - R_G \sin \theta = 0$$

$$\text{or} \quad R_F = R_G \quad \dots(i)$$

Resolving forces vertically,

$$R_F \cos \theta + R_G \cos \theta = 2000$$

$$\text{or} \quad R_F \cos \theta + R_F \cos \theta = 2000 \quad (\because R_F = R_G)$$

$$\therefore R_F = \frac{2000}{2 \times \cos \theta} = \frac{1000}{\cos 41.836^\circ} = 1342.179 \text{ N} \quad \dots(ii)$$

Equilibrium of cylinder 1

The cylinder 1 has points of contact at D and F . Also the cylinder 1 is connected to cylinder 2 by a string AB . To draw the free-body diagram of cylinder 1, there will be reactions R_F and R_D at points F and D as shown in Fig. 4.19 (d). Also there will be a force S in the direction of the string AB .

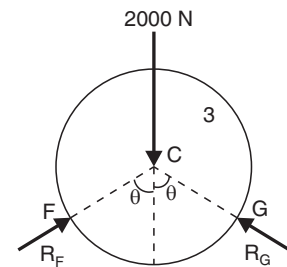


Fig. 4.19 (c)

For $\Sigma F_x = 0$, we have

$$S - R_F \sin \theta = 0$$

$$\begin{aligned} \therefore S &= R_F \sin \theta \\ &= 1342.179 \times \sin 41.836^\circ \\ [\because \text{From equation (ii), } R_F &= 1342.179] \\ &= \mathbf{895.2 \text{ N. Ans.}} \end{aligned}$$

For $\Sigma F_y = 0$, we have

$$R_D - 1000 - R_F \cos \theta = 0$$

$$\begin{aligned} \text{or } R_D &= 1000 + R_F \cos \theta \\ &= 1000 + 1342.179 \times \cos 41.836^\circ \\ &= 1999.99 \approx \mathbf{2000 \text{ N. Ans.}} \end{aligned}$$

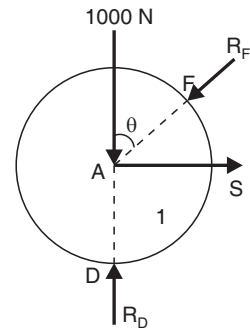


Fig. 4.19 (d)

The equilibrium of cylinders 1, 2 and 3 taken together

The three cylinders taken together have points of contact at D and E. The free-body diagram is shown in Fig. 4.19 (e). In this case only vertical forces exist. Hence resultant force in y-direction should be zero.

$$\therefore R_D + R_E - 1000 - 2000 - 1000 = 0$$

$$\text{or } R_E = 1000 + 2000 + 1000 - R_D$$

$$\begin{aligned} \text{or } R_E &= 4000 - R_D \\ &= 4000 - 2000 \quad (\because R_D = 2000) \\ &= \mathbf{2000 \text{ N. Ans.}} \end{aligned}$$

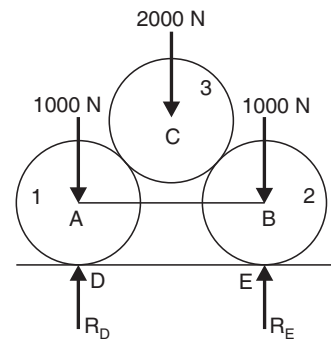


Fig. 4.19 (e)

Problem 4.17. A roller of radius 40 cm, weighing 3000 N is to be pulled over a rectangular block of height 20 cm as shown in Fig. 4.20, by a horizontal force applied at the end of a string wound round the circumference of the roller. Find the magnitude of the horizontal force which will just turn the roller over the corner of the rectangular block. Also determine the magnitude and direction of reactions at A and B. All surfaces may be taken as smooth.

Sol. Given :

Radius of roller = 40 cm

Weight, $W = 3000 \text{ N}$

Height of block = 20 cm

Find horizontal force P , reaction R_A and reaction R_B when the roller just turns over the block.

When the roller is about to turn over the corner of the rectangular block, the roller lifts at the point A and then there will be no contact between the roller and the point A. Hence reaction R_A at point A will become zero.

Now the roller will be in equilibrium under the action of the following three forces :

- (i) its weight W acting vertically downward
- (ii) horizontal force P
- (iii) reaction R_B at point B. The direction of R_B is unknown.

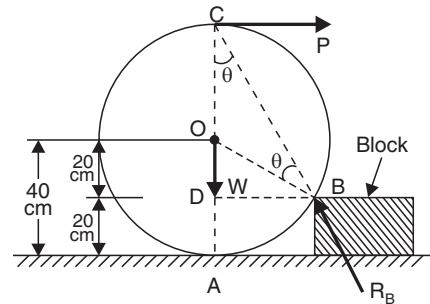


Fig. 4.20

For the equilibrium, these three forces should pass through a common point. As the force P and weight W is passing through point C , hence the reaction R_B must also pass through the point C . Therefore, the line BC gives the direction of the reaction R_B .

In $\triangle BOD$, $BO = \text{Radius} = 40 \text{ cm}$,

$$OD = OA - AD = 40 - 20 = 20 \text{ cm}$$

$$\therefore BD = \sqrt{BO^2 - OD^2} = \sqrt{40^2 - 20^2} = \sqrt{1200} = 34.64$$

$$\text{Now in } \triangle BCD, \quad \tan \theta = \frac{BD}{CD} = \frac{34.64}{CO + OD} = \frac{34.64}{(40 + 20)} = 0.5773$$

$$\therefore \theta = \tan^{-1} 0.5773 = 29.999^\circ \approx 30^\circ$$

Resolving forces horizontally, we get $P - R_B \sin \theta = 0$

$$\text{or} \quad P = R_B \sin \theta = R_B \times \sin 30^\circ = 0.5 R_B \quad \dots(i)$$

Resolving forces vertically, we get

$$W - R_B \cos \theta = 0 \quad \text{or} \quad 3000 - R_B \times \cos 30^\circ = 0$$

$$\text{or} \quad R_B = \frac{3000}{\cos 30^\circ} = 3464.2 \text{ N. Ans.}$$

Substituting this value of R_B in equation (i), we get

$$P = 0.5 \times 3464.2 = 1732.1 \text{ N. Ans.}$$

2nd Method

This problem can also be solved by taking moments of all the three forces about the point B (i.e., corner of the rectangular block) as shown below :

$$P \times CD = W \times BD$$

$$\text{or} \quad P \times 60 = 3000 \times 34.64 \quad (\because BD = 34.64)$$

$$\therefore P = \frac{3000 \times 34.64}{60} = 1732.0 \text{ N. Ans.}$$

Problem 4.18. If in the problem 4.17, the force P is applied horizontally at the centre of the roller, what would be the magnitude of this force? Also determine the least force and its line of action at the roller centre, for turning the roller over the rectangular block.

(Similar Question U.P.T.U., May 2009)

Sol. Given :

Radius of roller = 40 cm

Weight, $W = 3000 \text{ N}$

Height of block = 20 cm

When the roller is about to turn over the corner of the rectangular block, the roller lifts at the point A , and then there will be no contact between the roller and the point A . Hence reaction R_A at point A will be zero.

Now the roller will be in equilibrium under the action of the following three forces :

- (i) Weight of the roller W acting vertically downward through point O .
- (ii) Horizontal force P applied at the centre of the roller.
- (iii) Reaction R_B at point B . The direction of reaction R_B is unknown.

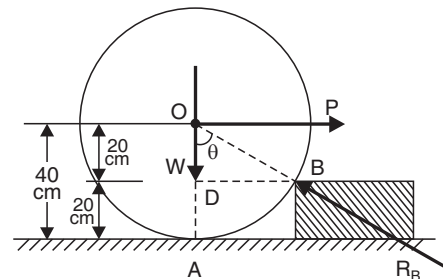


Fig. 4.21

The above three forces should pass through a common point, as roller is in equilibrium. But the weight W and force P is passing through point O , hence R_B should also pass through point O . Hence line joining B to O , gives the direction of R_B as shown in Fig. 4.21.

$$\text{From } \triangle BOD, \text{ we have } \cos \theta = \frac{OD}{BO} = \frac{20}{40} = 0.5$$

$$\therefore \theta = \cos^{-1} 0.5 = 60^\circ$$

Now resolving forces vertically,

$$R_B \cos \theta - W = 0$$

$$\text{or } R_B = \frac{W}{\cos \theta} = \frac{3000}{\cos 60^\circ} = 6000 \text{ N} \quad \dots(i)$$

Resolving forces horizontally, we get $P - R_B \sin \theta = 0$

$$\text{or } P = R_B \sin \theta = 6000 \times \sin 60^\circ = \mathbf{5196 \text{ N. Ans.}}$$

Method of Moments

The force P can also be calculated by the method of moments. Taking the moments of all the forces about the point B , we get

$$P \times OD - W \times BD = 0$$

$$\text{or } P = \frac{W \times BD}{OD} \quad \dots(ii)$$

$$(\because BD = \sqrt{BO^2 - OD^2} = \sqrt{40^2 - 20^2} = 34.64)$$

$$= \frac{3000 \times 34.64}{20} = \mathbf{5196 \text{ N. Ans.}}$$

Least force and its line of action

Let P_{min} = Least force applied as shown in Fig. 4.21 (a).

α = Angle of the least force

From $\triangle OBC$, $BC = BO \sin \alpha$

Taking moments of all forces about point B , we get $P_{min} \times BC - W \times BD = 0$

$$\text{or } P_{min} = \frac{W \times BD}{BC}$$

$$\text{or } P_{min} = \frac{3000 \times 34.64}{BO \sin \alpha} = \frac{3000 \times 34.64}{40 \times \sin \alpha}$$

$$(\because BO = \text{Radius} = 40)$$

The force P will be minimum when $\sin \alpha$ is maximum. But $\sin \alpha$ will be maximum, when $\alpha = 90^\circ$ or $\sin \alpha = 1$. Substituting this value of $\sin \alpha$ in the above equation, we get minimum force.

$$\therefore P_{min} = \frac{3000 \times 34.64}{40 \times 1} = \mathbf{2598 \text{ N. Ans.}}$$

The direction of least force (i.e., P_{min}) is at right angles to the line BO .

Problem 4.18 (A). The cross-section of a block is an equilateral triangle. It is hinged at A and rests on a roller at B . It is pulled by means of a string attached at C . If the weight of the block is $M \times g$ and the string is horizontal, determine the force P which should be applied through string to just lift the block off the roller. (U.P. Tech. University, 2002–2003)

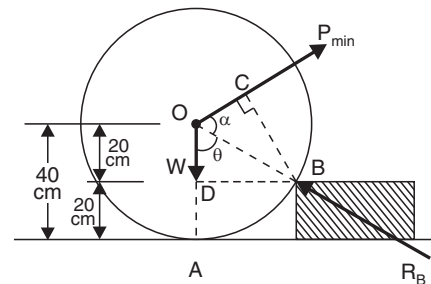


Fig. 4.21 (a)

Sol. Let P = force applied horizontally at C and the given block just lifts off the roller at B .

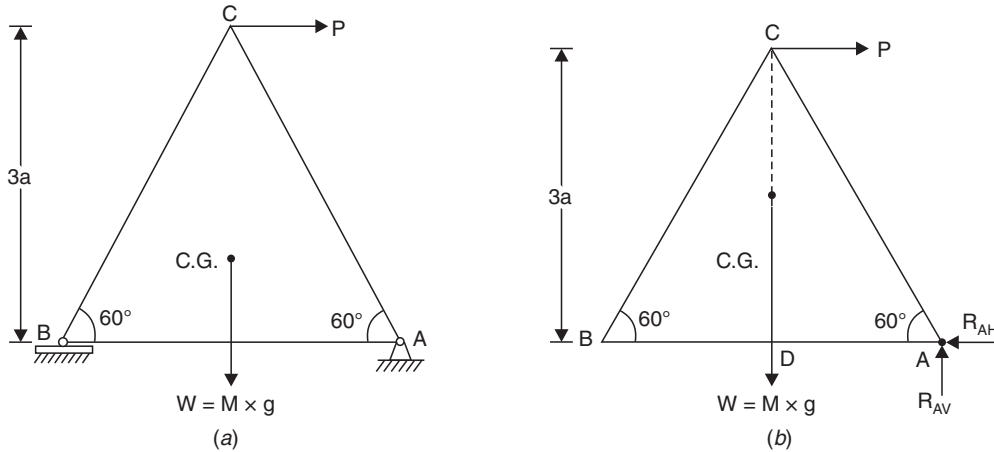


Fig. 4.22

When the given block just lifts off the roller at B , there will be no contact between the roller and point B . Hence reaction R_B at B will be zero. The given block will be then in equilibrium under the action of following three forces [Refer to Fig. 4.22 (b)] :

- (i) Weight ($M \times g$) acting downward
 - (ii) Horizontal force P and
 - (iii) Reaction R_A at A .
(R_{AV} and R_{AH} are components of R_A)
- From triangle ACD ,

$$\tan 60^\circ = \frac{CD}{AD} = \frac{3a}{AD}$$

$$\therefore AD = \frac{3a}{\tan 60^\circ} = \frac{3a}{\sqrt{3}} = \sqrt{3} \times a$$

Taking the moments about A , $\Sigma M_A = 0$,

$$W \times AD = P \times 3a$$

$$\text{or } P = \frac{W \times \sqrt{3} \times a}{3a} \quad (\because AD = \sqrt{3} \times a)$$

$$= \frac{W}{\sqrt{3}} = \frac{M \times g}{\sqrt{3}} \quad \text{Ans.}$$

Problem 4.19. A L -shaped body ABC is hinged at A with a force F acting at its end C . Determine the angle θ which this force should make with the horizontal to keep the edge AB of the body vertical.

Sol. Given :

Length $BC = 25$ cm

Length $AB = 40$ cm.

Let us assume that force F is making an angle θ with the horizontal as shown in Fig. 4.23.

The forces acting on the given body ABC are :

(i) Force F acting at point C .

(ii) Reaction R_A at a point A .

Only two forces are acting on the body and the body is in equilibrium. But the two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and have the same line of action. As the force F is acting at point C , hence reaction R_A should also pass through point C . Therefore, the line of action of reaction R_A is along line AC . The line of action of F should also be along AC , i.e., the force F will be making an angle α with the horizontal.

$$\text{Now from } \triangle ABC, \tan \alpha = \frac{AB}{BC} = \frac{40}{25} = 1.6$$

$$\therefore \alpha = \tan^{-1} 1.6 = 57.99^\circ$$

$$\therefore \theta = \alpha = 57.99^\circ. \text{ Ans.}$$

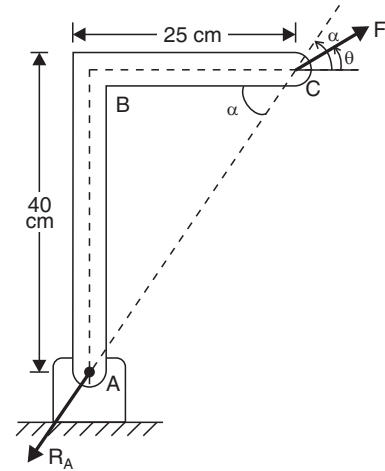


Fig. 4.23

Problem 4.20. A horizontal force 200 N is applied to the sloping bar BCD whose bottom rests on a horizontal plane, as shown in Fig. 4.24. Its upper end is pinned at B to the horizontal bar AB which has a pinned support at A . What couple M must be applied to AB to hold the system in equilibrium? What is the magnitude of the pin reaction at B ? Assume the bars to be weightless and pins at A and B to be smooth.

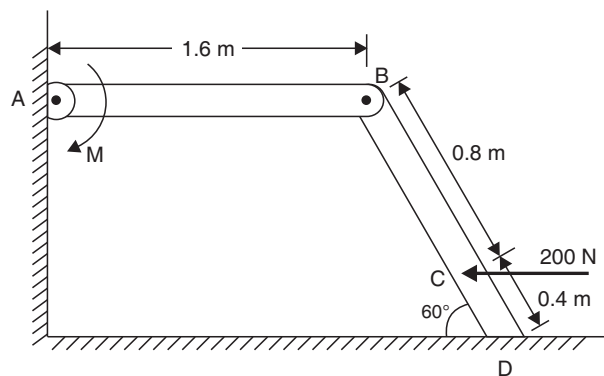


Fig. 4.24

Sol. Given :

Length $AB = 1.6 \text{ m}$, Length $BD = 1.2 \text{ m}$, $BC = 0.8 \text{ m}$ and $CD = 0.4 \text{ m}$

Horizontal force at $C = 200 \text{ N}$

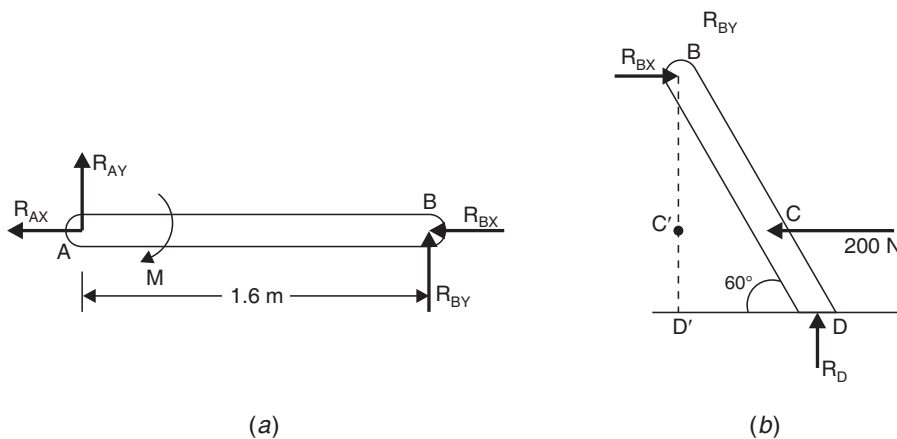


Fig. 4.25

Let M = Couple applied A to bar AB to keep the system in equilibrium.

R_B = Reaction at B

R_{BX} and R_{BY} are the horizontal and vertical components of reaction R_B .

The free-body diagram of the sloping bar BCD is shown in Fig. 4.25 (b). The reaction R_D at point D is vertical. Also show R_{BX} and R_{BY} at B .

Since the bar is in equilibrium, first apply $\Sigma F_x = 0$, which gives

$$R_{BX} = 200 \text{ N}$$

Now apply $\Sigma F_y = 0$, then $R_{BY} = R_D$... (i)

The magnitude of R_{BY} and R_D is unknown. To find their values apply ΣM at the pin B .

Taking moments of all forces at point B , we get

$$R_D \times DD' = 200 \times BC'$$

$$\text{or } R_D \times BD \cos 60^\circ = 200 \times BC \sin 60^\circ \quad (\because DD' = BD \cos 60^\circ \text{ and } BC' = BC \sin 60^\circ)$$

$$\text{or } R_D \times 1.2 \cos 60^\circ = 200 \times 0.8 \sin 60^\circ$$

$$\text{or } R_D = \frac{200 \times 0.8 \times 0.866}{1.2 \times \cos 60^\circ} = 230.93 \text{ N}$$

$\therefore$ From equation (i), $R_{BY} = R_D = 230.93 \text{ N}$

Reaction at B , $R_B = \sqrt{R_{BX}^2 + R_{BY}^2} = \sqrt{200^2 + 230.93^2} = 305.4 \text{ N. Ans.}$

Now draw the free-body diagram of the horizontal bar AB as shown in Fig. 4.25 (a). Show the reactions R_{BX} and R_{BY} at B . Also show the reactions R_{AY} and R_{AX} at A . The applied couple M is also shown in the figure.

Apply $\Sigma M_A = 0$

$$M = R_{BY} \times 1.6 = 230.93 \times 1.6 = 369.44 \text{ Nm. Ans.}$$

4.4. EQUILIBRIUM OF A BODY UNDER THREE FORCES

The three forces acting on a body which is in equilibrium may be either concurrent or parallel. Let us first consider that the body is in equilibrium when three forces, acting on the body, are concurrent. This is shown in Fig. 4.26.

(a) *When three forces are concurrent.* The three concurrent forces F_1 , F_2 and F_3 are acting on a body at point O and the body is in equilibrium. The resultant of F_1 and F_2 is given by R . If the force F_3 is collinear, equal and opposite to the resultant R , then the body will be in equilibrium. The force F_3 which is equal and opposite to the resultant R is known as *equilibrant*. Hence for three concurrent forces acting on a body when the body is in equilibrium, the resultant of the two forces should be equal and opposite to the third force.

(b) *When three forces are parallel.* Fig. 4.26 (a) shows a body on which three parallel forces F_1 , F_2 and F_3 are acting and the body is in equilibrium. If three forces F_1 , F_2 and F_3 are acting in the same direction, then there will be a resultant $R = F_1 + F_2 + F_3$ and body will not be in equilibrium. The three forces are acting in opposite direction and their magnitude is so adjusted that there is no resultant force and body is in equilibrium. Let us suppose that F_2 is acting in opposite direction as shown in Fig. 4.26 (a).

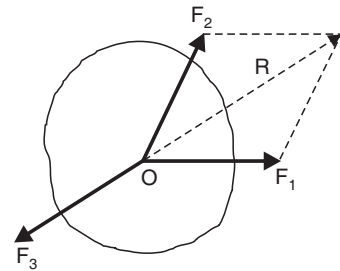


Fig. 4.26

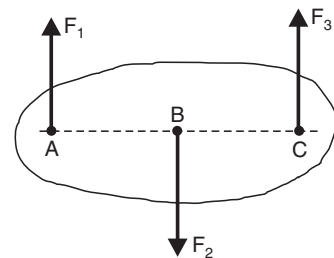


Fig. 4.26(a)

Now let us apply the three conditions of equilibrium :

- (i) $\Sigma F_x = 0$ as there is no horizontal force acting on the body
- (ii) $\Sigma F_y = 0$ i.e., $F_1 + F_3 = F_2$
- (iii) $\Sigma M = 0$ about any point.

Taking the moments of F_1 , F_2 and F_3 about point A,

$$\Sigma M_A = -F_2 \times AB + F_3 \times AC$$

(Moment of F_3 is anti-clockwise whereas moment of F_2 is clockwise)

For equilibrium, ΣM_A should be zero

i.e.,
$$-F_2 \times AB + F_3 \times AC = 0$$

If the distances AB and AC are such that the above equation is satisfied, then the body will be in equilibrium under the action of three parallel forces.

4.5. LAMI'S THEOREM

It states that, "If three forces acting at a point are in equilibrium, each force will be proportional to the sine of the angle between the other two forces."

Suppose the three forces P , Q and R are acting at a point O and they are in equilibrium as shown in Fig. 4.27.

- Let α = Angle between force P and Q
 β = Angle between force Q and R
 γ = Angle between force R and P .

Then according to Lami's theorem,

$P \propto \sin$ of angle between Q and $R \propto \sin \beta$.

$$\therefore \frac{P}{\sin \beta} = \text{constant}$$

Similarly $\frac{Q}{\sin \gamma} = \text{constant}$ and $\frac{R}{\sin \alpha} = \text{constant}$

or
$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha} \quad \dots(4.5)$$

Proof of Lami's Theorem. The three forces acting on a point, are in equilibrium and hence they can be represented by the three sides of the triangle taken in the same order. Now draw the force triangle as shown in Fig. 4.27 (a).

Now applying sine rule, we get

$$\frac{P}{\sin (180^\circ - \beta)} = \frac{Q}{\sin (180^\circ - \gamma)} = \frac{R}{\sin (180^\circ - \alpha)}$$

This can also be written

$$\frac{P}{\sin \beta} = \frac{Q}{\sin \gamma} = \frac{R}{\sin \alpha}$$

This is same equation as equation (4.5).

Note. All the three forces should be acting either towards the point or away from the point.

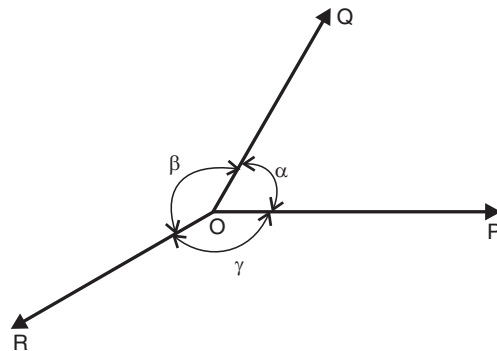


Fig. 4.27

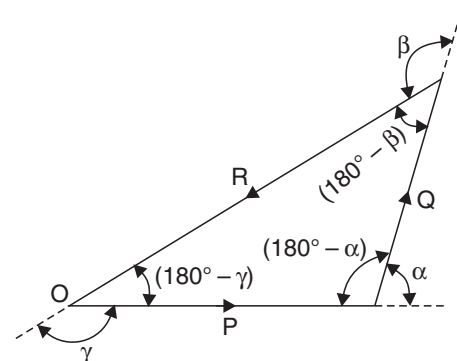


Fig. 4.27(a)

Problem 4.21. A body weighing 2000 N is suspended with a chain AB 2 m long. It is pulled by a horizontal force of 320 N as shown in Fig. 4.28. Find the force in the chain and the lateral displacement (i.e., x) of the body.

Sol. Given :

Weight suspended at B = 2000 N

Length AB = 2 m

Horizontal force at B = 320 N

Find : Force in AB and value of x

Let F = Force in chain AB

θ = Angle made by AB with horizontal.

The free-body diagram of the point B is shown in Fig. 4.28 (b).

The point B is in equilibrium under the action of three forces. Hence using Lami's theorem, we get

$$\frac{F}{\sin 90^\circ} = \frac{2000}{\sin (180^\circ - \theta)} = \frac{320}{\sin (90^\circ + \theta)}$$

$$\text{or } \frac{F}{1} = \frac{2000}{\sin \theta} = \frac{320}{\cos \theta} \quad [\because \sin (180^\circ - \theta) = \sin \theta, \sin (90^\circ + \theta) = \cos \theta]$$

$$\text{or } F \sin \theta = 2000 \quad \dots(i)$$

$$\text{and } F \cos \theta = 320 \quad \dots(ii)$$

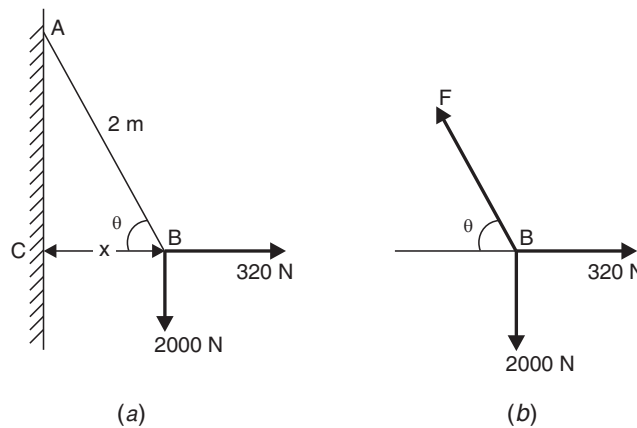


Fig. 4.28

Dividing equation (i) by equation (ii), we get

$$\tan \theta = \frac{2000}{320} = 6.25$$

$$\therefore \theta = \tan^{-1} 6.25 = 80.9^\circ$$

Substituting this value of θ in equation (i), we get

$$F \sin 80.9^\circ = 2000 \quad \text{or} \quad F = \frac{2000}{\sin 80.9^\circ} = 2025.5 \text{ N. Ans.}$$

$$\begin{aligned} \text{Now from Fig. 4.23 (a), } \cos \theta &= \frac{x}{2} \quad \text{or} \quad x = 2 \times \cos \theta = 2 \times \cos 80.9^\circ \\ &= 0.3163 \text{ m. Ans.} \end{aligned}$$

HIGHLIGHTS

1. The principle of equilibrium states that a stationary body will be in equilibrium if the algebraic sum of all the forces is zero and also the algebraic sum of moments of all the external forces is zero.
2. The conditions of equilibrium are written mathematically as $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. The sign Σ is known as sigma and this sign represents the algebraic sum.
3. When a body is subjected to two forces, the body will be in equilibrium if the two forces are collinear, equal and opposite.
4. Two equal and opposite parallel forces produces a couple whose moment is equal to either force multiplied by their perpendicular distance.
5. If three concurrent forces are acting on a body and the body is in equilibrium, then the resultant of two forces should be equal and opposite to the third force.
6. Free body diagram of a body is a diagram in which the body is completely isolated from its support and the supports are replaced by the reactions which these supports exert on the body.

EXERCISE 4

(A) Theoretical Questions

1. Define and explain the terms: Principle of equilibrium, force law of equilibrium and moment law of equilibrium.
2. A number of forces are acting on a body. What are conditions of equilibrium, so that the body is in equilibrium ?
3. Two forces are acting on a body and the body is in equilibrium. What conditions should be fulfilled by these two forces ?
4. How will you prove that a body will not be in equilibrium when the body is subjected to two forces which are equal and opposite but are parallel ?
5. Explain the statement “Two equal and opposite parallel forces produces a couple”.
6. (a) What conditions must be fulfilled by a set of three parallel forces which are acting on a body and body is in equilibrium ?
(b) State the graphical conditions that must be satisfied for the equilibrium of a system of coplanar forces.
(c) What are the conditions that must be satisfied in order that a body may have plane motion ?
(d) Discuss the various laws governing the equilibrium of coplanar forces.
7. Three concurrent forces are acting on a body which is in equilibrium, then the resultant of the two forces should be equal and opposite to the third force. Prove this statement.
8. State and explain the Lami's theorem.
9. What do you mean by action and reaction ? Give examples.
10. Explain and define the term : ‘Free-body Diagram’. Draw the free-body diagram of a ball of weight W , placed on a horizontal surface.
11. What is a free-body diagram ? Explain with suitable examples.

(U.P. Tech. University, May 2009)

(B) Numerical Problems

1. Three forces F_1 , F_2 and F_3 are acting on a body as shown in Fig. 4.29 and the body is in equilibrium. If the magnitude of force F_3 is 250 N, find the magnitudes of force F_1 and F_2 .

[Ans. $F_1 = 125$ N and $F_2 = 215.6$ N]

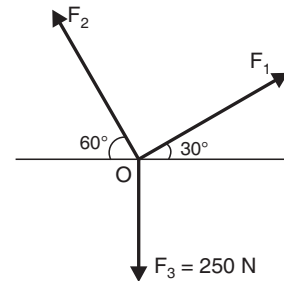


Fig. 4.29

2. Three forces of magnitudes P , 100 N and 200 N are acting at a point O as shown in Fig. 4.30. Determine the magnitude and direction of the force P .

[Ans. $P = 147$ N and $\theta = 76.8^\circ$]

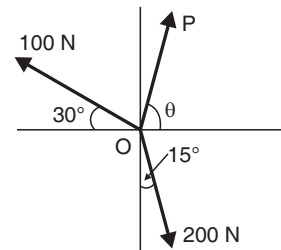


Fig. 4.30

3. Three parallel forces F_1 , F_2 and F_3 are acting on a body as shown in Fig. 4.31 and the body is in equilibrium. If force $F_1 = 300$ N and $F_3 = 1000$ N and the distance between F_1 and $F_2 = 2.0$ m, then determine the magnitude of force F_2 and distance of F_3 from force F_2 .

[Ans. 1300 N, 0.6 m]

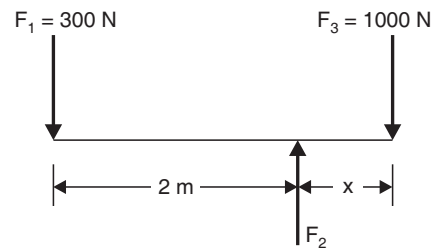


Fig. 4.31

4. Three forces of magnitude 40 kN, 15 kN and 20 kN are acting at a point O . The angles made by 40 kN, 15 kN and 20 kN forces with x -axis are 60° , 120° and 240° respectively. Determine the magnitude and direction of the resultant force.
[Ans. 30.41 kN and 85.28° with x -axis]
5. A lamp weighing 10 N is suspended from the ceiling by a chain. It is pulled aside by a horizontal cord until the chain makes an angle of 60° with the ceiling. Find the tensions in the chain and the cord by applying Lami's theorem and also by graphical method.
[Ans. 11.54 N and 5.77 N]
6. Draw the free-body diagram of a ball of weight W supported by a string AB and resting on a smooth horizontal surface at C when a horizontal force is applied to the ball as shown in Fig. 4.32.

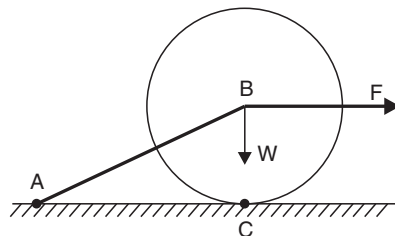


Fig. 4.32

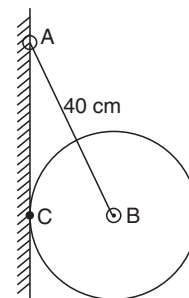


Fig. 4.33

7. A circular roller of weight 1000 N and radius 20 cm hangs by a tie rod $AB = 40$ cm and rests against a smooth vertical wall at C as shown in Fig. 4.33. Determine the tension in the tie rod and reaction R_C at point C .
[Ans. 1154.7 N and 577.3 N]
8. In problem 6 if radius of ball = 5 cm, length of string $AB = 10$ cm, weight of ball $W = 40$ N and the horizontal force $F = 30$ N, then find the tension in the string and vertical reaction R_C at point C .
[Ans. 34.64 N and 57.32 N]
9. A smooth circular cylinder of weight 1000 N and radius 10 cm rests in a right-angled groove whose sides are inclined at an angle of 30° and 60° to the horizontal as shown in Fig. 4.34. Determine the reaction R_A and R_C at the points of contact.
[Ans. $R_A = 500$ N, $R_C = 866.6$ N]

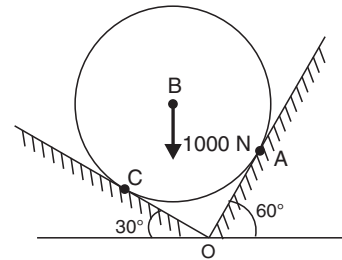


Fig. 4.34

10. If in the above problem, the sides of the groove makes an angle of 45° with the horizontal, then find the reactions R_A and R_C .
[Ans. $R_A = R_C = 707$ N]
11. Two identical rollers, each of weight 50 N, are supported by an inclined plane and a vertical wall as shown in Fig. 4.35. Find the reactions at the points of supports A , B and C . Assume all the surfaces to be smooth.
[Ans. $R_A = 43.3$ N, $R_B = 72$ N, $R_C = 57.5$ N]

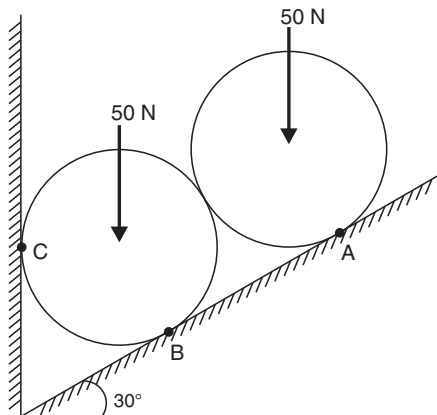


Fig. 4.35

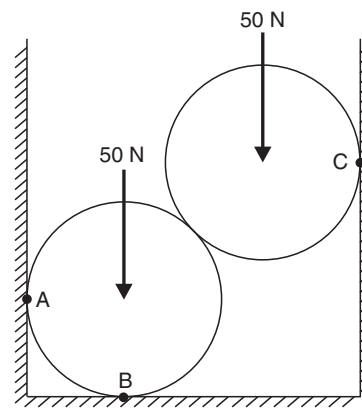


Fig. 4.36

12. Two spheres, each of weight 50 N and of radius 10 cm rest in a horizontal channel of width 36 cm as shown in Fig. 4.36. Find the reactions on the points of contact A , B and C .
[Ans. $R_A = R_C = 66.67$ N, $R_B = 100$ N]

Determination of Support Reactions

5.1. INTRODUCTION

When a number of forces are acting on a body, and the body is supported on another body, then the second body exerts a force known as reactions on the first body at the points of contact so that the first body is in equilibrium. The second body is known as support and the force, exerted by the second body on the first body, is known as support reactions.

5.2. TYPES OF SUPPORTS

Though there are many types of supports, yet the following are important from the subject point of view :

- (a) Simple supports or knife edge supports
- (b) Roller support
- (c) Pin-joint (or hinged) support
- (d) Smooth surface support
- (e) Fixed or built-in support.

5.2.1. Simple Support or Knife Edge Support. A beam supported on the knife edges A and B is shown in Fig. 5.1 (a). The reactions at A and B in case of knife edge support will be normal to the surface of the beam. The reactions R_A and R_B with free-body diagram of the beam is shown in Fig. 5.1 (b).

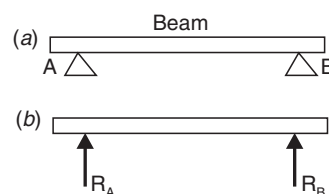


Fig. 5.1

5.2.2. Roller Support. A beam supported on the rollers at points A and B is shown in Fig. 5.2 (a). The reactions in case of roller supports will be normal to the surface on which rollers are placed as shown in Fig. 5.2 (b).

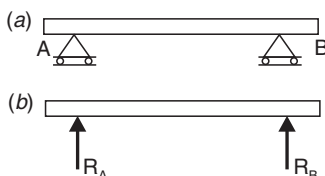


Fig. 5.2

5.2.3. Pin Joint (or hinged) Support. A beam, which is hinged (or pin-joint) at point A , is shown in Fig. 5.3. The reaction at the hinged end may be *either vertical or inclined* depending upon the type of loading. If the load is vertical, then the

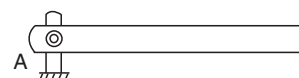


Fig. 5.3

reaction will also be vertical. But if the load is inclined, then the reaction at the hinged end will also be inclined.

5.2.4. Smooth Surface Support. Fig. 5.4 shows a body in contact with a smooth surface. The reaction will always act normal to the support as shown in Fig. 5.4 (a) and Fig. 5.4 (b).

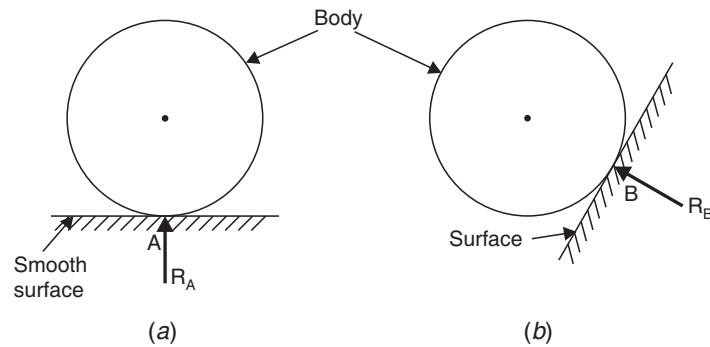


Fig. 5.4

Fig. 5.5 shows a rod AB resting inside a sphere, whose surface are smooth. Here the rod becomes body and sphere becomes surface. The reactions on the ends of the rod (*i.e.*, at point A and B) will be normal to the sphere surface at A and B . The normal at any point on the surface of the sphere will always pass through the centre of the sphere. Hence reactions R_A and R_B will have directions AO and BO respectively as shown in Fig. 5.5.

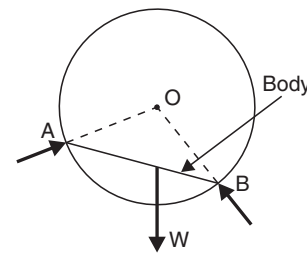


Fig. 5.5

5.2.5. Fixed or Built-in Support. Fig. 5.6 shows the end A of a beam, which is fixed. Hence the support at A is known as a fixed support. The fixed support prevents the vertical movement and rotation of the beam. Hence at the fixed support there can be horizontal reaction and vertical reaction. Also there will be fixing moment at the fixed end. In case of fixed support, the reaction can be inclined. Also the fixed support will provide a couple.

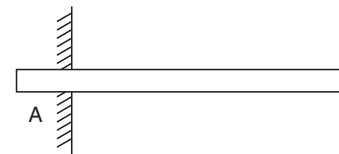


Fig. 5.6

5.3. TYPES OF LOADING

The following are the important types of loading :

- (a) Concentrated or point load,
- (b) Uniformly distributed load, and
- (c) Uniformly varying load.

5.3.1. Concentrated or Point Load. Fig. 5.7 shows a beam AB , which is simply supported at the ends A and B . A load W is acting at the point C . This load is known as point load (or concentrated load). Hence any load acting at a point on a beam, is known as point load.

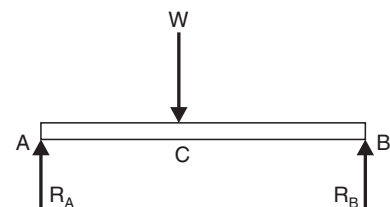


Fig. 5.7

In actual practice, it is not possible to apply a load at a point (*i.e.*, at a mathematical point) as it must have some contact area. But this area in comparison to the length of the beam is very very small (or area is negligible).

5.3.2. Uniformly Distributed Load. If a beam is loaded in such a way, that each unit length of the beam carries same intensity of the load, then that type of load is known as uniformly distributed load (which is written as U.D.L.). Fig. 5.8 shows a beam AB , which carries a uniformly distributed load.

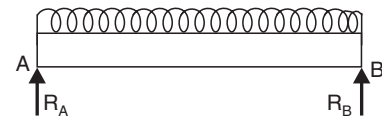


Fig. 5.8

For finding the reactions the total uniformly distributed load is assumed to act at the C.G. of the load.

5.3.3. Uniformly Varying Load. Fig. 5.9 shows a beam AB , which carries load in such a way that the rate of loading on each unit length of the beam varies uniformly. This type of load is known as uniformly varying load. The total load on the beam is equal to the area of the load diagram. The total load acts at the C.G. of the load diagram.

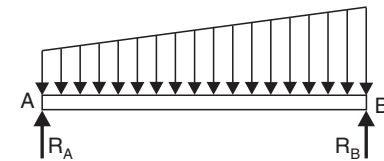


Fig. 5.9

5.4. DETERMINATION OF REACTIONS OF A BEAM

The following are the methods of finding out the reactions at the two supports of a beam :

1. Analytical method, and
2. Graphical method.

5.4.1. Analytical Method. Fig. 5.10 shows a beam AB of length L and is simply supported at the ends A and B . The beam carries two point loads W_1 and W_2 at a distance L_1 and L_2 from the end A .

Let R_A = Reaction at A
and R_B = Reaction at B

As the beam is in equilibrium, the equations of the equilibrium, *i.e.*, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$ should be satisfied. In this case there is no horizontal force, hence the equations of equilibrium are $\Sigma F_y = 0$ and $\Sigma M = 0$.

For $\Sigma F_y = 0$, we have

$$R_A + R_B = W_1 + W_2 \quad \dots(i)$$

For $\Sigma M = 0$, the moments about any point of all the forces should be zero.

Taking the moments about point A , we get

$$W_1 \times L_1 + W_2 \times L_2 - R_B \times L = 0$$

or
$$W_1 \times L_1 + W_2 \times L_2 = R_B \times L \quad \text{or} \quad R_B = \frac{W_1 \times L_1 + W_2 \times L_2}{L}$$

As W_1 , W_2 , L_1 , L_2 and L are given, hence value of R_B can be calculated.

Now from equation (i), we have $R_A = (W_1 + W_2) - R_B$.

5.4.2. Graphical Method for Finding out the Reactions of a Beam. The graphical method consists of the following steps :

- (a) Construction of space diagram ;

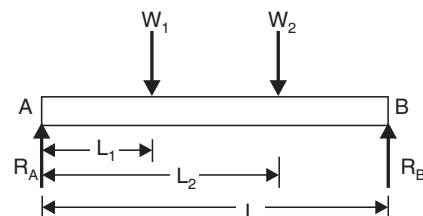


Fig. 5.10

(b) Use of Bow's notations ; and

(c) Construction of vector diagram.

The given beam is drawn to a suitable scale along with the loads and the reactions R_A and R_B . This step is known as construction of space diagram.

The different loads and forces (*i.e.*, reactions R_A and R_B) are named by two capital letters, placed on their either side of the space diagram as shown in Fig. 5.11. This step is known as *Bow's notation*. The load W_1 is named by PQ , W_2 by QR , reaction R_B by SR and reaction R_A by SP .

Now the vector diagram is drawn according to the following steps :

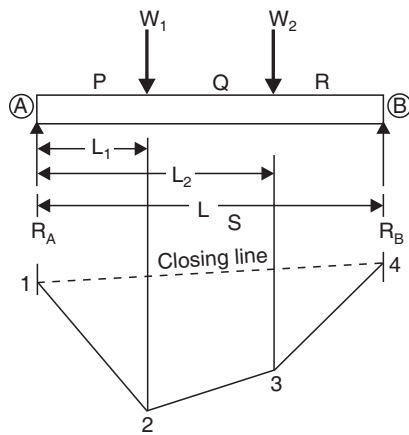
(i) Choose a suitable scale to represent the various loads. Now take any point p and draw pq parallel and equal to the load PQ (*i.e.*, W_1) vertically downward to the same scale.

(ii) Now through q , draw qr parallel and equal to QR vertically downward to the same scale.

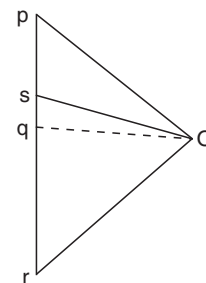
(iii) Select any suitable point O . Now join the point O to points p , q and r as shown in Fig. 5.11 (b).

(iv) Now in Fig. 5.11 (a), extend the lines of action of the loads and the two reactions. Take any point 1, on the line of action of the reaction R_A . Through 1, draw the line 1-2 parallel to pO , intersecting the line of action of load W_1 at 2.

(v) Now from point 2, draw line 2-3 parallel to qO , intersecting the line of action of load W_2 at 3. Similarly, from point 3, draw the line 3-4 parallel to rO , intersecting the line of action of reaction R_B at point 4.



(a) Space diagram



(b) Vector diagram

Fig. 5.11

(vi) Now join the point 1 to point 4. The line 1-4 is known as closing line. Now from point O (*i.e.*, from vector diagram) draw line Os parallel to line 1-4.

(vii) Now in the vector diagram the length sp represents the magnitude of reaction R_A to the same scale. Similarly, the length rs represents the magnitude of reaction R_B to the same scale.

5.4.3. Reactions of Simply Supported Beams

Problem 5.1. A simply supported beam AB of span 6 m carries point loads of 3 kN and 6 kN at a distance of 2 m and 4 m from the left end A as shown in Fig. 5.12. Find the reactions at A and B analytically and graphically.

Sol. Given :

Span of beam = 6 m

Let R_A = Reaction at A

R_B = Reaction at B

(a) **Analytical Method.** As the beam is in equilibrium, the moments of all the forces about any point should be zero.

Now taking the moment of all forces about A, and equating the resultant moment to zero, we get

$$R_B \times 6 - 3 \times 2 - 6 \times 4 = 0$$

$$\text{or} \quad 6R_B = 6 + 24 = 30$$

$$\text{or} \quad R_B = \frac{30}{6} = 5 \text{ kN. Ans.}$$

Also for equilibrium, $\Sigma F_y = 0$

$$\therefore R_A + R_B = 3 + 6 = 9$$

$$\therefore R_A = 9 - R_B = 9 - 5 = 4 \text{ kN. Ans.}$$

(b) **Graphical Method.** First of all draw the space diagram of the beam to a suitable scale. Let 1 cm length in space diagram represents 1 m length of beam. Hence take $AB = 6$ cm, distance of load 3 kN from A = 2 cm and distance of 6 kN from A = 4 cm as shown in Fig. 5.13 (a).

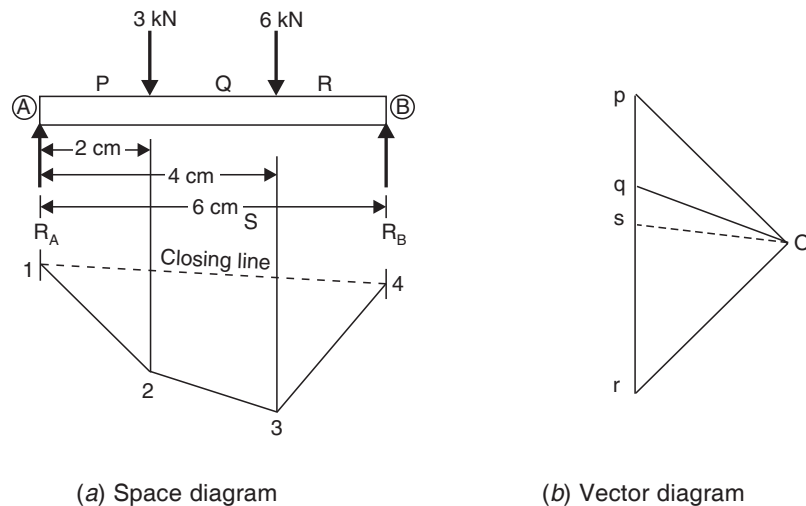


Fig. 5.13

Now name all the loads and reactions according to Bow's notation *i.e.*, load 3 kN is named by PQ , load 6 kN by QR , reaction R_B by SR and reaction R_A by SP .

Now the vector diagram is drawn according to the following steps : [Refer to Fig. 5.13 (b)].

1. Choose a suitable scale to represent various loads. Let 1 cm represents 1 kN load. Hence load PQ (i.e., 3 kN) will be equal to 3 cm and load QR (i.e., 6 kN) 6 cm.

2. Now take any point p and draw line pq parallel to load PQ (i.e., 3 kN). Take $pq = 3$ cm to represent the load of 3 kN.

3. Through q , draw line qr parallel to load QR (i.e., 6 kN). Cut qr equal to 6 cm to represent the load of 6 kN.

4. Now take any point O . Join the point O to the points p , q and r as shown in Fig. 5.13 (b).

5. Now in Fig. 5.13 (a), extend the lines of action of the loads (3 kN and 6 kN), and the two reactions. Take any point 1, on the line of action of the reaction R_A . Through 1, draw the line 1-2 parallel to pO , intersecting the line of action of load 3 kN at point 2.

6. From point 2, draw line 2-3 parallel to qO , intersecting the line of action of load 6 kN at 3. Similarly, from point 3, draw a line 3-4 parallel to rO , intersecting the line of action of reaction R_B at point 4.

7. Join 1 to 4. The line 1-4 is known as closing line. From the vector diagram, from point O , draw line Os parallel to line 1-4.

8. Measure the length sp and rs . The length sp represents the reaction R_A and length rs represents the reaction R_B .

By measurement, $sp = 4$ cm and $rs = 5$ cm

$\therefore R_A = \text{Length } sp \times \text{scale} = 4 \times 1 \text{ kN} = 4 \text{ kN. Ans.}$

$R_B = \text{Length } rs \times \text{scale} = 5 \times 1 \text{ kN} = 5 \text{ kN. Ans.}$

Problem 5.2. A simply supported beam AB of length 9 m, carries a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Calculate the reactions at A and B .

Sol. Given :

Length of beam = 9 m

Rate of U.D.L. = 10 kN/m

Length of U.D.L. = 6 m

Total load due to U.D.L.

$$= (\text{Length of U.D.L.}) \times \text{Rate of U.D.L.}$$

$$= 6 \times 10 = 60 \text{ kN}$$

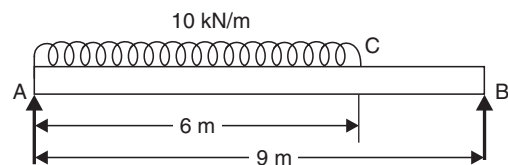


Fig. 5.14

This load of 60 kN will be acting at the middle point of AC i.e., at a distance of $\frac{6}{2} = 3$ m from A .

Let $R_A = \text{Reaction at } A$ and $R_B = \text{Reaction at } B$

Taking the moments of all forces about point A , and equating the resultant moment to zero, we get

$$R_B \times 9 - (6 \times 10) \times 3 = 0 \quad \text{or} \quad 9R_B - 180 = 0$$

$$\therefore R_B = \frac{180}{9} = 20 \text{ kN. Ans.}$$

Also for equilibrium, $\Sigma F_y = 0$

$$\text{or} \quad R_A + R_B = 6 \times 10 = 60$$

$$\therefore R_A = 60 - R_B = 60 - 20 = 40 \text{ kN. Ans.}$$

Problem 5.3. A simply supported beam of length 10 m, carries the uniformly distributed load and two point loads as shown in Fig. 5.15. Calculate the reactions R_A and R_B .

Sol. Given :

Length of beam = 10 m
 Length of U.D.L. = 4 m
 Rate of U.D.L. = 10 kN/m

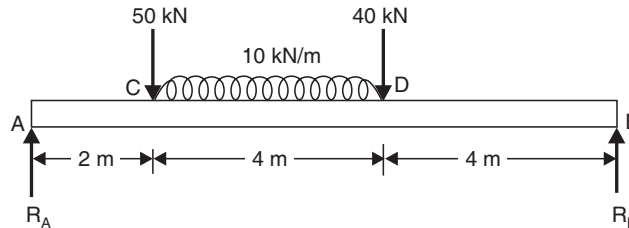


Fig. 5.15

∴ Total load due to U.D.L. = $4 \times 10 = 40$ kN

This load of 40 kN due to U.D.L. will be acting at the middle point of CD, i.e., at a distance of $\frac{4}{2} = 2$ m from C (or at a distance of $2 + 2 = 4$ m from point A).

Let R_A = Reaction at A

and R_B = Reaction at B

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 10 - 50 \times 2 - 40 \times (2 + 4) - (10 \times 4) \left(2 + \frac{4}{2}\right) = 0$$

$$\text{or } 10R_B - 100 - 240 - 160 = 0$$

$$\text{or } 10R_B = 100 + 240 + 160 = 500$$

$$\therefore R_B = \frac{500}{10} = 50 \text{ kN. Ans.}$$

Also for equilibrium of the beam, $\Sigma F_y = 0$

$$\therefore R_A + R_B = \text{Total load on the beam} = 50 + 10 \times 4 + 40 = 130$$

$$\therefore R_A = 130 - R_B = 130 - 50 = 80 \text{ kN. Ans.}$$

Problem 5.3 (A). Find the support reactions in the beam shown in Fig. 5.15 (a).

(U.P. Tech. University, 2000–2001)

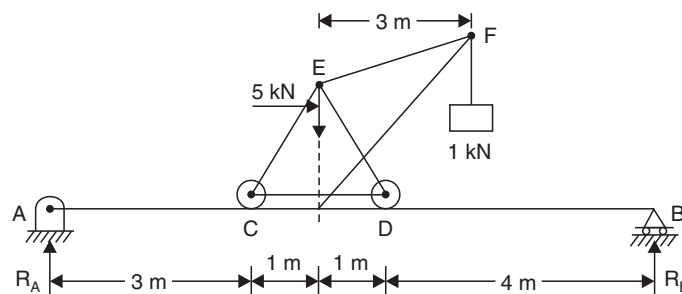


Fig. 5.15 (a)

Sol. Let R_A = Reaction at A and

R_B = Reaction at B

Vertical load at E = 5 kN and at F = 1 kN

As the load at E and F are vertical, hence reactions R_A and R_B will be vertical.

For equilibrium, the moment of all forces about any point should be zero.

Taking moments about A and equating to zero, we get ($M_A = 0$)

$$5 \times 4 + 1 \times 7 - R_B \times 9 = 0$$

$$\text{or} \quad 20 + 7 - 9R_B = 0$$

$$\text{or} \quad 9R_B = 20 + 7 = 27$$

$$\therefore R_B = \frac{27}{9} = 3 \text{ kN. Ans.}$$

$$\text{Also } \Sigma F_y = 0 \quad \therefore R_A + R_B = 5 + 1 = 6 \text{ kN}$$

$$\therefore R_A = 6 - R_B = 6 - 3 = 3 \text{ kN. Ans.}$$

Problem 5.4. A simply supported beam of span 9 m carries a uniformly varying load from zero at end A to 900 N/m at end B. Calculate the reactions at the two ends of the support.

Sol. Given :

Span of beam = 9 m

Load at end A = 0

Load at end B = 900 N/m

Total load on the beam

= Area of right-angled triangle ABC

$$= \frac{AB \times BC}{2} = \frac{9 \times 900}{2} = 4050 \text{ N}$$

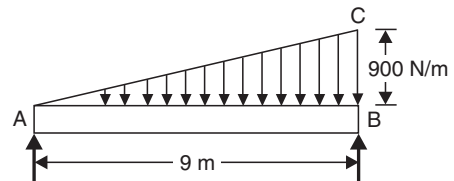


Fig. 5.16

This load will be acting at the C.G. of the $\triangle ABC$, i.e., at a distance of $\frac{2}{3} \times AB = \frac{2}{3} \times 9 = 6 \text{ m}$ from end A.

Let R_A = Reaction at A

and R_B = Reaction at B.

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 9 = (\text{Total load on beam}) \times \text{Distance of total load from A} \\ = 4050 \times 6$$

$$\therefore R_B = \frac{4050 \times 6}{9} = 2700 \text{ N. Ans.}$$

Also for equilibrium of the beam, $\Sigma F_y = 0$

$$\text{or} \quad R_A + R_B = \text{Total load on beam} = 4050$$

$$\therefore R_A = 4050 - R_B = 4050 - 2700 = 1350 \text{ N. Ans.}$$

Problem 5.5. A simply supported beam of length 5 m carries a uniformly increasing load of 800 N/m at one end to 1600 N/m at the other end. Calculate the reactions at both ends.

Sol. Given :

Length of beam = 5 m

Load at A = 800 N/m

Load at B = 1600 N/m

Total load on the beam

$$\begin{aligned}
 &= \text{Area of load diagram } ABDC \\
 &= \text{Area of rectangle } ABEC \\
 &\quad + \text{Area of } \triangle CED \\
 &= AB \times AC + \frac{CE \times ED}{2} \\
 &= 5 \times 800 + \frac{1}{2} \times 5 \times 800 \\
 &\quad (\because CE = AB = 5 \text{ m}, \\
 &\quad \quad ED = 1600 - 800 = 800) \\
 &= 4000 + 2000 = 6000 \text{ N}
 \end{aligned}$$

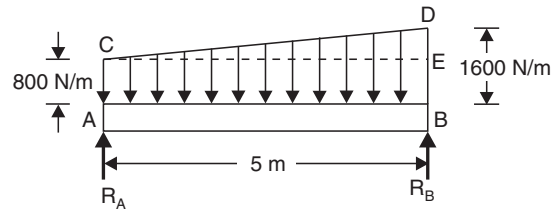


Fig. 5.17

The C.G. of the rectangle $ABEC$ will be at a distance of $\frac{5}{2} = 2.5$ m from A , whereas the C.G. of the triangle CED will be at a distance of $\frac{2}{3} \times 5 = 3.33$ m from A .

Let R_A = Reaction at A
and R_B = Reaction at B .

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 5 - (\text{Load due to rectangle}) \times \text{Distance of C.G. of rectangle from } A - (\text{Load due to triangle}) \times \text{Distance of C.G. of triangle from } A = 0$$

$$\text{or } 5R_B - (5 \times 800) \times 2.5 - \left(\frac{1}{2} \times 5 \times 800\right) \times \left(\frac{2}{3} \times 5\right) = 0$$

$$\text{or } 5R_B - 10000 - 6666.66 = 0$$

$$\text{or } 5R_B = 10000 + 6666.66 = 16666.66$$

$$\text{or } R_B = \frac{16666.66}{5} = 3333.33 \text{ N. Ans.}$$

Also for the equilibrium of the beam, $\Sigma F_y = 0$

$$\therefore R_A + R_B = \text{Total load on the beam} = 6000 \quad (\because \text{Total load on beam} = 6000 \text{ N})$$

$$\therefore R_A = 6000 - R_B = 6000 - 3333.33 = 2666.67 \text{ N. Ans.}$$

Problem 5.5 (A). Determine the reactions at A , B and D of the system shown in Fig. 5.17 (a).
(U.P. Tech. University, 2001–2002)

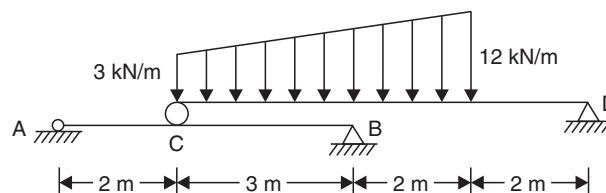


Fig. 5.17 (a)

Sol. Since the load on the beam CD is vertical, the reaction at C and D will be vertical. Also the reaction at A and B will be vertical.

(i) Reaction at D

First consider the free-body diagram of beam CD as shown in Fig. 5.17 (b). The distributed load is divided into a rectangle and a triangle as shown in Fig. 5.17 (b).

Load due to rectangle
 = Area of rectangle
 = $3 \times 5 = 15 \text{ kN}$

This load will be acting through the C.G. of the rectangle which is at a distance of 2.5 m from C .

Load due to triangle

$$= \text{Area of triangle} = \frac{1}{2} (5) \times (12 - 3)$$

$$= \frac{1}{2} \times 5 \times 9 = 22.5 \text{ kN}$$

This load will be acting at a distance of $\frac{2}{3}$ rd of 5 m from C , i.e., at a distance of $\frac{10}{3}$ m from C .

Taking the moments of all the forces about point C and equating the resultant moment to zero, we get

$$15 \times 2.5 + 22.5 \times \frac{10}{3} - R_D \times 7 = 0$$

or $37.5 + 75 - 7R_D = 0$

or $R_D = \frac{37.5 + 75}{7} = \frac{112.5}{7} = 16.07 \text{ kN}$

Also $\Sigma F_y = 0$ or $R_C + R_D = \text{Total load on beam}$
 $= 15 + 22.5 = 37.5 \text{ kN}$

$\therefore R_C = 37.5 - R_D = 37.5 - 16.07 = 21.43 \text{ kN}$.

(ii) Reaction at A and B

The reaction at C on beam CD will act as a concentrated load on beam AB . This reaction will be in opposite direction, i.e., downward for beam AB at point C as shown in Fig. 5.17 (c).

Consider the free-body diagram (or equilibrium) of beam AB as shown in Fig. 5.17 (c).

Taking moments about point A , we get

$$\Sigma M_A = 0$$

$\therefore R_B \times 5 - 21.43 \times 2 = 0$

or $5R_B = 42.86$

or $R_B = \frac{42.86}{5} = 8.57 \text{ kN}$. **Ans.**

$\Sigma F_y = 0 \therefore R_A + R_B = 21.43$

$\therefore R_A = 21.43 - R_B = 21.43 - 8.57 = 12.86 \text{ kN}$. **Ans.**

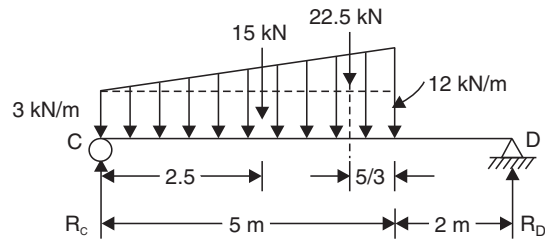


Fig. 5.17 (b)

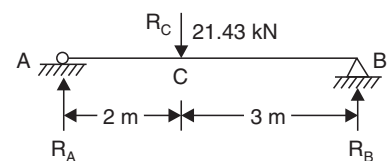


Fig. 5.17 (c)

5.4.4. Problems on Overhanging Beams. If the end portion of a beam is extended beyond the support, then the beam is known as overhanging beam. Overhanging portion may be at one end of the beam or at both ends of the beam as shown in Fig. 5.18.

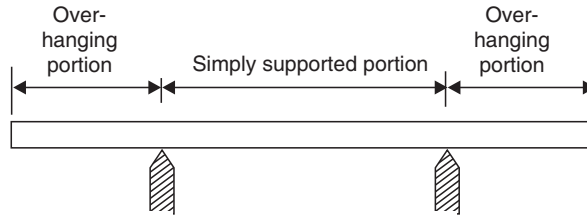


Fig. 5.18

Problem 5.6. A beam AB of span 8 m, overhanging on both sides, is loaded as shown in Fig. 5.19. Calculate the reactions at both ends.

Sol. Given :

Span of beam = 8 m

Let R_A = Reaction at A

and R_B = Reaction at B.

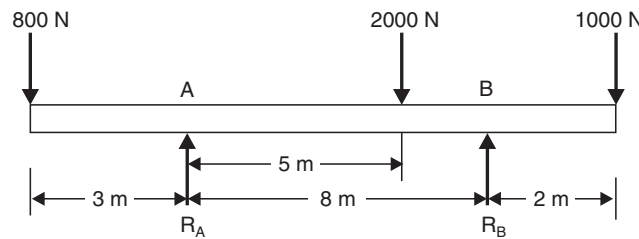


Fig. 5.19

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 8 + 800 \times 3 - 2000 \times 5 - 1000 \times (8 + 2) = 0$$

$$\text{or } 8R_B + 2400 - 10000 - 10000 = 0$$

$$\text{or } 8R_B = 20000 - 2400 = 17600$$

$$\therefore R_B = \frac{17600}{8} = 2200 \text{ N. Ans.}$$

Also for the equilibrium of the beam, we have

$$R_A + R_B = 800 + 2000 + 1000 = 3800$$

$$\therefore R_A = 3800 - R_B = 3800 - 2200 = 1600 \text{ N. Ans.}$$

Problem 5.7. A beam AB of span 4 m, overhanging on one side upto a length of 2 m, carries a uniformly distributed load of 2 kN/m over the entire length of 6 m and a point load of 2 kN/m as shown in Fig. 5.20. Calculate the reactions at A and B.

Sol. Given :

Span of beam = 4 m

Total length = 6 m

Rate of U.D.L. = 2 kN/m

Total load due to U.D.L. = $2 \times 6 = 12$ kN

The load of 12 kN (*i.e.*, due to U.D.L.) will act at the middle point of AC, *i.e.*, at a distance of 3 m from A.

Let R_A = Reaction at A

and R_B = Reaction at B.

Taking the moments of all forces about point A and equating the resultant moment to zero, we get

$$R_B \times 4 - (2 \times 6) \times 3 - 2 \times (4 + 2) = 0$$

$$\text{or } 4R_B - 36 - 12 = 0$$

$$\text{or } 4R_B = 36 + 12 = 48$$

$$\therefore R_B = \frac{48}{4} = 12 \text{ kN. Ans.}$$

$$\text{Also for equilibrium, } \Sigma F_y = 0 \text{ or } R_A + R_B = 12 + 2 = 14$$

$$\therefore R_A = 14 - R_B = 14 - 12 = 2 \text{ kN. Ans.}$$

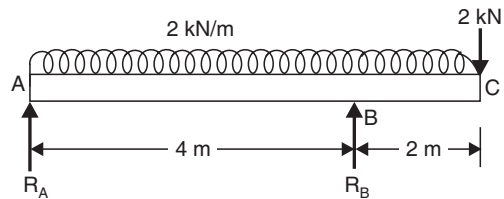


Fig. 5.20

5.4.5. Problems on Roller and Hinged Supported Beams. In case of roller supported beams, the reaction on the roller end is always *normal* to the support. All the steel trusses of the bridges is generally having one of their ends supported on rollers. The main advantage of such a support is that beam, due to change in temperature, can move easily towards left or right, on account of expansion or contraction.

In case of a hinged supported beam, the reaction on the hinged end may be either vertical or inclined, depending upon the type of loading. If the load is acting vertically downwards, the reaction at the hinge will be vertical. But if acting load is inclined to the beam then reaction will also be inclined. This reaction will have horizontal and vertical component. The main advantage of a hinged end is that the beam remains stable. Hence all the steel trusses of the bridges, have one of their end on rollers and the other end as hinged.

Problem 5.8. A beam AB 1.7 m long is loaded as shown in Fig. 5.21. Determine the reactions at A and B.

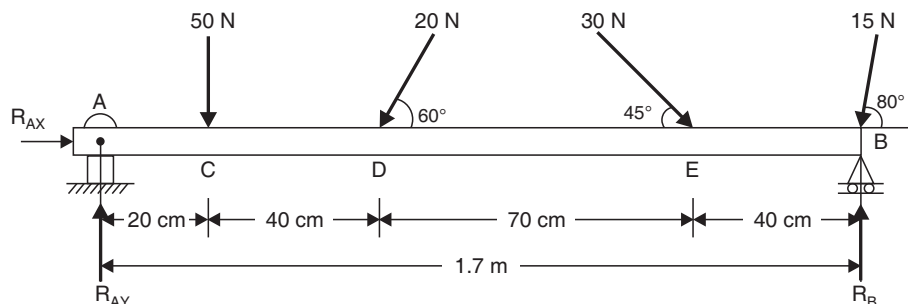


Fig. 5.21

Sol. Given :

Length of beam = 1.7 m

Let R_A = Reaction at A

and R_B = Reaction at B.

Since the beam is supported on rollers at B, therefore the reaction R_B will be vertical.

The beam is hinged at A, and is carrying inclined load, therefore the reaction R_A will be inclined. This means reaction R_A will have two components, *i.e.*, vertical component and horizontal component.

Let R_{AX} = Horizontal component of reaction R_A
 R_{AY} = Vertical component of reaction R_A .

First resolve all the inclined loads into their vertical and horizontal components.

(i) Vertical component of load at D

$$= 20 \sin 60^\circ = 20 \times 0.866 = 17.32 \text{ N}$$

and its horizontal component $= 20 \cos 60^\circ = 10 \text{ N} \leftarrow$

(ii) Vertical component of load at E

$$= 30 \sin 45^\circ = 21.21 \text{ N}$$

and its horizontal component $= 30 \cos 45^\circ = 21.21 \text{ N} \rightarrow$

(iii) Vertical component of load at B

$$= 15 \sin 80^\circ = 14.77 \text{ N}$$

and its horizontal component $= 15 \cos 80^\circ = 2.6 \text{ N} \leftarrow$

From condition of equilibrium, $\Sigma F_x = 0$

$$\text{or } R_{AX} - 10 + 21.21 - 2.6 = 0$$

$$\text{or } R_{AX} = 10 - 21.21 + 2.6 = -8.61 \text{ N}$$

-ve sign shows that the assumed direction of R_{AX} (*i.e.*, horizontal component of R_A) is wrong. Correct direction will be opposite to the assumed direction. Assumed direction of R_{AX} is towards right. Hence correct direction of R_{AX} will be towards left at A.

$$\therefore R_{AX} = 8.61 \text{ N} \leftarrow$$

To find R_B , take moments* of all forces about A.

For equilibrium, $\Sigma M_A = 0$

$$\therefore 50 \times 20 + (20 \sin 60^\circ) \times (20 + 40) + (30 \times \sin 45^\circ) \times (20 + 40 + 70) + (15 \sin 80^\circ) \times (170) - 170 R_B = 0$$

$$\text{or } 1000 + 1039.2 + 2757.7 + 2511 - 170 R_B = 0$$

$$\text{or } 7307.9 - 170 R_B = 0$$

$$\therefore R_B = \frac{7307.9}{170} = 42.98 \text{ N. Ans.}$$

To find R_{AY} , apply condition of equilibrium, $\Sigma F_y = 0$

$$\text{or } R_{AY} + R_B = 50 + 20 \sin 60^\circ + 30 \sin 45^\circ + 15 \sin 80^\circ$$

$$\text{or } R_{AY} + 42.98 = 50 + 17.32 + 21.21 + 14.77 = 103.3$$

$$\therefore R_{AY} = 103.3 - 42.98 = 60.32 \text{ N} \uparrow$$

$$\begin{aligned} \therefore \text{Reaction at A, } R_A &= \sqrt{R_{AX}^2 + R_{AY}^2} \\ &= \sqrt{8.61^2 + 60.32^2} = 60.92 \text{ N} \end{aligned}$$

The angle made by R_A with x-direction is given by

$$\tan \theta = \frac{R_{AY}}{R_{AX}} = \frac{60.32}{8.61} = 7.006$$

$$\therefore \theta = \tan^{-1} 7.006 = 81.87^\circ. \text{ Ans.}$$

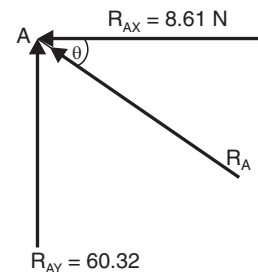


Fig. 5.21 (a)

*The moment of all horizontal components about point A, will be zero.

Problem 5.8 (A). A force $P = 5000 \text{ N}$ is applied at the centre C of the beam AB of length 5 m as shown in Fig. 5.21 (b). Find the reaction at the hinge and roller supports.

(U.P. Tech. University, 2001–2002)

Sol. The reaction R_B will be vertical as the beam is supported on rollers at end B .

The reaction R_A will be inclined, as the beam is hinged at A and carries inclined load.

Let R_{Ax} = Horizontal component of R_A and

R_{Ay} = Vertical component of R_A .

The inclined load of 5000 N is resolved into horizontal and vertical components as shown in Fig. 5.21 (c).

For equilibrium of beam,

$$\Sigma F_x = 0, \Sigma F_y = 0 \text{ and } \Sigma M = 0$$

$$\text{For } \Sigma F_x = 0, \quad R_{Ax} = 5000 \cos 30^\circ = 4330 \text{ N. Ans.}$$

$$\text{For } \Sigma F_y = 0, \quad R_{Ay} + R_B = 5000 \sin 30^\circ = 2500 \text{ N}$$

Taking moments about A , $\Sigma M_A = 0$, we get

$$R_B \times 5 = (5000 \sin 30^\circ) \times 2.5$$

$$\therefore R_B = \frac{2500 \times 2.5}{5} = 1250 \text{ N. Ans.}$$

Substituting this value of R_B in equation (i), we get

$$R_{Ay} + 1250 = 2500$$

$$\therefore R_{Ay} = 2500 - 1250 = 1250 \text{ N. Ans.}$$

Problem 5.9. A beam AB 6 m long is loaded as shown in Fig. 5.22. Determine the reactions at A and B by (a) analytical method, and (b) graphical method.

Sol. Given :

Length of beam = 6 m

Let R_A = Reaction at A

R_B = Reaction at B .

The reaction R_B will be vertical as the beam is supported on rollers at end B .

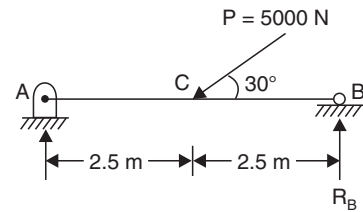


Fig. 5.21 (b)

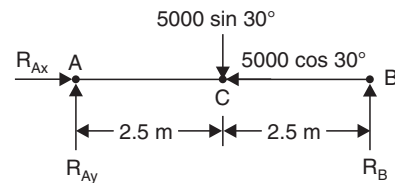


Fig. 5.21 (c)

...(i)

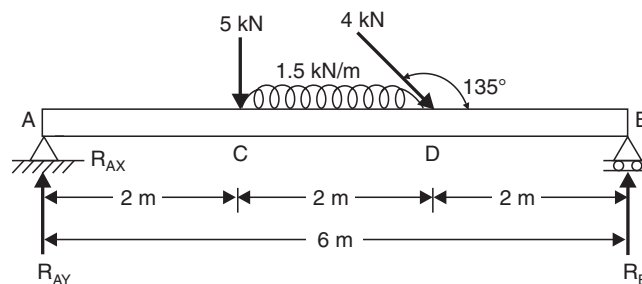


Fig. 5.22

The reaction R_A will be inclined, as the beam is hinged at A and carries inclined load.

Let R_{Ax} = Horizontal component of reaction R_A

R_{Ay} = Vertical component of reaction R_A .

(a) **Analytical Method.** First resolve the inclined load of 4 kN into horizontal and vertical components.

Horizontal component of 4 kN at D

$$= 4 \cos 45^\circ = 2.828 \text{ kN} \rightarrow$$

and its vertical component $= 4 \sin 45^\circ = 2.828 \text{ kN} \downarrow$

$$\text{For equilibrium, } \Sigma F_x = 0$$

$$\therefore -R_{AX} + 2.828 = 0$$

$$\text{or } R_{AX} = 2.828 \text{ N}$$

To find R_B , take the moments* of all forces about point A .

$$\therefore \text{For equilibrium, } \Sigma M_A = 0$$

$$R_B \times 6 - 5 \times 2 - (2 \times 1.5) \left(2 + \frac{2}{2} \right) - (4 \sin 45^\circ) (2 + 2) = 0$$

$$\text{or } 6R_B - 10 - 9 - 11.312 = 0$$

$$\text{or } 6R_B = 10 + 9 + 11.312 = 30.312$$

$$\therefore R_B = \frac{30.312}{6} = 5.052 \text{ kN. Ans.}$$

To find R_{AY} , apply the condition of equilibrium, $\Sigma F_y = 0$

$$\therefore R_{AY} + R_B - 5 - (1.5 \times 2) - 4 \sin 45^\circ = 0$$

$$\text{or } R_{AY} + 5.052 - 5 - 3 - 2.828 = 0$$

$$\therefore R_{AY} = -5.052 + 5 + 3 + 2.828 = 5.776 \text{ kN}$$

$\therefore$ Reaction at A is given by

$$\begin{aligned} R_A &= \sqrt{R_{AX}^2 + R_{AY}^2} \\ &= \sqrt{2.828^2 + 5.776^2} \\ &= \sqrt{7.997 + 33.362} \\ &= \sqrt{41.359} = 6.43 \text{ kN. Ans.} \end{aligned}$$

Let θ = Angle made by R_A with x -direction.

$$\therefore \tan \theta = \frac{R_{AY}}{R_{AX}} = \frac{5.776}{2.828} = 2.0424$$

$$\therefore \theta = \tan^{-1} 2.0424 = 63.9^\circ. \text{ Ans.}$$

(b) **Graphical Method.** First of all convert the uniformly distributed load (U.D.L.) into its equivalent point load acting at the C.G. of the portion on which U.D.L. is acting. Hence total load due to U.D.L. will be $1.5 \times 2 = 3 \text{ kN}$ acting at a distance of 3 m from point A :

(i) Now draw the space diagram of the beam according to some suitable scale, as shown in Fig. 5.23 (a).

(ii) Name all the loads and reactions according to Bow's notation. Now draw the vector diagram as shown in Fig. 5.23 (b). Choose any suitable scale for vector diagram.

(iii) Take any point p for drawing vector diagram. From p , draw line pq parallel and equal to load 5 kN (i.e., load PQ). From q , draw qr parallel and equal to 3 kN. From r , draw rs parallel and equal to 4 kN load.

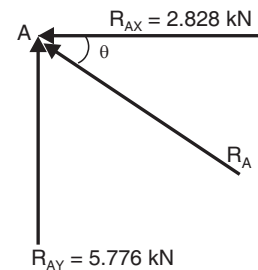


Fig. 5.22 (a)

*The moments of all horizontal components about point A , will be zero.

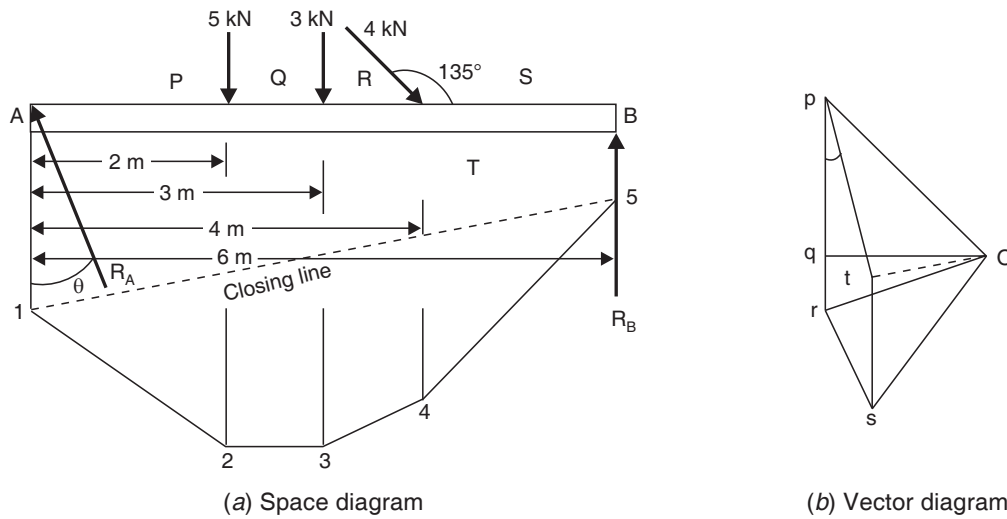


Fig. 5.23

(iv) Now take any point O , and join Op , Oq , Or and Os .

(v) Now in space diagram [i.e., Fig. 5.23 (a)], extend the lines of actions of loads PQ , QR , RS and reaction R_B .

(vi) Take any point 1, vertically below the point A as shown in Fig. 5.23 (a). From point 1, draw line 1-2 parallel to line pO , intersecting the line of action of 5 kN at point 2.

(vii) Similarly, draw lines 2-3, 3-4 and 4-5 parallel to qO , rO and sO respectively. Join point 1 to 5. Line 1-5 is the closing line in space diagram.

(viii) From O in vector diagram, draw a line parallel to closing line 1-5. Now through s , draw a line st vertical (as the reaction R_B is vertical), intersecting the line through O at t . Join t to p .

(ix) The length st represents the reaction R_B in magnitude and direction whereas the length tp gives the magnitude and direction of reaction R_A . At point A, draw a line parallel to tp as shown in Fig. 5.23 (a). By measurement, we get

$$R_A = \text{length } tp = 6.43 \text{ kN}$$

$$R_B = \text{length } st = 5.052 \text{ kN}$$

and

$$\theta = 26.1^\circ.$$

Problem 5.10. A beam AB 10 m long is hinged at A and supported on rollers over a smooth surface inclined at 30° to the horizontal at B. The beam is loaded as shown in Fig. 5.24. Determine reactions at A and B.

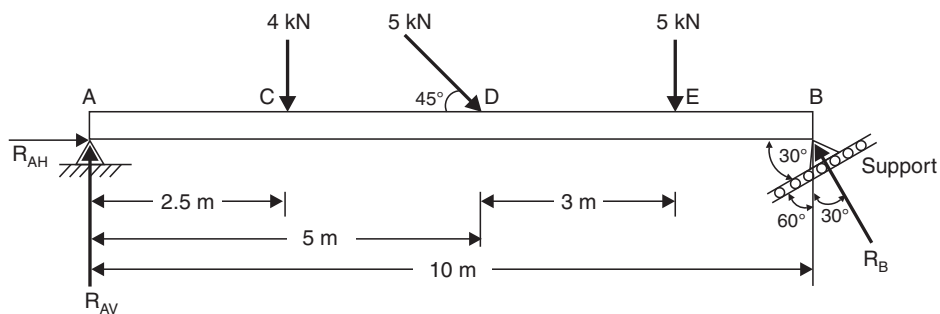


Fig. 5.24

Sol. Given :

Length of beam = 10 m

Let R_A = Reaction at A

and R_B = Reaction at B

The reaction R_B will be normal to the support as the beam at B is supported on the rollers. But the support at B is making an angle 30° with the horizontal or 60° with the vertical as shown in Fig. 5.24. Hence the reaction R_B is making an angle of 30° with the vertical.

The vertical component of R_B
 $= R_B \cos 30^\circ$

and horizontal component of R_B
 $= R_B \sin 30^\circ$

These components are shown in Fig. 5.24 (a).

Resolving the load of 5 kN acting at D into horizontal and vertical components, we get

Vertical component of 5 kN
 $= 5 \sin 45^\circ = 5 \times 0.707 = 3.535 \text{ kN}$

Horizontal component of 5 kN
 $= 5 \cos 45^\circ = 5 \times 0.707 = 3.535 \text{ kN}$

The reaction at A will be inclined, as the end A is hinged and beam carries inclined load.

Let R_{AH} = Horizontal component of reaction R_A

R_{AV} = Vertical component of reaction R_A

For equilibrium of the beam, the moments of all forces about any point should be zero.

Taking the moments* about point A,

$$(R_B \cos 30^\circ) \times 10 - 4 \times 2.5 - (5 \sin 45^\circ) \times 5 - 5 \times 8 = 0.$$

$$8.66 R_B - 10 - 17.675 - 40 = 0$$

or $R_B = \frac{10 + 17.675 + 40}{8.66} = 7.81 \text{ kN. Ans.}$

For equilibrium, $\Sigma F_x = 0$

or $R_{AH} + 5 \cos 45^\circ - R_B \sin 30^\circ = 0$

or $R_{AH} + 3.535 - 7.81 \times 0.5 = 0$

$\therefore R_{AH} = 7.81 \times 0.5 - 3.535 = 0.37 \text{ kN}$

For equilibrium, $\Sigma F_y = 0$

$\therefore R_{AV} + R_B \cos 30^\circ - 4 - 5 \sin 45^\circ - 5 = 0$

or $R_{AV} + 7.81 \times 0.866 - 4 - 3.535 - 5 = 0$

or $R_{AV} + 6.763 - 12.535 = 0$

or $R_{AV} = 12.535 - 6.763 = 5.77 \text{ kN}$

$\therefore$ Reaction at A, $R_A = \sqrt{R_{AH}^2 + R_{AV}^2} = \sqrt{0.37^2 + 5.77^2}$
 $= 5.78 \text{ kN. Ans.}$

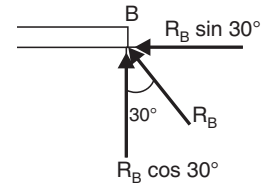


Fig. 5.24 (a)

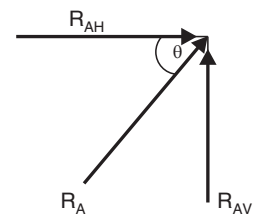


Fig. 5.24 (b)

*The moments of horizontal components of 5 kN at D and of reaction R_B will be zero about the point A.

The angle made by R_A with x -direction is given by

$$\tan \theta = \frac{R_{AV}}{R_{AH}} = \frac{5.77}{0.37} = 15.59$$

$$\therefore \theta = \tan^{-1} 15.59 = 86.33^\circ. \quad \text{Ans.}$$

Problem 5.10 (A). A beam 8 m long is hinged at A and supported on rollers over a smooth surface inclined at 30° to the horizontal at B. The beam is loaded as shown in Fig. 5.24 (c). Determine the support reactions.

(U.P. Tech. University, 2001–2003)

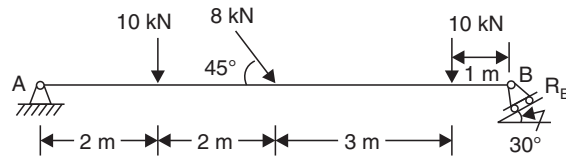


Fig. 5.24 (c)

Sol. From Fig. 5.24 (d), the reaction at B is normal to roller surface. The components of R_B in horizontal and vertical direction are $R_{BH} = R_B \sin 30^\circ$ and $R_{BV} = R_B \cos 30^\circ$

For equilibrium of beam, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$.

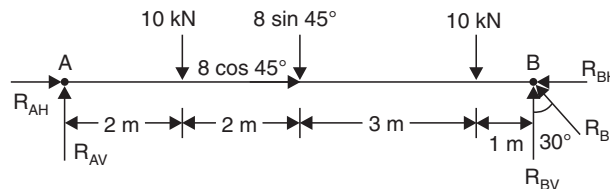


Fig. 5.24 (d)

(i) For $\Sigma F_x = 0$, we have

$$R_{AH} + 8 \cos 45^\circ - R_{BH} = 0$$

$$\text{or } R_{AH} + 8 \cos 45^\circ - R_B \sin 30^\circ = 0 \quad \dots(i)$$

(ii) For $\Sigma F_y = 0$, we have

$$R_{AV} - 10 - 8 \sin 45^\circ - 10 + R_{BV} = 0$$

$$\text{or } R_{AV} - 8 \sin 45^\circ - 20 + R_B \cos 30^\circ = 0 \quad \dots(ii)$$

(iii) For $\Sigma M = 0$. Take moments about B ($\Sigma M_B = 0$),

$$R_{AV} \times 8 - 10 \times 6 - (8 \sin 45^\circ) \times 4 - 10 \times 1 = 0$$

$$\text{or } 8R_{AV} - 60 - 22.627 - 10 = 0$$

$$\text{or } R_{AV} = \frac{60 + 22.627 + 10}{8} = 11.578 \text{ kN.} \quad \text{Ans.}$$

Substituting this value of R_{AV} in equation (ii),

$$11.578 - 8 \sin 45^\circ - 20 + R_B \cos 30^\circ = 0$$

$$\text{or } R_B \cos 30^\circ = 20 + 8 \sin 45^\circ - 11.578 = 14.078$$

$$\therefore R_B = \frac{14.078}{\cos 30^\circ} = 16.25 \text{ kN.} \quad \text{Ans.}$$

Substituting this value of R_B in equation (i),

$$R_{AH} + 8 \cos 45^\circ - 16.25 \sin 30^\circ = 0$$

$$\text{or } R_{AH} = 16.25 \sin 30^\circ - 8 \cos 45^\circ \\ = 8.125 - 5.656 = 2.47 \text{ kN.} \quad \text{Ans.}$$

Now

$$R_A = \sqrt{R_{AH}^2 + R_{AV}^2}$$

$$= \sqrt{2.47^2 + 11.578^2} = \sqrt{6.1 + 134.05} = 11.838 \text{ kN. Ans.}$$

Problem 5.11. Two rollers *C* and *D* produce vertical forces *P* and *Q* on the horizontal beam *AB*, as shown in Fig. 5.25. Determine the distance *x* of the load *P* from the support *A* if the reaction at *A* is twice as great as the reaction at *B*. The weight of the beam is to be neglected. Given : *P* = 18 kN, *Q* = 9 kN, *L* = 3.6 m and *C* = 0.9 m. (U.P. Tech. University, May 2009)

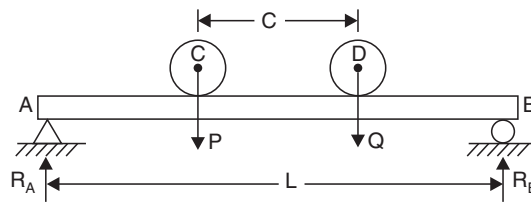


Fig. 5.25

Sol. Given :

$$P = 18 \text{ kN} ; \quad Q = 9 \text{ kN}, \quad L = 3.6 \text{ m}, \quad C = 0.9 \text{ m}, \quad R_A = 2 R_B$$

Let *x* = Distance of load *P* from the support *A*.

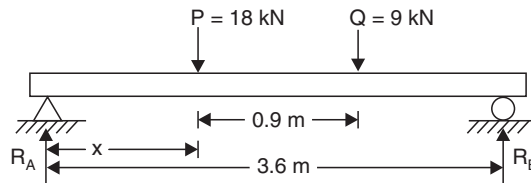


Fig. 5.26

Refer to Fig. 5.26

$$\Sigma F_y = 0,$$

$$\text{But } R_A = 2 R_B$$

$\therefore$

$$R_A + R_B = 18 + 9 = 27 \text{ kN}$$

$$2R_B + R_B = 27 \text{ kN}$$

or

$$R_B = \frac{27}{3} = 9 \text{ kN}$$

and

$$R_A = 27 - R_B = 27 - 9 = 18 \text{ kN}$$

Taking moments of all forces about *A* or $\Sigma M_A = 0$

$$\therefore R_B \times 3.6 = 18 \times x + 9(x + 0.9)$$

or

$$9 \times 3.6 = 18x + 9x + 9 \times 0.9$$

or

$$32.4 = 27x + 8.1$$

or

$$27x = 32.4 - 8.1 = 24.3$$

$\therefore$

$$x = \frac{24.3}{27} = 0.9 \text{ m. Ans.}$$

(Given)

Problem 5.12. What force and moment is transmitted to the supporting wall at A in the given cantilever beam shown in Fig. 5.27 ? (U.P. Tech. University, 2002–2003)

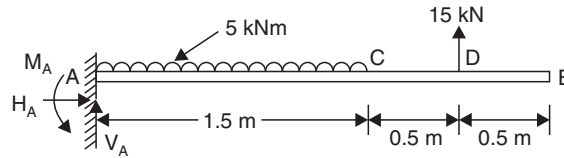


Fig. 5.27

Sol. As the end A is fixed. Hence there will be fixing moment at A and also there will be reactions at A.

Let M_A = Fixing moment at A (or moment transmitted at A). Assume this moment to be anti-clockwise.

V_A = Vertical reaction at A. Assume it is acting vertically up.

H_A = Horizontal reaction at A.

For equilibrium of cantilever, we have

$$\Sigma F_x = 0, \Sigma F_y = 0 \text{ and } \Sigma M = 0$$

(i) For $\Sigma F_x = 0$ (i.e., resultant force in horizontal direction), we have

$$H_A = 0 \quad \text{(No horizontal force on cantilever)}$$

(ii) For $\Sigma F_y = 0$ (i.e., resultant force in vertical direction), we have

$$V_A - (5 \times 1.5) + 15 = 0$$

$$\text{or} \quad V_A = 5 \times 1.5 - 15 = -7.5 \text{ kN}$$

Negative sign means, V_A is acting vertically downward (i.e., opposite to the assumed direction)

$$\therefore V_A = 7.5 \text{ kN. Ans.}$$

(iii) For $\Sigma M = 0$ (i.e., the resultant moment about any point should be zero), take moments about point B and equate the resultant moment to zero. We get [Refer to Fig. 5.28 (a)]

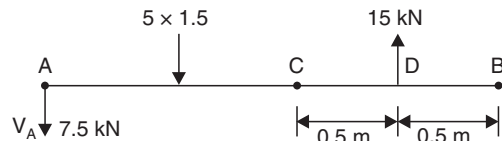


Fig. 5.28

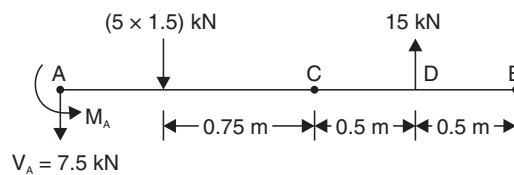


Fig. 5.28 (a)

$$-M_A - V_A \times 2.5 - 7.5 \times 1.75 + 15 \times 0.5 = 0$$

$$\begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{(A.C.W.)} & & \text{(A.C.W.)} & & \text{(A.C.W.)} & & \text{(C.W.)} \end{array}$$

A.C.W. = Anti-clockwise and C.W. = clockwise

$$\text{or} \quad -M_A - 7.5 \times 2.5 - 7.5 \times 1.75 + 7.5 = 0$$

$$\begin{aligned} \text{or} \quad M_A &= -7.5 \times 2.5 - 7.5 \times 1.75 + 7.5 \\ &= -7.5(2.5 + 1.75 - 1) = -7.5 \times 3.25 \\ &= -24.375 \text{ kNm} \end{aligned}$$

Negative sign means, moments at A will be in opposite direction to the assumed direction. This means M_A will be anti-clockwise as shown in Fig. 5.28 (b).

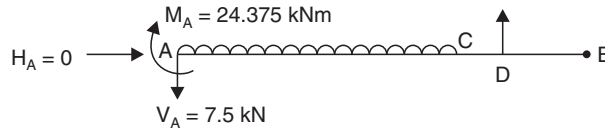


Fig. 5.28 (b)

Problem 5.13. Fig. 5.29 shows two vertical forces and a couple of moment 2000 Nm acting on a horizontal rod which is fixed at end A. Determine the resultant of the system. Also determine the equivalent force system through A. (U.P. Tech. University, 2000–2001)

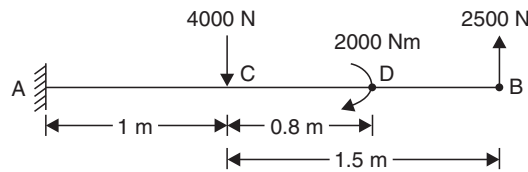


Fig. 5.29

Sol. The end A is a fixed one. The fixed end prevents vertical movement and rotation of the beam.

Let R_A = Reaction at A in which R_{AH} and R_{AV} are its horizontal and vertical components.

M_A = Fixing end moment at A.

The free-body diagram is shown in Fig. 5.29 (a). For equilibrium,

$$\Sigma F_x = 0, \Sigma F_y = 0 \text{ and } \Sigma M = 0$$

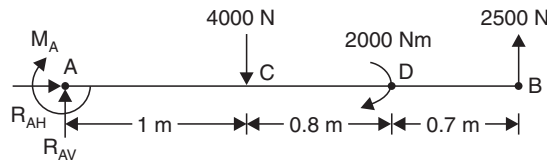


Fig. 5.29 (a)

(i) For $\Sigma F_x = 0$, we have, $R_{AH} = 0$

(ii) For $\Sigma F_y = 0$, we have

$$R_{AV} - 4000 + 2500 = 0$$

or

$$R_{AV} = 4000 - 2500 = 1500 \text{ N}$$

$$\therefore R_A = \sqrt{R_{AH}^2 + R_{AV}^2} = \sqrt{0^2 + 1500^2} = 1500 \text{ N. Ans.}$$

(iii) For $\Sigma M = 0$, take the moments about A, (i.e., $\Sigma M_A = 0$),

$$M_A + 4000 \times 1 + 2000 - 2500 \times 2.5 = 0$$

$$\begin{array}{ccccccc} \downarrow & \downarrow & \downarrow & & \downarrow & & \\ \text{(Clockwise)} & \text{(CW)} & \text{(CW)} & & \text{(ACW)} & & \\ & & & & \text{(CW)} & & \end{array}$$

or

$$\begin{aligned} M_A &= -4000 - 2000 + 2500 \times 2.5 \\ &= -6000 + 6250 = 250 \text{ Nm (clockwise)} \end{aligned}$$

The equivalent force system through A consists of reaction $R_A = 1500 \text{ N}$ ($\uparrow$) and fixing moment (or couple) at A equal to 250 Nm acting clockwise.

5.4.6. Problems when Beams are Subjected to Couples. In this section, the reactions of the beam will be calculated when beams are subjected to clockwise or anti-clockwise couple along with the other loads. While taking the moments about any point, the magnitude and sense of the couple is taken into consideration. But when the total load on the beam is calculated the magnitude and sense of the couple is not considered.

Problem 5.14. A simply supported beam AB of 7 m span is subjected to : (i) 4 kNm clockwise couple at 2 m from A, (ii) 8 kNm anti-clockwise couple at 5 m from A and (iii) a triangular load with zero intensity at 2 m from A increasing to 4 kN per m at a point 5 m from A. Determine reactions at A and B.

Sol. Given :

Span of beam = 7 m

Couple at C (i.e., at 2 m from A) = 4 kNm (clockwise)

Couple at D (i.e., at 5 m from A) = 8 kNm (anti-clockwise)

Triangular load from C to D with :

Vertical load at C = 0

Vertical load at D = 4 kN/m

$$\therefore \text{Total load on beam} = \text{Area of triangle } CDE = \frac{CD \times DE}{2} = \frac{3 \times 4}{2} = 6 \text{ kN}$$

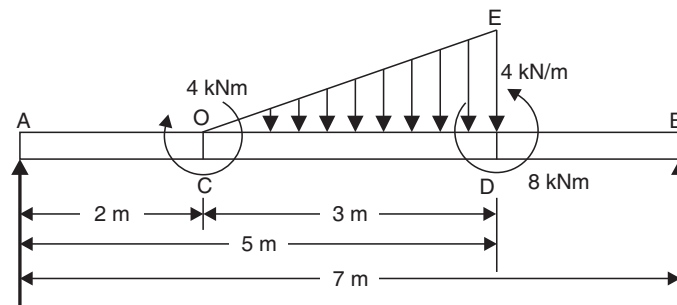


Fig. 5.30

This load will be acting at the C.G. of the $\triangle CDE$ i.e., at a distance of $\frac{2}{3} \times CD = \frac{2}{3} \times 3 = 2 \text{ m}$ from C or $2 + 2 + 4 \text{ m}$ from end A.

Let R_A = Reaction at A,

R_B = Reaction at B.

Taking the moments of all forces about point A and equating the resultant moment to zero (i.e., $\Sigma M_A = 0$ and considering clockwise moment positive and anti-clockwise moment negative), we get

$$-R_B \times 7 + 4^* - 8^{**} + (\text{Total load on beam}) \times (\text{Distance of total load from A}) = 0$$

or

$$-7 R_B + 4 - 8 + 6 \times 4 = 0$$

*The couple at C is 4 kNm clockwise. Hence its sense is positive.

**The couple at D is 8 kNm anti-clockwise. Hence its sense is negative. The couple is also moment.

$$-7 R_B + 4 - 8 + 24 = 0$$

or $20 = 7 R_B$

or $R_B = \frac{20}{7} \text{ kN. Ans.}$

Also for the equilibrium of the beam $\Sigma F_y = 0$

or $R_A + R_B = \text{Total load on the beam} = 6 \text{ kN}$

$$\therefore R_A = 6 - R_B = 6 - \frac{20}{7} = \frac{22}{7} \text{ kN. Ans.}$$

HIGHLIGHTS

1. The reaction at the knife edge support will be normal to the surface of the beam.
2. The reaction in case of roller support will be normal to the surface of roller base.
3. The reaction at the hinged end (or pinned end) will be either vertical or inclined depending upon the type of loading. If the load is vertical, then reaction will be vertical. But if the load is inclined, then the reaction will also be inclined.
4. For a smooth surface, the reaction is always normal to the support.
5. A load, acting at a point on a beam, is known as point load or concentrated load.
6. If each unit length of the beam carries same intensity of load, then that type of load is known as uniformly distributed load which is written as U.D.L.
7. The reactions of a beam can be determined by analytical method and graphical method.
8. The reactions by analytical method are obtained by using equations of equilibrium, *i.e.*, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M = 0$.
9. The reactions by graphical method are obtained by drawing a space diagram and a vector diagram.
10. If a beam is loaded with inclined loads, then the inclined loads are resolved normal to the beam and along the beam. Now the equations of equilibrium are used for finding reactions.

EXERCISE 5

(A) Theoretical Questions

1. Explain the term 'support reactions.' What are the different types of support ?
2. What is the difference between a roller support and a hinged support ?
3. What are the important types of loading on a beam ? Differentiate between uniformly distributed load and uniformly varying load on a beam.
4. Name the different methods of finding the reactions at the two supports of a beam.
5. A beam AB of length L is simply supported at the ends A and B . It carries two point loads W_1 and W_2 at a distance L_1 and L_2 from the end A respectively. How will you find the reactions R_A and R_B by analytical method ?
6. Describe in details the different steps involved in finding the reactions of a beam by graphical method.
7. Define and explain an overhanging beam.
8. What is the main advantage of roller support in case of the steel trusses of the bridges ?

(B) Numerical Problems

1. A simply supported beam of length 8 m carries point loads of 4 kN and 6 kN at a distance of 2 m and 4 m from the left end. Find the reactions at both ends analytically and graphically.

[Ans. 6 kN, 4 kN]

2. A simply supported beam of length 8 m carries a uniformly distributed load of 10 kN/m for a distance of 4 m, starting from a point which is at a distance of 1 m from the left end. Calculate the reactions at both ends. [Ans. 25 kN, 15 kN]
3. A beam 6 m long is simply supported at the ends and carries a uniformly distributed load of 1.5 kN/m and three concentrated loads 1 kN, 2 kN and 3 kN acting respectively at a distance of 1.5 m, 3 m and 4.5 m from the left end. Calculate the reactions at both ends. [Ans. 7 kN, 8 kN]
4. A simply supported beam of span 10 m carries a uniformly varying load from zero at the left end to 1200 N/m at the right end. Calculate the reactions at both ends of the beam. [Ans. 2000 N and 4000 N]
5. A simply supported beam AB is subjected to a distributed load increasing from 1500 N/m to 4500 N/m from end A to end B respectively. The span $AB = 6$ m. Determine the reactions at the supports. [Ans. $R_A = 7500$ N, $R_B = 10500$ N]
6. An overhanging beam carries the loads as shown in Fig. 5.31. Calculate the reactions at both ends. [Ans. $R_A = 1$ kN, $R_B = 6$ kN]

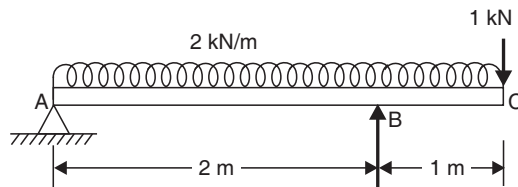


Fig. 5.31

7. An overhanging beam carries the loads as shown in Fig. 5.32. Calculate the reactions at both ends. [Ans. $R_A = 10$ kN, $R_B = 11$ kN]

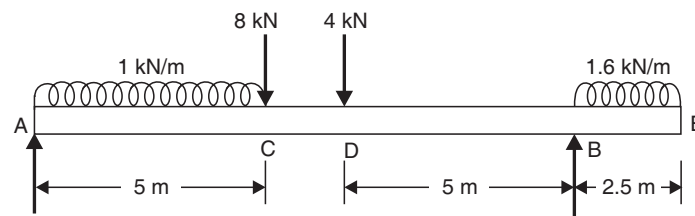


Fig. 5.32

8. A beam is loaded as shown in Fig. 5.33. Determine the reactions at both ends. [Ans. $R_{AV} = 2.875$ kN, $R_{AH} = 5.196$ kN $\leftarrow$, $R_B = 7.125$ kN]

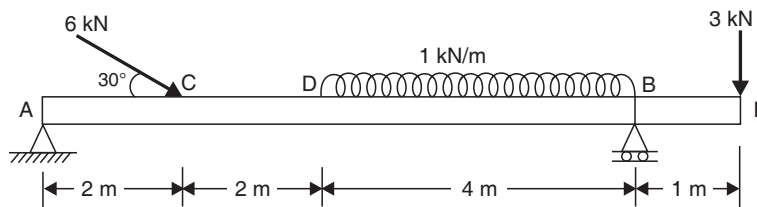


Fig. 5.33

9. A beam AB of span 6 m is hinged at A and supported on rollers at end B and carries load as shown in Fig. 5.34. Determine the reactions at A and B . [Ans. $R_{AV} = 5.87$ kN, $R_{AH} = 3.222$ kN $\leftarrow$, $R_B = 7.3$ kN]

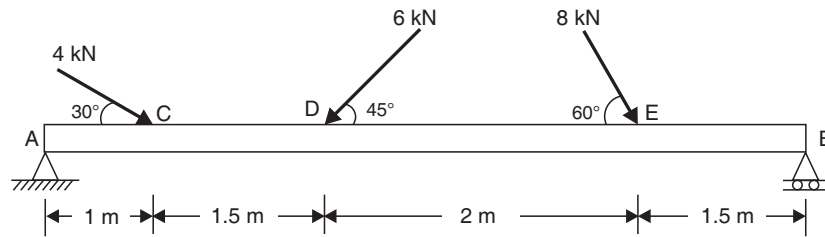


Fig. 5.34

10. A beam AB of span 8 m is subjected to the uniformly distributed load of 1 kN/m over the entire length and the moment 32 kN/m at C as shown in Fig. 5.35. Determine the reactions at both ends.
[Ans. $R_A = 0$, $R_B = 8$ kN]

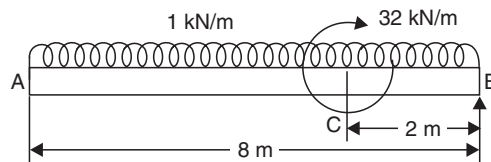


Fig. 5.35

11. A simply supported beam AB is subjected to a distributed load increasing from 1500 N/m to 4500 N/m from end A to end B . The span $AB = 6$ m. Determine the reactions at the supports.

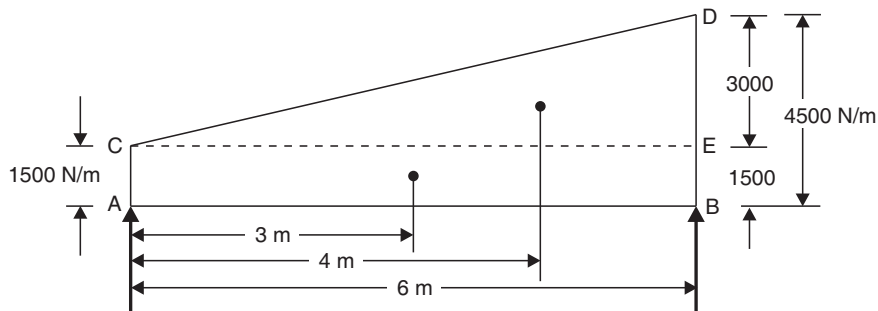


Fig. 5.36

[Hint. Area of rectangle = $1500 \times 6 = 9000$ N, Area of triangle = $\frac{3000 \times 6}{2} = 9000$ N. C.G. of rectangle from

$$A = 3 \text{ m, C.G. of triangle from } A = \frac{2}{3} \times 6 = 4 \text{ m.}$$

$$\Sigma M_A = 0, 6 R_B = 9000 \times 3 + 9000 \times 4 = 63000$$

$$\therefore R_B = 10500 \text{ N and } R_A = (9000 + 9000) - 10500 = 7500 \text{ N}]$$

6

Friction

6.1. INTRODUCTION

When a solid body slides over a stationary solid body, a force is exerted at the surface of contact by the stationary body on the moving body. This force is called the *force of friction* and is always acting in the direction opposite to the direction of motion. The property of the bodies by virtue of which a force is exerted by a stationary body on the moving body to resist the motion of the moving body is called *friction*. Friction acts parallel to the surface of contact and depends upon the nature of surface of contact.

6.1.1. Definitions. For defining the terms like co-efficient of friction (μ) and angle of friction (ϕ), consider a solid body placed on a horizontal plane surface as shown in Fig. 6.1.

Let W = Weight of body acting through C.G. downward,

R = Normal reaction of body acting through C.G. upward,

P = Force acting on the body through C.G. and parallel to the horizontal surface.

If P is small, the body will not move as the force of friction acting on the body in the direction opposite to P will be more than P . But if the magnitude of P goes on increasing, a stage comes, when the solid body is on the point of motion. At this stage, the force of friction acting on the body is called *limiting force of friction*. The limiting force of friction is denoted by F .

Resolving the forces on the body vertically and horizontally, we get

$$R = W$$

$$F = P.$$

If the magnitude of P is further increased the body will start moving. The force of friction, acting on the body when the body is moving, is called *kinetic friction*.

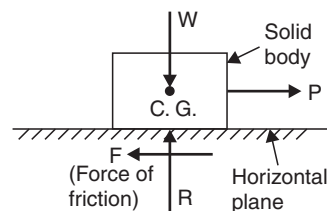


Fig. 6.1. Solid body on horizontal surface.

6.2. CO-EFFICIENT OF FRICTION (μ)

It is defined as the ratio of the limiting force of friction (F) to the normal reaction (R) between two bodies. It is denoted by the symbol μ . Thus

$$\mu = \frac{\text{Limiting force of friction}}{\text{Normal reaction}} = \frac{F}{R}.$$

$$\therefore F = \mu R \quad \dots(6.1)$$

6.3. ANGLE OF FRICTION (ϕ)

It is defined as the angle made by the resultant of the normal reaction (R) and the limiting force of friction (F) with the normal reaction (R). It is denoted by ϕ . Fig. 6.2 shows a solid body resting on a rough horizontal plane.

Let S = Resultant of the normal reaction (R) and limiting force of friction (F)

Then angle of friction = ϕ = Angle between S and R

From Fig. 6.2, we have

$$\begin{aligned}\tan \phi &= \frac{F}{R} = \frac{\mu R}{R} & [\because F = \mu R \text{ from (6.1)}] \\ &= \mu = \text{Co-efficient of friction} & \dots(6.2)\end{aligned}$$

Thus the tangent of the angle of friction is equal to the co-efficient of friction.

A block of weight W is placed on a rough horizontal plane surface as shown in Fig. 6.3 and a force P is applied at an angle θ with the horizontal such that the block just tends to move.

Let

R = Normal reaction

μ = Co-efficient of friction

F = Force of friction

$= \mu R$

In this case the normal reaction R will not be equal to weight of the body. The normal reaction is obtained by resolving the forces on the block horizontally and vertically. The force P is resolved in two components *i.e.*, $P \cos \theta$ in the horizontal direction and $P \sin \theta$ in the vertical direction.

Resolving forces on the block horizontally, we get

$$F = P \cos \theta$$

$$\text{or} \quad \mu R = P \cos \theta \quad \dots(i) \quad (\because F = \mu R)$$

Resolving forces on the block vertically, we get

$$R + P \sin \theta = W$$

$$\therefore R = W - P \sin \theta \quad \dots(ii)$$

From equation (ii), it is clear that normal reaction is not equal to the weight of the block.

If in equation (ii), the values of W , P and θ are known, then value of normal reaction (R) can be obtained. This value of R can be substituted in equation (i) to determine the value of co-efficient of friction μ .

Note. (i) The force of friction is always equal to μR (*i.e.*, $F = \mu R$).

(ii) The normal reaction (R) is not equal to the weight of the body always.

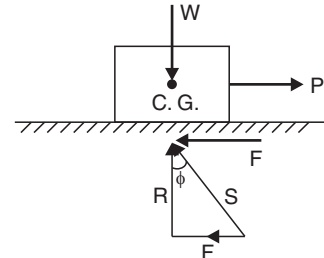


Fig. 6.2. Angle of friction.

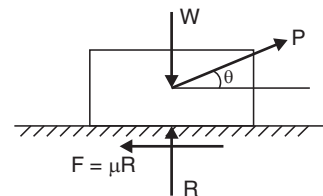


Fig. 6.3

6.4. CONE OF FRICTION

It is defined as the right circular cone with vertex at the point of contact of the two bodies (or surfaces), axis in the direction of normal reaction (R) and semi-vertical angle equal to angle of friction (ϕ). Fig. 6.4 shows the cone of friction in which,

O = Point of contact between two bodies

R = Normal reaction and also axis of the cone of friction

ϕ = Angle of friction.

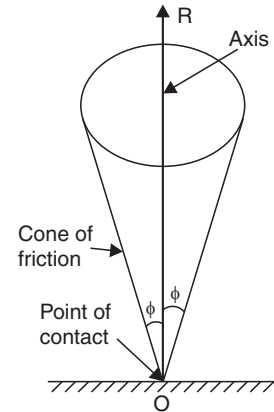


Fig. 6.4. Cone of friction.

6.5. TYPES OF FRICTION

The friction is divided into following two types depending upon the nature of the two surfaces in contact :

1. Static and dynamic friction
2. Wet and dry friction.

6.5.1. Static and Dynamic Friction. If the two surfaces, which are in contact, are at rest, the force experienced by one surface is called **static friction**. But if one surface starts moving and the other is at rest, the force experienced by the moving surface is called dynamic friction.

6.5.2. Wet and Dry Friction. If between two surfaces, which are in contact, lubrication (oil or grease) is used, the friction, that exists between two surfaces is known wet friction. But if no lubrication (oil or grease) is used, then the friction between two surfaces is called **Dry Friction** or **Solid Friction**.

6.6. COULOMB'S LAWS OF FRICTION

The friction, that exists between two surfaces which are not lubricated, is known as solid friction. The two surfaces may be at rest or one of the surface is moving and other surface is at rest. The following are the laws of solid friction :

1. The force of friction acts in the opposite direction in which surface is having tendency to move.
2. The force of friction is equal to the force applied to the surface, so long as the surface is at rest.
3. When the surface is on the point of motion, the force of friction is maximum and this maximum frictional force is called the limiting friction force.
4. The limiting frictional force bears a constant ratio to the normal reaction between two surfaces.
5. The limiting frictional force does not depend upon the shape and areas of the surfaces in contact.

6. The ratio between limiting friction and normal reaction is slightly less when the two surfaces are in motion.

7. The force of friction is independent of the velocity of sliding.

The above laws of solid friction are also called laws of static and dynamic friction.

Problem 6.1. A body of weight 100 Newtons is placed on a rough horizontal plane. Determine the co-efficient of friction if a horizontal force of 60 Newtons just causes the body to slide over the horizontal plane.

Sol. Given :

Weight of body, $W = 100 \text{ N}$

Horizontal force applied,

$$P = 60 \text{ N}$$

$\therefore$ Limiting force of friction,

$$F = P = 60 \text{ N}$$

Let μ = Co-efficient of friction.

The normal reaction of the body is given as

$$R = W = 100 \text{ N}$$

Using equation (6.1),

$$F = \mu R$$

or

$$\mu = \frac{F}{R} = \frac{60}{100} = 0.6. \text{ Ans.}$$

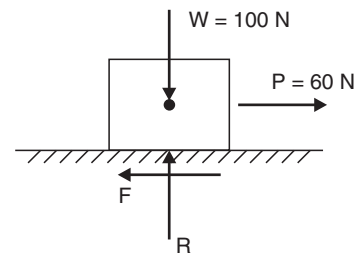


Fig. 6.5

Problem 6.2. A body of weight 200 N is placed on a rough horizontal plane. If the co-efficient of friction between the body and the horizontal plane is 0.3, determine the horizontal force required to just slide the body on the plane.

Sol. Given :

Weight of body, $W = 200 \text{ N}$

Co-efficient of friction, $\mu = 0.3$

Normal reaction, $R = W = 200 \text{ N}$

Let F = horizontal force which causes the body to just slide over the plane.

Using equation (6.1),

$$F = \mu R = 0.3 \times 200 = 60 \text{ N. Ans.}$$

Problem 6.3. The force required to pull a body of weight 50 N on a rough horizontal plane is 15 N. Determine the co-efficient of friction if the force is applied at an angle of 15° with the horizontal.

Sol. Given :

Weight of the body,

$$W = 50 \text{ N}$$

Force applied,

$$p = 15 \text{ N}$$

Angle made by the force P , with horizontal, $\theta = 15^\circ$

Let the co-efficient of friction

$$= \mu$$

Normal reaction

$$= R$$

When a force equal to 15 N is applied to the body at an angle 15° to the horizontal, the body is on the point of motion in the forward direction. Hence a force of friction equal to μR will be acting in the backward direction. The body is in equilibrium under the action of the forces shown in Fig. 6.6.

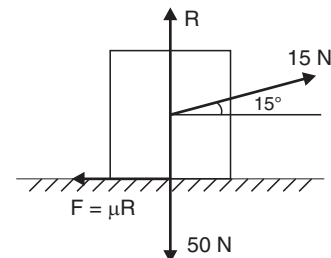


Fig. 6.6

Resolving the forces along the plane, $\mu R = 15 \cos 15^\circ$... (i)

Resolving the forces normal to the plane

$$R + 15 \sin 15^\circ = 50$$

$$\therefore R = 50 - 15 \sin 15^\circ = 50 - 15 \times 0.2588 = 46.12 \text{ N}$$

Substituting the value of R in equation (i), we get

$$\mu \times 46.12 = 15 \cos 15^\circ$$

$$\therefore \mu = \frac{15 \cos 15^\circ}{46.12} = \frac{15 \times 0.9659}{46.12} = 0.314. \text{ Ans.}$$

Problem 6.4. The force required to pull the body of weight 50 N on a rough horizontal surface is 20 N when it is applied at an angle of 25° with the horizontal [Fig. 6.6 (a)].

Determine the co-efficient of friction and magnitude of reaction R between the body and the horizontal surface. Does the reaction pass through the C.G. of the body?

(U.P. Tech. University, 2002–2003)

Sol. Given :

Weight, $W = 50 \text{ N}$

Force applied, $P = 20 \text{ N}$

Let $\mu =$ Co-efficient of friction

$R =$ Normal reaction

Force of friction, $F = \mu R$

For equilibrium of the body, $\Sigma F_x = 0$ and $\Sigma F_y = 0$

For $\Sigma F_x = 0$, we have

$$\mu R = 20 \times \cos 25^\circ = 18.126 \quad \dots (i)$$

For $\Sigma F_y = 0$, we have

$$R + 20 \sin 25^\circ - 50 = 0$$

$$\text{or } R + 8.452 - 50 = 0$$

$$\text{or } R = 50 - 8.452 = 41.548 \text{ N. Ans.}$$

Substituting this value of R in equation (i),

$$\mu \times 41.548 = 18.126$$

$$\therefore \mu = \frac{18.126}{41.548} = 0.436. \text{ Ans.}$$

The normal reaction will pass through the C.G. of the body so that the body is in equilibrium position.

Problem 6.5. A body of weight 70 N is placed on a rough horizontal plane. To just move the body on the horizontal plane, a push of 20 N inclined at 20° to the horizontal plane is required. Find the co-efficient of friction.

Sol. Given :

Weight of body, $W = 70 \text{ N}$

Force applied, $P = 20 \text{ N}$

Inclination of P , $\theta = 20^\circ$.

Let $\mu =$ Co-efficient of friction

$R =$ Normal reaction

$F =$ Force of friction $= \mu R$.

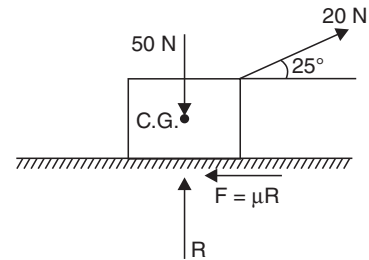


Fig. 6.6 (a)

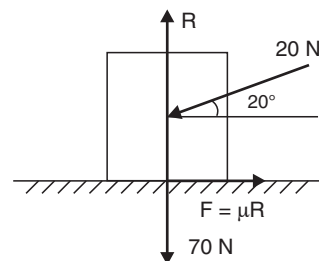


Fig. 6.7

When a push of 20 N at an angle 20° to the horizontal is applied to the body, the body is just to move towards left. Hence a force of friction $F = \mu R$, will be acting towards right as shown in Fig. 6.7.

$$\text{Resolving forces along the plane, } \mu R = 20 \cos 20^\circ \quad \dots(i)$$

$$\begin{aligned} \text{Resolving forces normal to the plane, } R &= 70 + 20 \sin 20^\circ \\ &= 70 + 20 \times 0.342 = 70 + 6.84 = 76.84 \end{aligned}$$

Substituting the value of R in equation (i),

$$\mu \times 76.84 = 20 \cos 20^\circ$$

$$\therefore \mu = \frac{20 \cos 20^\circ}{76.84} = \frac{20 \times 0.9397}{76.84} = \mathbf{0.244. \text{ Ans.}}$$

Problem 6.6. A block of weight W is placed on a rough horizontal plane surface as shown in Fig. 6.8 and a force P is applied at an angle θ with the horizontal such that the block just tends to move. Prove that the force P will be the least if the angle θ is equal to the angle of friction ϕ .

Sol. Given :

Weight of block $= W$

Force applied $= P$

Inclination of force $= \theta$

Let R = Normal reaction

μ = Co-efficient of friction

F = Force of friction $= \mu R$.

The forces acting on the block are shown in Fig. 6.8.

Resolving forces vertically, we get

$$R + P \sin \theta = W$$

$$\text{or } R = W - P \sin \theta \quad \dots(i)$$

Resolving forces horizontally, we get

$$P \cos \theta = F = \mu R \quad (\because F = \mu R) \quad \dots(ii)$$

Substituting the value of R from equation (i), the equation (ii) becomes as

$$P \cos \theta = \mu [W - P \sin \theta] \quad \dots(iii)$$

But from equation (6.2), we know

$$\mu = \tan \phi$$

where ϕ = angle of friction.

Substituting the value of μ in equation (iii), we get

$$P \cos \theta = \tan \phi (W - P \sin \theta)$$

$$= \frac{\sin \phi}{\cos \phi} (W - P \sin \theta)$$

$$\text{or } P \cos \theta \cos \phi = W \sin \phi - P \sin \theta \sin \phi$$

$$\text{or } P \cos \theta \cos \phi + P \sin \theta \sin \phi = W \sin \phi$$

$$\text{or } P(\cos \theta \cos \phi + \sin \theta \sin \phi) = W \sin \phi$$

$$\text{or } P \cos (\theta - \phi) = W \sin \phi \quad [\because \cos \theta \cos \phi + \sin \theta \sin \phi = \cos (\theta - \phi)]$$

$$\therefore P = \frac{W \sin \phi}{\cos (\theta - \phi)}$$

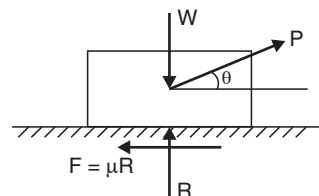


Fig. 6.8

The force P will be least, if the denominator *i.e.*, $\cos(\theta - \phi)$ is maximum. But $\cos(\theta - \phi)$ will be maximum, if

$$\begin{aligned} \cos(\theta - \phi) &= 1 \\ \text{or } \theta - \phi &= 0 \\ \text{or } \theta &= \phi \end{aligned}$$

$$\therefore P_{\text{least}} = W \sin \phi \text{ or } W \sin \theta.$$

Hence the force P will be least if the angle of inclination of P with the horizontal is equal to the angle of friction ϕ .

Problem 6.7. A man wishing to slide a stone block of weight 1000 N over a horizontal concrete floor, ties a rope to the block and pulls it in a direction inclined upward at an angle of 20° to the horizontal. Calculate the minimum pull necessary to slide the block if the co-efficient of friction $\mu = 0.6$. Calculate also the pull required if the inclination of the rope with the horizontal is equal to the angle of friction and prove that this is the least force required to slide the block.

Sol. Given :

Weight, $W = 1000 \text{ N}$

Angle with horizontal, $\theta = 20^\circ$

Co-efficient of friction, $\mu = 0.6$

Let $P = \text{Force applied}$

$R = \text{Normal reaction}$

$F = \text{Force of friction} = \mu R$

The forces acting on the block are shown in Fig. 6.9.

Resolving forces horizontally,

$$P \cos \theta = \mu R$$

$$\text{or } P \cos 20^\circ = 0.6 \times R \quad \dots(i)$$

Resolving forces vertically,

$$R + P \sin \theta = W$$

$$\text{or } R + P \sin 20^\circ = 1000$$

$$\text{or } R = 1000 - P \sin 20^\circ \quad \dots(ii)$$

Substituting the value of R in equation (i), we get

$$\begin{aligned} P \cos 20^\circ &= 0.6 (1000 - P \sin 20^\circ) \\ &= 600 - 0.6 P \sin 20^\circ \end{aligned}$$

$$\text{or } P \cos 20^\circ + 0.6 P \sin 20^\circ = 600$$

$$\text{or } P (\cos 20^\circ + 0.6 \sin 20^\circ) = 600 \quad \dots(iii)$$

$$\begin{aligned} \therefore P &= \frac{600}{(\cos 20^\circ + 0.6 \sin 20^\circ)} = \frac{600}{(0.9397 + 0.6 \times 0.342)} \\ &= \frac{600}{1.1449} = 524 \text{ N. Ans.} \end{aligned}$$

Pull required if the inclination of the rope with the horizontal is equal to angle of friction.

Let $\phi = \text{Angle of friction}$

= The angle made by rope with horizontal (given) = 20°

If in equation (iii), the angle 20° is replaced by angle ϕ , then we get the force required to pull the body as,

$$P(\cos \phi + 0.6 \sin \phi) = 600$$

$$\therefore P = \frac{600}{(\cos \phi + 0.6 \sin \phi)} \quad \dots(iv)$$

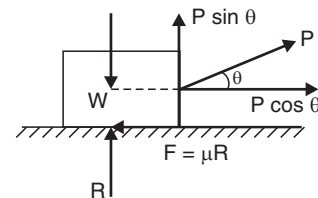


Fig. 6.9

The force P will be minimum, if $(\cos \phi + 0.6 \sin \phi)$ is maximum. But $(\cos \phi + 0.6 \sin \phi)$ will be maximum if

$$\frac{d}{d\phi} (\cos \phi + 0.6 \sin \phi) = 0 \quad \text{or} \quad -\sin \phi + 0.6 \cos \phi = 0$$

or $0.6 \cos \phi = \sin \phi \quad \text{or} \quad 0.6 = \frac{\sin \phi}{\cos \phi} = \tan \phi$

But $0.6 = \mu$. Hence force P will be minimum if $\tan \phi = \mu = 0.6$ (Proved)

Now $\tan \phi = 0.6$

$$\therefore \phi = \tan^{-1} 0.6 = 30.96^\circ$$

Substituting this value of ϕ in equation (iv), we get

$$\begin{aligned} P &= \frac{600}{(\cos 30.96^\circ + 0.6 \sin 30.96^\circ)} = \frac{600}{(0.8575 + 0.6 \times 0.5144)} \\ &= \frac{600}{1.1661} = 514.5 \text{ N. Ans.} \end{aligned}$$

6.7. ANGLE OF REPOSE

The angle of repose is defined as the maximum inclination of a plane at which a body remains in equilibrium over the inclined plane by the assistance of friction only.

Consider a body of weight W , resting on a rough inclined plane as shown in Fig. 6.10.

Let R = Normal reaction acting at right angle to the inclined plane.

α = Inclination of the plane with the horizontal

F = Frictional force acting upward along the plane.

Let the angle of inclination (α) be gradually increased, till the body just starts sliding down the plane. This angle of inclined plane, at which a body just begins to slide down the plane, is called angle of repose.

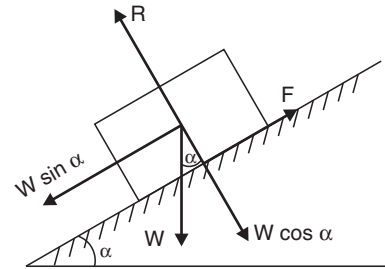


Fig. 6.10

Resolving the forces along the plane, we get

$$W \sin \alpha = F \quad \dots(i)$$

Resolving the forces normal to the plane, we get

$$W \cos \alpha = R \quad \dots(ii)$$

Dividing equation (i) by equation (ii),

$$\frac{W \sin \alpha}{W \cos \alpha} = \frac{F}{R} \quad \text{or} \quad \tan \alpha = \frac{F}{R} \quad \dots(iii)$$

But from equation (6.2), we know

$$\tan \phi = \frac{F}{R} \quad \dots(iv)$$

where ϕ = Angle of friction.

Hence from equations (iii) and (iv), we have

$$\tan \alpha = \tan \phi$$

or $\alpha = \phi$

or Angle of repose = Angle of friction.

6.8. EQUILIBRIUM OF A BODY LYING ON A ROUGH INCLINED PLANE

In Art. 6.7 we have studied that if the inclination of the plane, with the horizontal, is less than the angle of friction, the body will remain in equilibrium without any external force. If the body is to be moved upwards or downwards in this condition an external force is required. But if the inclination of the plane is more than the angle of friction, the body will not remain in equilibrium. The body will move downward and an upward external force will be required to keep the body in equilibrium.

Such problems are solved by resolving the forces along the plane and perpendicular to the planes. The force of friction (F), which is always equal to μR is acting opposite to the direction of motion of the body.

Problem 6.8. *Prove that the angle of friction (ϕ) is equal to the angle made by an inclined plane with the horizontal when a solid body, placed on the inclined plane, is about to slide down.*

Sol. A solid body of weight, W is placed on an inclined plane AC as shown in Fig. 6.11.

Let α = Angle of the inclined plane AC with horizontal plane AB , such that body just starts moving downward.

The body is in equilibrium under the action of following forces :

1. Weight of the body (W) acting vertically downwards.
2. Normal reaction (R), acting perpendicular to the inclined plane, AC .
3. The force of friction, $F = \mu R$, acting up the plane as the body is about to slide down the plane.

The weight, W can be resolved in two component one along the plane and other perpendicular to the plane. The components are $W \sin \alpha$ and $W \cos \alpha$ respectively.

As the body is in equilibrium, the forces along and perpendicular to the inclined plane are:

$$W \sin \alpha = F = \mu R$$

$$W \cos \alpha = R$$

$$\text{Dividing} \quad \frac{W \sin \alpha}{W \cos \alpha} = \frac{\mu R}{R} = \mu$$

or

$$\tan \alpha = \mu$$

But from equation (6.2), we have

$$\tan \phi = \mu$$

where ϕ = Angle of friction

$$\therefore \tan \alpha = \tan \phi = \mu$$

or

$$\alpha = \phi.$$

The above relation shows that the angle of friction is equal to angle of the inclined plane when a solid body, placed on the inclined plane is about to slide down.

Problem 6.9. *Find the least force required to drag a body of weight W , placed on a rough inclined plane having inclination α to the horizontal. The force is applied to the body in such a way that it makes an angle θ to the inclined plane and the body is (a) on the point of motion up the plane and (b) on the point of motion down the plane.*

Sol. Given :

Weight of body = W

Inclination of plane = α

Force applied = P

Angle made by force P with inclined plane = θ .

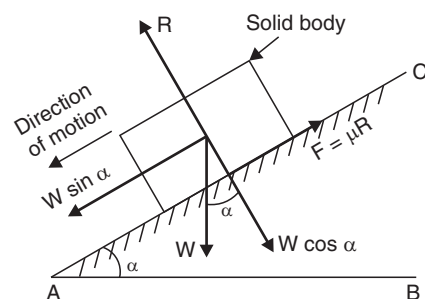


Fig. 6.11

(a) *Least force when the body is on the point of motion up the plane.*

When the body is on the point of motion up the plane, the force of friction ($F = \mu R$) is acting down the plane. The body is in equilibrium under the action of following forces as shown in Fig. 6.12.

1. Weight (W) of the body acting vertically downwards,
2. Normal reaction (R), perpendicular to the inclined plane,
3. The force of friction, $F = \mu R$ acting down the plane, and
4. Force P , inclined at an angle θ to the plane.

Resolving the forces along the plane, we get

$$W \sin \alpha + \mu R = P \cos \theta \quad \dots(i)$$

Resolving force perpendicular to the inclined plane,

$$W \cos \alpha = R + P \sin \theta$$

or

$$R = W \cos \alpha - P \sin \theta \quad \dots(ii)$$

Substituting the value of R in equation (i), we get

$$W \sin \alpha + \mu (W \cos \alpha - P \sin \theta) = P \cos \theta$$

or

$$W \sin \alpha + \mu W \cos \alpha - \mu P \sin \theta = P \cos \theta$$

or

$$W \sin \alpha + \mu W \cos \alpha = P \cos \theta + \mu P \sin \theta$$

or

$$W (\sin \alpha + \mu \cos \alpha) = P (\cos \theta + \mu \sin \theta)$$

$\therefore$

$$\begin{aligned} P &= \frac{W(\sin \alpha + \mu \cos \alpha)}{(\cos \theta + \mu \sin \theta)} \\ &= \frac{W(\sin \alpha + \tan \phi \cos \alpha)}{(\cos \theta + \tan \phi \sin \theta)} \quad (\because \mu = \tan \phi) \\ &= \frac{W \left(\sin \alpha + \frac{\sin \phi}{\cos \phi} \cos \alpha \right)}{\left(\cos \theta + \frac{\sin \phi}{\cos \phi} \sin \theta \right)} = \frac{W(\sin \alpha \cos \phi + \sin \phi \cos \alpha)}{(\cos \theta \cos \phi + \sin \phi \sin \theta)} \\ &= \frac{W(\sin \alpha + \phi)}{\cos (\theta - \phi)} \quad \dots(6.3) \end{aligned}$$

The force P will be least if the denominator $\cos (\theta - \phi)$ is maximum. But the maximum value of $\cos (\theta - \phi)$ will be equal to one.

$$\therefore \cos (\theta - \phi) = 1 = \cos \theta$$

$$\therefore \theta - \phi = 0 \text{ or } \theta = \phi.$$

Substituting this value of $\theta = \phi$ in equation (6.3), we get

$$P_{min} = W \sin (\alpha + \theta) \quad \dots(6.4)$$

Thus the force P will be minimum if the angle of inclination of the force with the inclined plane is equal to the angle of friction.

(b) *Least force when the body is on the point of motion down the plane.*

When the body is on the point of motion down the plane, the force of friction $F = \mu R$ is acting up the plane as shown in Fig. 6.13.

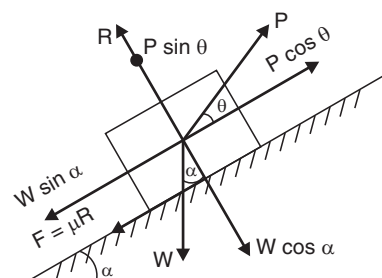


Fig. 6.12. Body moving up.

Resolving the forces along the plane

$$W \sin \alpha = P \cos \theta + \mu R \quad \dots(iii)$$

Resolving forces perpendicular to the inclined plane

$$R + P \sin \theta = W \cos \alpha$$

or

$$R = W \cos \alpha - P \sin \theta$$

Substituting the value of R in equation (iii), we get

$$\begin{aligned} W \sin \alpha &= P \cos \theta + \mu (W \cos \alpha - P \sin \theta) \\ &= P \cos \theta + \mu W \cos \alpha - \mu P \sin \theta \end{aligned}$$

or

$$W \sin \alpha - \mu W \cos \alpha = P \cos \theta - \mu P \sin \theta$$

or

$$W [\sin \alpha - \mu \cos \alpha] = P [\cos \theta - \mu \sin \theta]$$

or

$$\begin{aligned} P &= \frac{W(\sin \alpha - \mu \cos \alpha)}{(\cos \theta - \mu \sin \theta)} \\ &= \frac{W(\sin \alpha - \tan \phi \cos \alpha)}{(\cos \theta - \tan \phi \sin \theta)} \quad (\because \mu = \tan \phi) \\ &= \frac{W \left(\sin \alpha - \frac{\sin \phi}{\cos \phi} \cos \alpha \right)}{\left(\cos \theta - \frac{\sin \phi}{\cos \phi} \sin \theta \right)} = \frac{W(\sin \alpha \cos \phi - \sin \phi \cos \alpha)}{(\cos \theta \cos \phi - \sin \phi \sin \theta)} \\ &= \frac{W \sin (\alpha - \phi)}{\cos (\theta + \phi)} \quad \dots(6.5) \end{aligned}$$

Problem 6.10. A body of weight 500 N is pulled up an inclined plane, by a force of 350 N. The inclination of the plane is 30° to the horizontal and the force is applied parallel to the plane. Determine the co-efficient of friction.

Sol. Given :

Weight of body, $W = 500 \text{ N}$

Force applied, $P = 350 \text{ N}$

Inclination, $\alpha = 30^\circ$

Let $\mu =$ Co-efficient of friction

$R =$ Normal reaction

$F =$ Force of friction $= \mu R$.

The body is in equilibrium under the action of the forces shown in Fig. 6.14.

Resolving the forces along the plane,

$$500 \sin 30^\circ + F = 350$$

or

$$500 \sin 30^\circ + \mu R = 350 \quad (\because F = \mu R) \quad \dots(i)$$

Resolving forces normal to the plane,

$$R = 500 \cos 30^\circ = 500 \times .866 = 433 \text{ N}$$

Substituting the value of R in equation (i), we get

$$500 \sin 30^\circ + \mu \times 433 = 350$$

or

$$500 \times 0.5 + 433 \mu = 350$$

or

$$433 \mu = 350 - 500 \times 0.5 = 350 - 250 = 100$$

$$\therefore \mu = \frac{100}{433} = 0.23. \quad \text{Ans.}$$

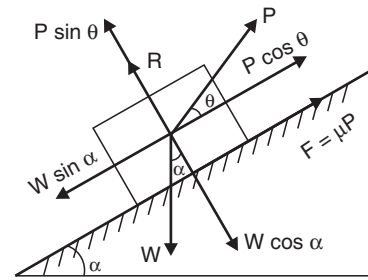


Fig. 6.13. Body moving down.

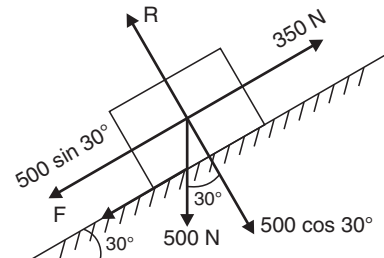


Fig. 6.14. Body moving up.

Problem 6.11. A body of weight 450 N is pulled up along an inclined plane having inclination 30° to the horizontal at a steady speed. Find the force required if the co-efficient of friction between the body and the plane is 0.25 and force is applied parallel to the inclined plane. If the distance travelled by the body is 10 m along the plane, find the work done on the body.

Sol. Given :

Weight of body, $W = 450 \text{ N}$

Inclination of plane, $\alpha = 30^\circ$

Co-efficient of friction, $\mu = 0.25$

Distance travelled by body = 10 m

Let the force required = P .

The body is in equilibrium under the action of forces shown in Fig. 6.15.

Resolving forces along the plane,

$$P = W \sin 30^\circ + \mu R = 450 \times 0.5 + 0.25 \times R$$

or

$$P = 225 + 0.25 R \quad \dots(i)$$

Resolving forces normal to the plane,

$$R = W \cos 30^\circ = 450 \times 0.866 = 389.7 \text{ N}$$

Substituting the value of R in equation (i),

$$P = 225 + 0.25 \times 389.7 = \mathbf{322.425 \text{ N. Ans.}}$$

Work done on the body = Force $\times$ Distance travelled in the direction of force

$$= 322.525 \times 10 \text{ Nm} = 3224.25 \text{ Nm}$$

$$= \mathbf{3224.25 \text{ J (where J = Joules = Nm). Ans.}}$$

Problem 6.12. A rough inclined plane, co-efficient of friction = μ , rises 1 cm for every 5 cm of its length. Calculate the effort require to drag a body weighing 100 N up the plane :

(i) when the effort is applied horizontally, and

(ii) when the effort is applied parallel to the plane.

Sol. Given :

Co-efficient of friction = μ

Rise of plane is 1 cm for every 5 cm of its length.

See Fig. 6.16.

$$\therefore \sin \theta = \frac{1}{5} = 0.2$$

or

$$\theta = \sin^{-1} 0.2 \\ = 11.53^\circ$$

Weight of body, $W = 100 \text{ N}$.

(i) Find the effort when it is applied horizontally

The body is in equilibrium under the action of the forces shown in Fig. 6.17.

Resolving the forces along the plane,

$$P \cos \theta = W \sin \theta + \mu R \quad \dots(i)$$

Resolving the forces normal to the plane,

$$R = P \sin \theta + W \cos \theta \quad \dots(ii)$$

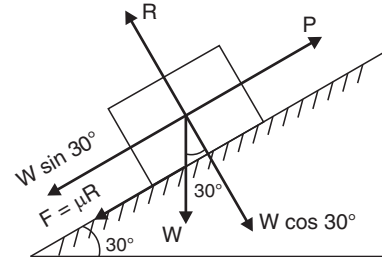


Fig. 6.15. Body moving up.

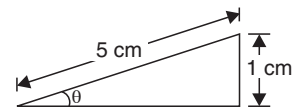


Fig. 6.16

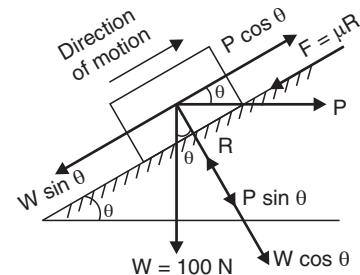


Fig. 6.17

Substituting the value of R in equation (i),

$$\begin{aligned} P \cos \theta &= W \sin \theta + \mu \times (P \sin \theta + W \cos \theta) \\ &= W \sin \theta + \mu P \sin \theta + \mu W \cos \theta \end{aligned}$$

or $P \cos \theta - \mu P \sin \theta = W \sin \theta + \mu W \cos \theta$

or $P (\cos \theta - \mu \sin \theta) = W (\sin \theta + \mu \cos \theta)$

or
$$\begin{aligned} P &= \frac{W(\sin \theta + \mu \cos \theta)}{(\cos \theta - \mu \sin \theta)} \\ &= \frac{100(\sin 11.53^\circ + \mu \cos 11.53^\circ)}{(\cos 11.53^\circ - \mu \sin 11.53^\circ)} = \frac{100(0.2 + \mu \times 0.9798)}{(0.9798 - \mu \times 0.2)} \\ &= \frac{100(0.2 + 0.9798 \mu)}{(0.9798 - 0.2 \mu)} \text{ N. Ans.} \end{aligned}$$

(ii) Find the effort when it is applied parallel to the plane

Let P = Effort applied parallel to the plane. The body is in equilibrium under the action of forces shown in Fig. 6.18.

Resolving the forces along the plane,

$$P = W \sin \theta + \mu \times R \quad \dots(iii)$$

Resolving the forces normal to the plane,

$$R = W \cos \theta \quad \dots(iv)$$

Substituting the value of R in equation (iii),

$$\begin{aligned} P &= W \sin \theta + \mu \times W \cos \theta \\ &= W (\sin \theta + \mu \cos \theta) \\ &= 100 (\sin 11.53^\circ + \mu \cos 11.53^\circ) \\ &= 100 (0.2 + \mu \times 0.9798) \\ &= 100 (0.2 + 0.9798 \times \mu) \text{ N. Ans.} \end{aligned}$$

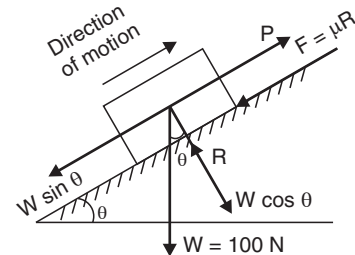


Fig. 6.18

Problem 6.13. (a) Define co-efficient of friction and limiting friction. (b) Block A weighing 15 N is a rectangular prism resting on a rough inclined plane as shown in Fig. 6.19. The block is tied up by a horizontal string which has a tension of 5 N. Find :

(i) The frictional force on the block,

(ii) The normal reaction of the inclined plane, and

(iii) The co-efficient of friction between the surfaces of contact.

Sol. (a) For definition of co-efficient of friction and limiting friction, please refer to Art. 6.2 and Art. 6.1.1.

(b) Given :

Weight of block, $W = 15 \text{ N}$

Tension in string, $T = 5 \text{ N}$

Inclination of plane, $\alpha = 45^\circ$

Let F = Frictional force,
 R = Normal reaction, and
 μ = Co-efficient of friction.

Since there is tension in string which means if string is removed, the block will slide down the plane. Hence force of friction will be acting in the upward direction.

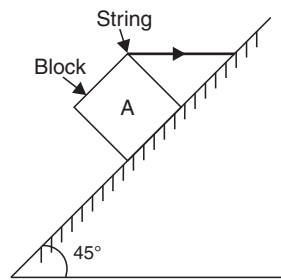


Fig. 6.19

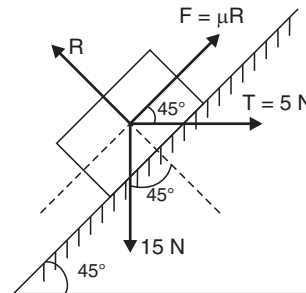


Fig. 6.20. Forces acting on the Block.

The block A is in equilibrium under the action of the forces shown in Fig. 6.20. The forces are :

1. The weight of block, $W = 15 \text{ N}$
2. Horizontal tension in the string, $T = 5 \text{ N}$
3. Normal reaction, R
4. Force of friction, $F = \mu R$, acting upward. Resolving forces along the inclined plane,

$$15 \sin 45^\circ = F + 5 \cos 45^\circ$$

$$\therefore F = 15 \sin 45^\circ - 5 \cos 45^\circ = 15 \times .707 - 5 \times .707 \\ = 10 \times .707 = 7.07 \text{ N}$$

Resolving forces normal to inclined plane,

$$R = 15 \cos 45^\circ + T \cos 45^\circ = 15 \cos 45^\circ + 5 \cos 45^\circ \\ = 15 \times .707 + 5 \times .707 = 20 \times .707 = 14.14 \text{ N}$$

Using equation (6.1), we get

$$F = \mu R$$

$$\therefore \mu = \frac{F}{R} = \frac{7.07}{14.14} = 0.5.$$

$\therefore$ (i) Frictional force on the block, $F = 7.07 \text{ N}$. **Ans.**

(ii) Normal reaction of the inclined plane, $R = 14.14 \text{ N}$. **Ans.**

(iii) Co-efficient of friction, $\mu = 0.5$. **Ans.**

Problem 6.14. Find the force required to move a load of 30 N up a rough inclined plane, the force being applied parallel to the plane. The inclination of the plane is such that when the same body is kept on a perfectly smooth plane inclined at that angle, a force of 6 N applied at an inclination of 30° to the plane keeps the same in equilibrium. Assume co-efficient of friction between the rough plane and the load is equal to 0.3 .

Sol. Given :

Load, $W = 30 \text{ N}$

Co-efficient of friction between the rough plane and load,

$$\mu = 0.3$$

Let $\alpha =$ Inclination of the plane with horizontal

$P_1 =$ Force required to move the load up a rough inclined plane, when the force is applied parallel to the plane.

The force applied when same body is kept on a smooth inclined plane, $P_2 = 6 \text{ N}$.

Inclination of the force with the inclined plane, $\theta = 30^\circ$.

1st Case. Consider the body of weight 30 N placed on a smooth inclined plane as shown in Fig. 6.21.

The forces acting on the body are :

- (i) The weight ($W = 30$ N) vertically downward.
- (ii) The force $P_2 (= 6$ N) at an angle of 30° with the inclined plane.
- (iii) The normal reaction R .

Resolving forces normal to the inclined plane.

$$R + P_2 \sin 30^\circ = W \cos \alpha$$

$$\text{or} \quad R + 6 \times \frac{1}{2} = 30 \cos \alpha \quad \dots(i)$$

$$(\because P_2 = 6 \text{ N}, W = 30 \text{ N})$$

Resolving forces along the inclined plane,

$$P_2 \cos 30^\circ = W \sin \alpha$$

$$\text{or} \quad 6 \times \frac{\sqrt{3}}{2} = 30 \sin \alpha$$

$$\therefore \sin \alpha = 6 \times \frac{\sqrt{3}}{2} \times \frac{1}{30} = \frac{\sqrt{3}}{10} = 0.1732$$

$$\therefore \alpha = 9.974^\circ.$$

2nd Case. The body of weight 30 N is placed on a rough inclined plane having inclination $\alpha (= 9.974^\circ)$ with the horizontal as shown in Fig. 6.22.

The forces acting on the body are :

- (i) The weight $W (= 30$ N) vertically downward.
- (ii) The force P_1 , parallel to the plane.
- (iii) The normal reaction R^* .
- (iv) Force of friction $F = \mu R^*$.

Resolving the forces along the inclined plane,

$$W \sin \alpha + F = P_1$$

$$\text{or} \quad 30 \sin 9.974^\circ + \mu R^* = P_1 \quad (\because F = \mu R^* \text{ and } \alpha = 9.974^\circ) \quad \dots(ii)$$

Resolving the forces normal to the inclined plane

$$R^* = W \cos \alpha$$

$$= 30 \times \cos 9.974^\circ$$

$$= 30 \times 0.9848 = 29.544.$$

$$(\because \alpha = 9.974^\circ)$$

Substituting the value of R^* in equation (ii), we get

$$30 \times \sin 9.974^\circ + \mu \times 29.544 = P_1$$

$$\text{or} \quad 30 \times 0.1732 + 0.3 \times 29.544 = P_1$$

$$\text{or} \quad 5.196 + 8.8632 = P_1$$

$$\text{or} \quad P_1 = 14.059 \text{ N. Ans.}$$

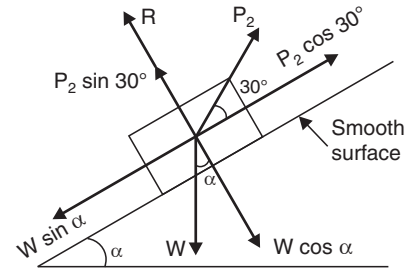


Fig. 6.21

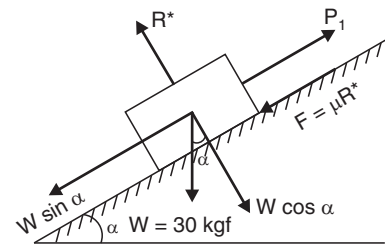


Fig. 6.22

Problem 6.15. A cord connects two bodies of weights 300 N and 800 N. The two bodies are placed on an inclined plane and cord is parallel to inclined plane. The co-efficients of friction for the weight of 400 N is 0.15 and that for 800 N is 0.4. Determine the inclination of the plane

to the horizontal and the tension in the cord when the motion is about to take place, down the inclined plane. The body weighing 400 N is below the body weighing 800 N.

Sol. Weight of first body, $W_1 = 400 \text{ N}$

Weight of second body, $W_2 = 800 \text{ N}$

Co-efficient of friction for first body, $\mu_1 = 0.15$

Co-efficient of friction for second body, $\mu_2 = 0.40$

Let $\alpha =$ Inclination of plane

$T =$ Tension in cord

$R_1 =$ Normal reaction for 1st body

$R_2 =$ Normal reaction for 2nd body

$F_1 =$ Force of friction between 1st body and plane $= \mu_1 R_1$

$F_2 =$ Force of friction between 2nd body and plane $= \mu_2 R_2$.

As the motion of the two bodies is about to take place down the inclined plane, the force of frictions F_1 and F_2 will be acting upward. The two bodies are in equilibrium under the action of forces shown in Fig. 6.23.

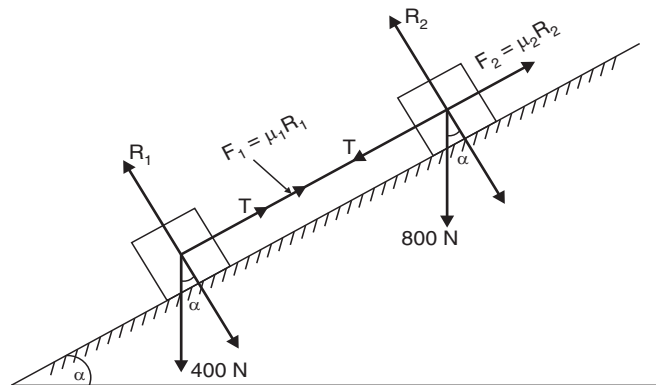


Fig. 6.23

Forces on the first body

Resolving forces along the plane,

$$400 \sin \alpha = T + F_1 = T + \mu_1 R_1 = T + 0.15 R_1 \quad \dots(i)$$

Resolving forces normal to the plane,

$$400 \cos \alpha = R_1$$

Substituting the value of R_1 in equation (i),

$$\begin{aligned} 400 \sin \alpha &= T + 0.15 \times 400 \cos \alpha \\ &= T + 60 \cos \alpha \end{aligned}$$

$$\therefore T = 400 \sin \alpha - 60 \cos \alpha \quad \dots(ii)$$

Forces on the second body

Resolving forces along the plane,

$$800 \sin \alpha + T = F_2 = \mu_2 R_2 = 0.40 R_2 \quad \dots(iii)$$

Resolving forces normal to the plane,

$$R_2 = 800 \cos \alpha$$

Substituting the values of R_2 in equation (iii),

$$800 \sin \alpha + T = 0.40 \times 800 \cos \alpha = 320 \cos \alpha$$

or

$$T = 320 \cos \alpha - 800 \sin \alpha$$

...(iv)

Equating the values of T , given by equations (ii) and (iv),

$$400 \sin \alpha - 60 \cos \alpha = 320 \cos \alpha - 800 \sin \alpha$$

or

$$400 \sin \alpha + 800 \sin \alpha = 320 \cos \alpha + 60 \cos \alpha$$

or

$$1200 \sin \alpha = 380 \cos \alpha$$

$$\text{or} \quad \frac{\sin \alpha}{\cos \alpha} = \frac{380}{1200} = 0.3166 \quad \text{or} \quad \tan \alpha = 0.3166$$

$$\therefore \alpha = \tan^{-1} 0.3166 = 17.56^\circ \quad \text{or} \quad 17^\circ 34'. \quad \text{Ans.}$$

Substituting the value of α in equation (ii),

$$\begin{aligned} T &= 400 \sin 17.56^\circ - 60 \cos 17.56^\circ \\ &= 400 \times 0.3017 - 60 \times 0.9534 \\ &= 120.68 - 57.20 = 63.48 \text{ N.} \quad \text{Ans.} \end{aligned}$$

Problem 6.16. Two blocks A and B are connected by a horizontal rod and are supported on two rough planes as shown in Fig. 6.24. If the weight of block B is 1500 N and co-efficient of friction of block A and B are 0.25 and 0.35 respectively, find the smallest weight of block A for which equilibrium can exist.

Sol. Given : Weight of block B,

$$W_B = 1500 \text{ N}$$

Co-efficient of friction for block A,

$$\mu_A = 0.25$$

Co-efficient of friction for block B,

$$\mu_B = 0.35$$

Let the smallest weight of block A for equilibrium
= W_A .

If the weight of block A is less than the value required for equilibrium, the block B will be slide downwards. But the block A and B are connected by a horizontal rod of fixed length. Now when block B starts moving in the downward direction, the block A starts moving towards left. Hence a force of friction F_A equal to $\mu_A R_A$ will be acting on block A towards right as shown in Fig. 6.24. On block B, the force of friction F_B equal to $\mu_B R_B$ will be acting in the upward direction.

For block A

Resolving force normal to plane, $R_A = W_A$.

Force of friction, $F_A = \mu_A R_A = \mu \times 0.25 \times W_A = 0.25 W_A$.

This force will be transmitted to block B through rod AB.

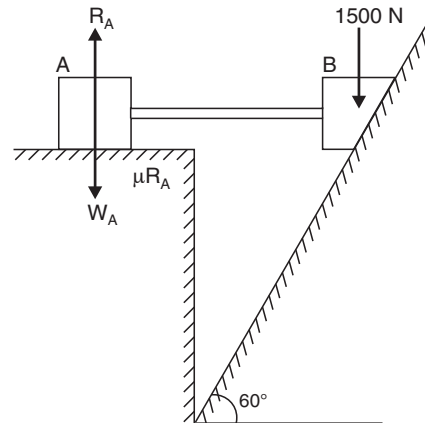


Fig. 6.24

For block B

The block B will be in equilibrium under the action of the forces, shown in Fig. 6.25. The forces are :

- (i) The weight of block $B = 1500$ N acting vertically downwards.
- (ii) The normal reaction R_B of the plane.
- (iii) The horizontal force $= 0.25 W_A$, transmitted to block B through rod AB .
- (iv) Force of friction $F_B = \mu_B R_B = 0.35 R_B$ acting up the inclined plane.

In this case, the forces are resolved horizontally and vertically instead of along the inclined plane and normal to the plane. For an equilibrium state, the forces acting in any direction must be zero.

Resolving forces horizontally,

$$0.25 W_A + F_B \cos 60^\circ = R_B \cos 30^\circ$$

$$0.25 W_A + 0.35 R_B \cos 60^\circ = R_B \cos 30^\circ \quad (\because F_B = 0.35 R_B)$$

$$0.25 W_A + 0.35 \times 0.5 R_B = R_B \times .866$$

$$0.25 W_A + 0.175 R_B = 0.866 R_B$$

$$\text{or} \quad 0.25 W_A = 0.866 R_B - 0.175 R_B = 0.691 R_B \quad \dots(i)$$

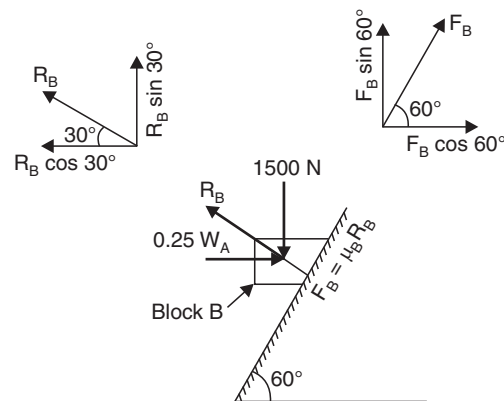


Fig. 6.25

Resolving forces vertically,

$$R_B \sin 30^\circ + F_B \sin 60^\circ = 1500$$

$$R_B \times 0.5 + 0.35 R_B \times .866 = 1500 \quad (\because F_B = 0.35 R_B)$$

$$0.5 R_B + 0.303 R_B = 1500 = 0.803 R_B = 1500$$

$$\therefore R_B = \frac{1500}{0.803} = 1868 \text{ N.}$$

Substituting this value of R_B in (i), we get

$$0.25 W_A = 0.691 \times 1868$$

$$\therefore W_A = \frac{0.691 \times 1868}{0.25} = 5163 \text{ N. Ans.}$$

Problem 6.17. Referring to the Fig. 6.26 given below, determine the least value of the force P to cause motion to impend rightwards. Assume the co-efficient of friction under the blocks to be 0.2 and pulley to be frictionless.

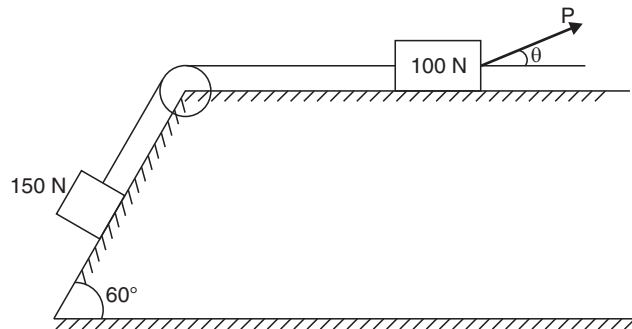


Fig. 6.26

Sol. Given :

Co-efficient of friction under both blocks, $\mu = 0.2$

Pulley is frictionless. Motion of block of weight 100 N is towards right. Find least value of P .

1st Case

Consider the equilibrium of block of weight 150 N

As the block of weight 100 N tends to move rightwards, the block of weight 150 N will tend to move upwards. Hence force of friction will act downwards as shown in Fig. 6.27.

Let T = Tension in the string
 R = Normal reaction
 F = Force of friction = μR
 $= 0.2R$

The weight 150 N is acting vertically downwards. The body is in equilibrium under the action of forces shown in Fig. 6.27. Resolving the forces along the plane,

$$T = 150 \sin 60^\circ + \mu R \quad \dots(i)$$

Resolving the forces normal to inclined plane,

$$R = 150 \cos 60^\circ = 150 \times \frac{1}{2} = 75 \text{ N} \quad \dots(ii)$$

Substituting the value of R in equation (i),

$$T = 150 \sin 60^\circ + 0.2 \times 75 \quad (\because \mu = 0.2)$$

$$= 144.9 \text{ N}$$

2nd Case

Now consider the equilibrium of block of weight 100 N

The block of weight 100 N tends to move rightwards, hence force of friction will be acting towards left as shown in Fig. 6.28. Also the pulley is frictionless hence the tension in the string which is attached to the block of weight 100 N will be 144.9 N. The body will be in equilibrium under the action of forces shown in Fig. 6.28.

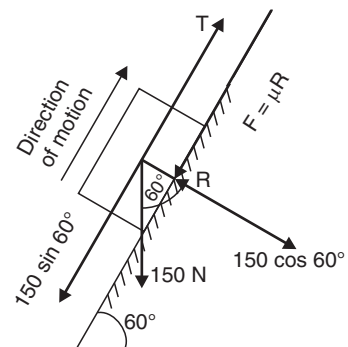


Fig. 6.27

Resolving forces along the plane, (*i.e.*, horizontally),

$$P \cos \theta = T + 0.2 R^* \\ = 144.9 + 0.2 R^* \quad \dots(iii)$$

Resolving the forces normal to the plane (*i.e.*, vertically)

$$R^* + P \sin \theta = 100$$

$$\text{or} \quad R^* = 100 - P \sin \theta \quad \dots(iv)$$

Substituting the value of R^* in equation (iii),

$$P \cos \theta = 144.9 + 0.2 \times (100 - P \sin \theta) \\ = 144.9 + 20 - 0.2 P \sin \theta$$

$$\text{or} \quad P \cos \theta + 0.2 P \sin \theta = 164.9$$

$$\text{or} \quad P (\cos \theta + 0.2 \sin \theta) = 164.9$$

$$\text{or} \quad P = \frac{164.9}{(\cos \theta + 0.2 \sin \theta)} \quad \dots(v)$$

The force P will be minimum, if $(\cos \theta + 0.2 \sin \theta)$ is maximum. But $(\cos \theta + 0.2 \sin \theta)$ will be maximum if

$$\frac{d}{d\theta} (\cos \theta + 0.2 \sin \theta) = 0$$

$$\text{or} \quad -\sin \theta + 0.2 \cos \theta = 0$$

$$\text{or} \quad 0.2 \cos \theta = \sin \theta$$

$$\text{or} \quad 0.2 = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$\text{or} \quad \theta = \tan^{-1} 0.2 = 11.309^\circ$$

Substituting the value of θ in equation (v), the least value of P will be obtained.

$$\therefore P_{(least)} = \frac{164.9}{(\cos 11.309^\circ + 0.2 \times \sin 11.309^\circ)} = 161.88 \text{ N. Ans.}$$

Problem 6.18. What should be the value of the angle θ in Fig. 6.29 so that the motion of the 90 N block impends down the plane? The co-efficient of friction μ for all the surfaces is $1/3$.

Sol. Given :

Co-efficient of friction for all surfaces, $\mu = 1/3$

Motion of weight 90 N impends down the plane. Find the value of θ .

First consider the equilibrium of weight 30 N.

As the weight 90 N tends to move downwards, there will be a rubbing action between the surfaces of weight 90 N and 30 N. Hence a force of friction will be acting between these two surfaces.

The weight 30 N is tied to a string, the other end of the string is fixed to the plane. When the weight 90 N tends to move downwards, the weight 30 N with respect to weight 90 N will move upwards. Hence the force of friction on the lower surface of the weight 30 N will act downward as shown in Fig. 6.29 (a). The weight 30 N will be in equilibrium under the action of the forces shown in Fig. 6.29 (a) in which

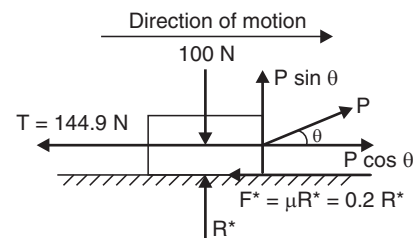


Fig. 6.28

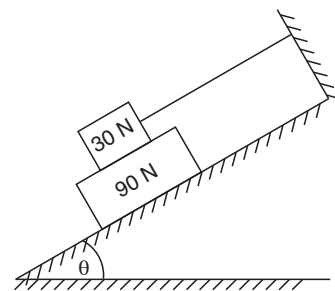


Fig. 6.29

T = Tension in the string
 R_1 = Normal reaction on the
 lower surface of weight 30 N
 F_1 = Force of friction = μR_1

Resolving the forces along the plane,

$$T = 30 \sin \theta + \mu R_1$$

$$= 30 \sin \theta + \frac{1}{3} R_1 \quad \dots(i)$$

Resolving the forces normal to the plane,

$$R_1 = 30 \cos \theta \quad \dots(ii)$$

Substituting the value of R_1 in equation (i),

$$T = 30 \sin \theta + \frac{1}{3} \times 30 \cos \theta$$

$$= 30 \sin \theta + 10 \cos \theta \quad \dots(iii)$$

Now consider the equilibrium of weight 90 N

The weight 90 N will be in equilibrium under the action of forces shown in Fig. 6.29 (b).

Resolving the forces along the plane,

$$90 \sin \theta = \mu R_1 + \mu R_2$$

$$= \frac{1}{3} R_1 + \frac{1}{3} R_2$$

$$= \frac{1}{3} \times 30 \cos \theta + \frac{1}{3} R_2$$

$$(\because R_1 = 30 \cos \theta)$$

$$= 10 \cos \theta + \frac{1}{3} R_2 \quad \dots(iv)$$

Resolving the forces normal to the plane,

$$R_2 = R_1 + 90 \cos \theta$$

$$= 30 \cos \theta + 90 \cos \theta$$

$$(\because R_1 = 30 \cos \theta)$$

$$= 120 \cos \theta$$

Substituting the value of R_2 in equation (iv), we get

$$90 \sin \theta = 10 \cos \theta + \frac{1}{3} \times 120 \cos \theta$$

$$= 10 \cos \theta + 40 \cos \theta$$

$$= 50 \cos \theta$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{50}{90} = 0.5555$$

or

$$\therefore \tan \theta = 0.5555$$

$$\therefore \theta = \tan^{-1} 0.5555 = 29.05^\circ. \quad \text{Ans.}$$

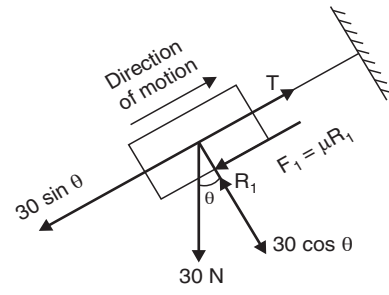


Fig. 6.29 (a)

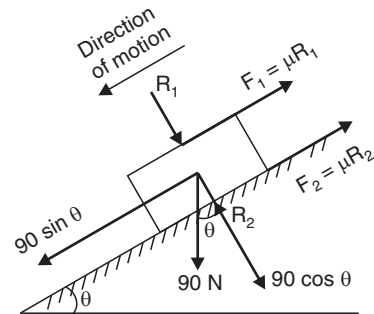


Fig. 6.29 (b)

... (v)

6.9. APPLICATIONS OF FRICTION

The followings are the important applications of friction :

- (i) Wedge Friction
- (ii) Belt Friction
- (iii) Brakes and Screws Friction.

6.10. WEDGE FRICTION

A wedge is a piece of metal or wood which is usually of a triangular or trapezoidal in cross-section. It is used for either lifting loads or used for slight adjustments in the position of a body *i.e.*, for tightening fits or keys for shafts.

When lifting a heavy load the wedge is placed below the load and a horizontal force P is applied as shown in Fig. 6.30. If the force P is just sufficient to lift the load, the wedge will move towards left and load will move up. When the wedge moves towards left, the sliding of the surfaces AC and AB will take place. At the same time load moves up and sliding of the load takes place along GD . Thus for the wedge and load shown in Fig. 6.30 sliding takes place along surfaces AB , AC and DG . Hence there will be three normal reactions at AB , AC and DG .

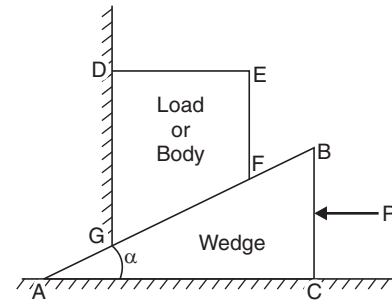


Fig. 6.30

The problems on wedges are generally the problems of equilibrium on inclined planes. Therefore, these problems are solved by equilibrium method or by applying Lami's Theorem.

Equilibrium Method. In this method, the equilibrium of the load (or the body placed on the wedge) and the equilibrium of the wedge are considered.

Equilibrium of Wedge

Consider the equilibrium of the wedge. The forces acting on the wedge are shown in Fig. 6.30(a). They are :

- (i) The force P applied horizontally on face BC .
- (ii) Reaction R_1 on the face AC (The reaction R_1 is the resultant of the normal reaction on the rubbing face AC and force of friction on surface AC). The reaction R_1 will be inclined at an angle ϕ_1 (when ϕ_1 is angle of friction) with the normal.
- (iii) Reaction R_2 on the face AB (The reaction R_2 is the resultant of normal reaction on the rubbing face AB and force of friction on surface AB). The reaction R_2 will be inclined at an angle ϕ_2 with the normal.

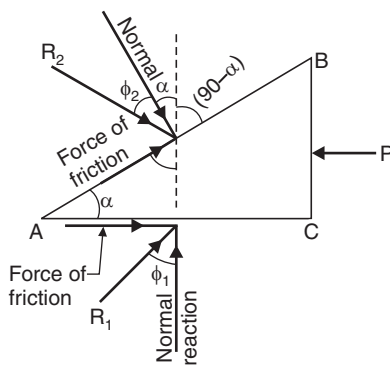


Fig. 6.30(a). Equilibrium of wedge

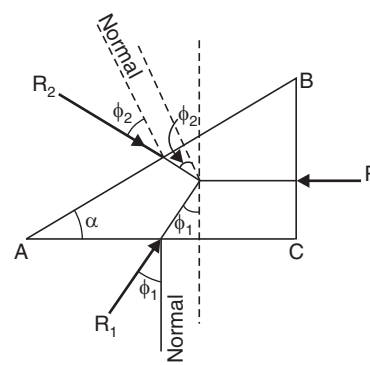


Fig. 6.30(b)

When the force P is applied on the wedge, the surface CA will be moving towards left and hence force of friction on this surface will be acting towards right. Similarly, the force of friction on face AB will be acting from A to B . These forces are shown in Fig. 6.30(a).

Resolving the forces horizontally, we get

$$R_1 \sin \phi_1 + R_2 \sin (\phi_2 + \alpha) = P.$$

Resolving the forces vertically, we get

$$R_1 \cos \phi_1 = R_2 \cos (\phi_2 + \alpha).$$

By Lami's Theorem

The wedge is in equilibrium under the action of three forces namely R_1 , R_2 and P . These forces, when produced, will meet at a point as shown in Fig. 6.30(b).

Applying Lami's theorem, we get

$$\begin{aligned} \frac{P}{\sin (180 - \phi_1 - \phi_2 - \alpha)} &= \frac{R_1}{\sin (90 + \alpha + \phi_2)} \\ &= \frac{R_2}{\sin (90 + \phi_1)} \end{aligned}$$

Equilibrium of Body placed on the Wedge

The forces acting on the body are shown in Fig. 6.31. They are :

- (i) The weight W of the body.
- (ii) Reaction R_3 on the face GD . (The reaction R_3 is the resultant of the normal reaction N_3 on the rubbing face GD and force of friction on surface GD).
- (iii) Reaction R_2 on the face GF (The reaction R_2 is the resultant of the normal reaction on the rubbing face GF and force of friction on surface GF).

These forces are shown in Fig. 6.31.

Resolving the forces R_2 , R_3 and W horizontally, we get

$$R_3 \cos \phi_3 = R_2 \sin (\alpha + \phi_2)$$

Resolving forces vertically,

$$W + R_3 \sin \phi_3 = R_2 \cos (\alpha + \phi_2)$$

By Lami's Theorem

The forces R_3 , R_2 and W are produced to meet at a point as shown in Fig. 6.31.

The body is in equilibrium under the action of these three forces.

Hence applying Lami's theorem, we get

$$\frac{W}{\sin (90 + \phi_3 + \alpha + \phi_2)} = \frac{R_2}{\sin (90 - \phi_3)} = \frac{R_3}{\sin [180 - (\alpha + \phi_1)]}.$$

Problem 6.19. A block over lying a 10° wedge on a horizontal floor and leaning against a vertical wall and weighing 1500 N is to be raised by applying a horizontal force to the wedge. Assuming co-efficient of friction between all the surfaces in contact to be 0.3 , determine the minimum horizontal force to be applied to raise the block.

Sol. Given :

Angle of the wedge, $\alpha = 10^\circ$

Weight of the block, $W = 1500 \text{ N}$

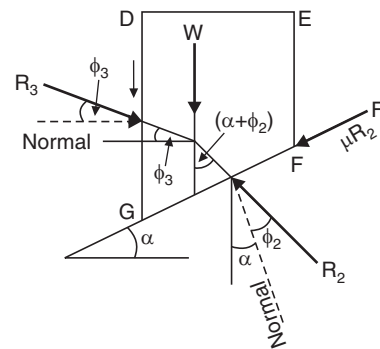


Fig. 6.31

Co-efficient of friction for all surfaces in contact,

$$\mu = 0.3$$

But $\mu = \tan \phi = 0.3$

$$\therefore \phi = \tan^{-1} 0.3 = 16^\circ 42'$$

Let P = Minimum horizontal force applied on wedge to raise the block.

Fig. 6.32 shows the block which is to be raised by the wedge.

1st Method

Considering the equilibrium of the block* as shown in Fig. 6.33(a).

Let R_2 = Reaction on face GF

R_3 = Reaction on face GD

Resolving the forces acting on the block horizontally, we get

$$R_3 \cos \phi = R_2 \sin (\alpha + \phi)$$

$$\text{or } R_3 \cos (16^\circ 42') = R_2 \sin (10^\circ + 16^\circ 42')$$

$$\text{or } R_2 \times 0.9578 = R_3 \times 0.4493$$

$$\therefore R_2 = \frac{0.9578}{0.4493} R_3 = 2.1317 R_3 \quad \dots(i)$$

Now resolving the forces vertically

$$R_3 \sin \phi + W = R_2 \cos (\alpha + \phi)$$

$$\text{or } R_3 \sin (16^\circ 42') + 1500 = R_2 \cos (10^\circ + 16^\circ 42')$$

$$\text{or } R_3 \times 0.2874 + 1500 = R_2 \times 0.8934$$

$$= 2.1317 \times R_3 \times 0.8934 \quad (\because \text{From (i), } R_2 = 2.1317 R_3)$$

$$\text{or } 0.2874 R_3 + 1500 = 1.9044 R_3$$

$$\text{or } 1.9044 R_3 - 0.2874 R_3 = 1500$$

$$\text{or } (1.9044 - 0.2874) R_3 = 1500$$

$$\text{or } 1.617 R_3 = 1500$$

$$\therefore R_3 = \frac{1500}{1.617} = 927.54 \text{ N} \quad \dots(ii)$$

From equation (ii), $R_2 = 2.1317 R_3$

$$= 2.1317 \times 927.64 = 1977.45 \text{ N} \quad \dots(iii)$$

Now considering the equilibrium of the wedge as shown in Fig. 6.33(b).

Resolving the forces vertically, we get

$$R_2 \cos (\alpha + \phi) = R_1 \cos \phi$$

$$\text{or } R_1 = \frac{R_2 \cos (\alpha + \phi)}{\cos \phi}$$

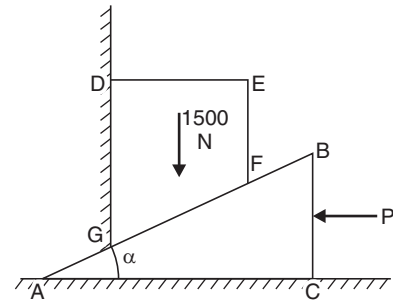


Fig. 6.32

*If the value of load is given, then consider the equilibrium of block first. But if the value of P is given, then consider the equilibrium of wedge first.

$$\begin{aligned}
 &= 1977.45 \times \frac{\cos (10^\circ + 16^\circ 42')}{\cos (16^\circ 42')} \\
 &= 1977.45 \times \frac{0.8943}{0.9578} = 1844.49 \text{ N}
 \end{aligned}$$

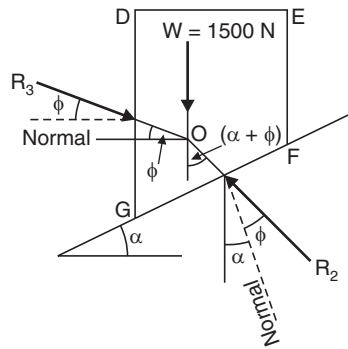


Fig. 6.33(a)

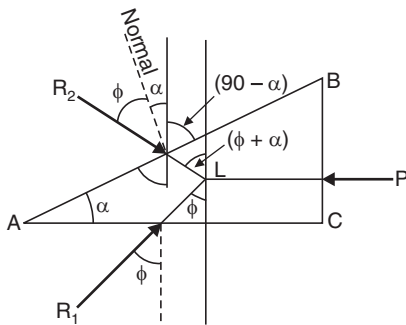


Fig. 6.33(b)

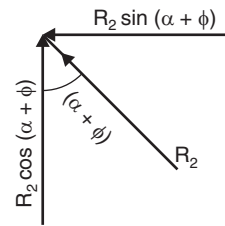


Fig. 6.33(c)

Resolving the forces horizontally, we get

$$\begin{aligned}
 P &= R_1 \sin \phi + R_2 \sin (\alpha + \phi) \\
 &= 1844.49 \times \sin (16^\circ 42') + 1977.45 \times \sin (10^\circ + 16^\circ 42') \\
 &= 1844.49 \times 0.2874 + 1977.45 \times 0.4493 \\
 &= 530.106 + 888.468 = \mathbf{1418.57 \text{ N.}} \quad \text{Ans.}
 \end{aligned}$$

2nd Method by Lami's Theorem

The block shown in Fig. 6.33 (a) is in equilibrium under the action of three forces W , R_3 and R_2 . These forces when produced are meeting at the point O .

Applying Lami's theorem to the point O , we get

$$\begin{aligned}
 \frac{W}{\sin (\phi + 90 + \alpha + \phi)} &= \frac{R_3}{\sin [180 - (\alpha + \phi)]} = \frac{R_2}{\sin (90 - \phi)} \\
 \text{or } \frac{1500}{\sin (16^\circ 42' + 90 + 10^\circ + 16^\circ 42')} &= \frac{R_2}{\sin [180 - (10^\circ + 16^\circ 42')]} \\
 &= \frac{R_2}{\sin (90 - 16^\circ 42')}
 \end{aligned}$$

$$\text{or} \quad \frac{1500}{\sin(133^\circ 24')} = \frac{R_3}{\sin(153^\circ 18')} = \frac{R_2}{\sin(72^\circ 18')}$$

$$\text{or} \quad \frac{1500}{0.7265} = \frac{R_3}{0.4493} = \frac{R_2}{0.9578}$$

$$\therefore R_3 = 1500 \times \frac{0.4493}{0.7265} = 927.66 \text{ N}$$

$$\text{and} \quad R_2 = 1500 \times \frac{0.9578}{0.7265} = 1977.56 \text{ N.}$$

Now consider the equilibrium of the wedge shown in Fig. 6.33(b). Three forces R_1 , R_2 and P when produced are meeting at the point L . Applying Lami's theorem to the point L , we get

$$\frac{R_1}{\sin(90^\circ + \alpha + \phi)} = \frac{R_2}{\sin(90^\circ + \phi)} = \frac{P}{\sin[180^\circ - \phi - (\phi + \alpha)]}$$

$$\begin{aligned} \text{or} \quad \frac{R_1}{\sin(90^\circ + 10^\circ + 16^\circ 42')} &= \frac{R_2}{\sin(90^\circ + 16^\circ 42')} \\ &= \frac{P}{\sin[180^\circ - 16^\circ 42' - (16^\circ 42' + 10^\circ)]} \end{aligned}$$

or From the last two parts, we get

$$\frac{R_2}{\sin(106^\circ 42')} = \frac{P}{\sin(136^\circ 36')}$$

$$\begin{aligned} \therefore P &= R_2 \times \frac{\sin(136^\circ 36')}{\sin(106^\circ 42')} = 1977.56 \times \frac{0.687}{0.9578} \\ &= 1418.44. \quad \text{Ans.} \end{aligned}$$

6.11. BELT FRICTION

A belt is passing over a pulley and hence the belt is in contact with the surface of the pulley. If the surface of the pulley is perfectly smooth, the tension in the belt on both sides* of the pulley will be same (*i.e.* the tension throughout the belt will be constant). Also for the perfectly smooth surface, there will be no frictional resistance and hence no driving torque** will be developed.

But if the surface of the pulley is rough, the tension in the belt will not be constant. The tension will vary throughout the length of the belt which is in contact with pulley. This variation in tension is due to frictional resistance. The frictional resistance depends on the co-efficient of friction (*i.e.* value of μ) between the belt and pulley surface. It will be shown in the next articles that

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

where T_1 = Tension in the belt on the tight side,
 T_2 = Tension in the belt on slack side,
 μ = Co-efficient of friction, and
 θ = Angle of contact in radians.

*This means that $T_1 = T_2 = T = \text{constant}$, where T_1 and T_2 are tensions on both sides of the belt.

**Torque = $(T_1 - T_2) \times r$ where r is radius. As in this case $T_1 = T_2$ and hence torque will be zero.

6.11.1. Ratio of Belt Tensions. Fig. 6.34 shows a driver pulley A and driven pulley B rotating in the clockwise direction. Fig. 6.35 shows only the driven pulley B . Consider the driven pulley B .

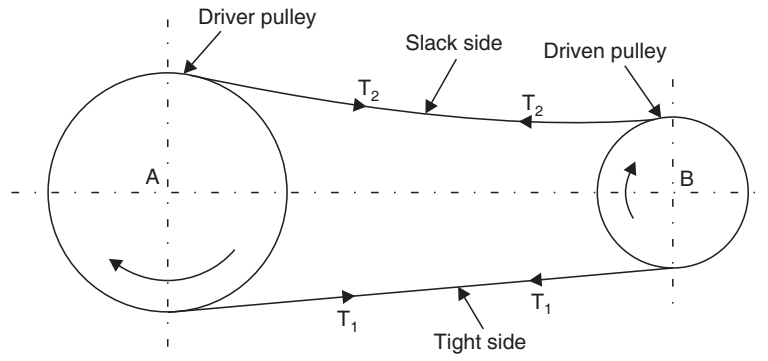


Fig. 6.34

Let T_1 = Tension in the belt on the tight side

T_2 = Tension in the belt on the slack side

θ = Angle of contact, *i.e.*, the angle subtended by the arc EF at the centre of the driven pulley

μ = Co-efficient of friction between the belt and pulley.

The ratio of the two tensions may be found by considering an elemental piece of the belt MN subtending an angle $\delta\theta$ at the centre of the pulley B as shown in Fig. 6.35. The various forces which keep the elemental piece MN in equilibrium are :

- (i) Tension T in the belt at M acting tangentially,
- (ii) Tension $T + \delta T$ in the belt at N acting tangentially,
- (iii) Normal reaction R acting radially outward at P , where P is the middle point of MN ,
- (iv) Frictional force $F = \mu R$ acting at right angles to R and in the opposite direction of the motion of pulley.

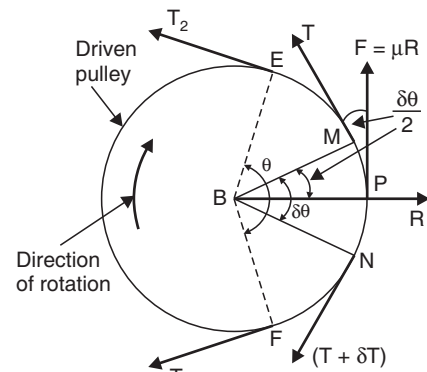


Fig. 6.35

Now angle $PBM = \frac{\delta\theta}{2}$. Also angle $TPF = \frac{\delta\theta}{2}$.

Resolving all the forces acting on the belt MN in the horizontal direction, we get

$$R = T \sin \frac{\delta\theta}{2} + (T + \delta T) \sin \frac{\delta\theta}{2}$$

Since the angle $\delta\theta$ is very small, $\sin \frac{\delta\theta}{2}$ can be written as $\frac{\delta\theta}{2}$. Hence the above equation becomes as

$$\begin{aligned} R &= T \times \frac{\delta\theta}{2} + (T + \delta T) \times \frac{\delta\theta}{2} \\ &= T \times \frac{\delta\theta}{2} + T \times \frac{\delta\theta}{2} + \delta T \times \frac{\delta\theta}{2} \end{aligned}$$

$$\begin{aligned}
 &= T \times \delta\theta + \frac{\delta T \times \delta\theta}{2} \\
 &= T \times \delta\theta \quad \dots(i)
 \end{aligned}
 \left(\text{Neglecting the small quantity } \frac{\delta T \times \delta\theta}{2} \right)$$

Now resolving all the forces vertically, we get

$$F = (T + \delta T) \cos \frac{\delta\theta}{2} - T \cos \frac{\delta\theta}{2}$$

Since $\delta\theta$ is very small, hence $\cos \frac{\delta\theta}{2}$ reduces to unity *i.e.*, 1. Hence the above equation becomes as

$$\begin{aligned}
 F &= (T + \delta T) - T = \delta T \\
 \text{or } \mu R &= \delta T & (\because F = \mu R) \\
 \text{or } R &= \frac{\delta T}{\mu} & \dots(ii)
 \end{aligned}$$

Equating the two values of R given by equations (i) and (ii), we get

$$\begin{aligned}
 T \times \delta\theta &= \frac{\delta T}{\mu} \\
 \text{or } \frac{\delta T}{T} &= \mu \cdot \delta\theta
 \end{aligned}$$

Integrating the above equation between the limits T_2 and T_1 , we get

$$\begin{aligned}
 \int_{T_2}^{T_1} \frac{\delta T}{T} &= \int \mu \cdot d\theta = \mu \int d\theta \\
 \text{or } \log_e \frac{T_1}{T_2} &= \mu \times \theta \\
 \text{or } \frac{T_1}{T_2} &= e^{\mu \times \theta} \quad \dots(6.6)
 \end{aligned}$$

In equation (6.6), θ should be taken in radians. Here θ is known as angle of contact. For an open belt or for a crossed belt the angle of contact is determined as given below.

Angle of contact for open belt drive. With an open belt drive, the belt will begin to slip on the smaller pulley, since the angle of lap is smaller on this pulley than on the large pulley. The angle θ should be taken as the minimum angle of contact. Hence in equation (6.6), the angle of contact of lap (θ) at the smaller pulley must be taken into consideration.

$$\text{Angle of contact, } \theta = (180 - 2\alpha) \quad \dots(6.7)$$

But the value of α is given by,

$$\sin \alpha = \frac{r_1 - r_2}{x} \quad \dots(6.8)$$

where r_1 = Radius of larger pulley,

r_2 = Radius of smaller pulley, and

x = Distance between the centres of two pulleys.

Angle of contact for crossed belt drive. For a crossed belt drive, the angle of lap on both the pulleys is same.

$$\therefore \text{Angle of contact, } \theta = (180 + 2\alpha) \quad \dots(6.9)$$

The value of α is given by, $\sin \alpha = \frac{r_1 + r_2}{x}$... (6.10)

where r_1 = Radius of larger pulley,

r_2 = Radius of smaller pulley, and

x = Distance between the centres of the two pulleys.

6.11.2. Power Transmitted by Belt

Let T_1 = Tension in the tight side of the belt

T_2 = Tension in the slack side of the belt

v = Velocity of the belt in metre/s.

The effective tension or force acting at the circumference of the driven pulley is the difference between the two tensions (*i.e.*, $T_1 - T_2$).

$\therefore$ Effective driving force = $(T_1 - T_2)$

$\therefore$ Work done per second = Force $\times$ Velocity
= $(T_1 - T_2) \times v$ Nm

$\therefore$ Power transmitted = $\frac{(T_1 - T_2) \times v}{1000}$ kW ... (6.11)

or $P = (T_1 - T_2) \times v$ watts ... (6.12)

Equation (6.11) gives the power in kW whereas equation (6.12) gives the power in watts. In case of equation (6.12), the tensions T_1 and T_2 are taken in Newtons.

Torque exerted on the driving pulley = $(T_1 - T_2) \times r_1$... (6.13)

and Torque exerted on the driven pulley = $(T_1 - T_2) \times r_2$... (6.14)

Problem 6.20. A belt is running over a pulley of diameter 120 cm at 200 r.p.m. The angle of contact is 165° and co-efficient of friction between the belt and pulley is 0.3. If the maximum tension in the belt is 3000 N, find the power transmitted by the belt.

Sol. Given :

Dia. of pulley, $d = 120$ cm = 1.2 m

Speed of pulley, $N = 200$ r.p.m.

Angle of contact, $\theta = 165^\circ = 165 \times \frac{\pi}{180}$ radians. $\left(\because 1^\circ = \frac{\pi}{180} \text{ rad.} \right)$

Co-efficient of friction, $\mu = 0.3$

Max. tension, $T_1 = 3000$ N

Velocity of belt, $v = \frac{\pi d N}{60} = \frac{\pi \times 1.2 \times 200}{60} = 12.56$ m/s.

Let T_2 = Tension on the slack side of the belt.

Using equation (6.6), we get

$$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

$$= e^{0.3 \times 165 \times \pi/180} = e^{0.8635} = 2.3714$$

or $\frac{3000}{T_2} = 2.3714 \quad \therefore T_2 = \frac{3000}{2.3714} = 1265$ N

Power transmitted is given by equation (6.11) as,

$$P = \frac{(T_1 - T_2) \times v}{1000} \text{ kW}$$

$$= \frac{(3000 - 1265) \times 12.56}{1000} = 21.79 \text{ kW. Ans.}$$

Problem 6.21. In the Fig. 6.36 (a), the co-efficient of friction is 0.2 between the rope and fixed pulley, and between other surfaces of contact, $\mu = 0.3$. Determine the minimum weight W to prevent the downward motion of the 100 N body. (U.P. Tech. University, 2001)

Sol. Given :

Co-efficient of friction between rope and pulley, $\mu = 0.2$

$$\tan \alpha = \frac{3}{4}$$

$$\therefore \cos \alpha = \frac{4}{5}$$

and

$$\sin \alpha = \frac{3}{5}$$

Co-efficient of friction between other surfaces,

$$\mu_1 = \mu_2 = 0.3.$$

Let us first draw the Free-Body Diagrams (FBD) of pulley, weight W and body of weight 100 N.

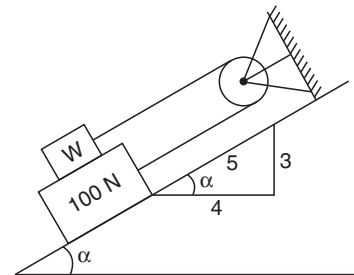
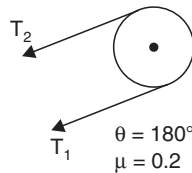
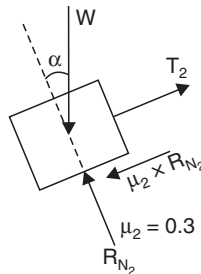


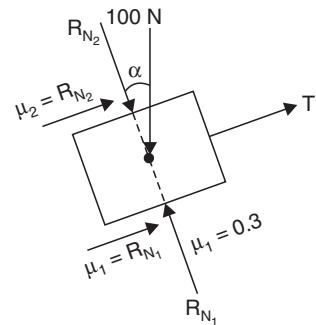
Fig. 6.36 (a)



(i) FBD of pulley



(ii) FBD of W



(iii) FBD of body of weight 100 N

Fig. 6.36 (b)

(i) Consider the FBD of pulley

Angle of contact, $\theta = 180^\circ = \pi$ radian

Using equation (6.6), $\frac{T_1}{T_2} = e^{\mu \times \theta}$

$$\therefore \frac{T_1}{T_2} = e^{0.2 \times \pi} = 1.874$$

$$\therefore T_1 = 1.874 T_2 \quad \dots(i)$$

(As the body of weight 100 N will have tendency of moving downwards, hence T_1 will be more than T_2)

(ii) Now consider the FBD of weight W

The forces acting on weight W are shown in Fig. 6.36 (b). They are :

- (a) Tension T_2
- (b) Weight W
- (c) Normal reaction on lower surface of contact, R_{N_2}
- (d) Force of friction, $\mu \times R_{N_2} = 0.3 \times R_{N_2}$

For equilibrium, the resultant force along the inclined plane and normal to the inclined plane should be zero.

$$\Sigma(\text{Forces normal to plane}) = 0$$

$$\therefore W \cos \alpha - R_{N_2} = 0$$

$$\therefore R_{N_2} = W \cos \alpha = W \times \frac{4}{5} \quad \left(\because \cos \alpha = \frac{4}{5} \right)$$

$$\Sigma(\text{Forces along the plane}) = 0$$

$$\therefore T_2 - \mu_2 \times R_{N_2} - W \sin \alpha = 0$$

$$\begin{aligned} \therefore T_2 &= \mu_2 \times R_{N_2} + W \sin \alpha \\ &= 0.3 \times R_{N_2} + W \times \frac{3}{5} \quad \left(\because \sin \alpha = \frac{3}{5} \right) \\ &= 0.3 \times \frac{4W}{5} + \frac{3W}{5} \quad \left(\because R_{N_2} = \frac{4W}{5} \right) \\ &= 0.24W + 0.6W = 0.84W \quad \dots(ii) \end{aligned}$$

(iii) Now consider the FBD of body of weight 100 N. The forces acting on the body of weight 100 N are shown in Fig. 6.36 (b) (iii). They are :

- (a) Tension, T_1 acting along the plane
- (b) Weight 100 N acting vertically downward
- (c) Normal reaction on the lower surface of contact, R_{N_1}
- (d) Force of friction on lower surface, $\mu_1 \times R_{N_1}$
- (e) Normal friction on the upper surface, R_{N_2}
- (f) Force of friction on upper surface, $\mu_2 \times R_{N_2}$

For equilibrium,

$$\Sigma(\text{Forces normal to plane}) = 0$$

$$R_{N_1} - R_{N_2} - 100 \cos \alpha = 0$$

$$\begin{aligned} \therefore R_{N_1} &= R_{N_2} + 100 \times \cos \alpha \\ &= \frac{4}{5} W + 100 \times \frac{4}{5} \quad \left(\because R_{N_2} = \frac{4}{5} W, \cos \alpha = \frac{4}{5} \right) \\ &= 0.8W + 80 \end{aligned}$$

$$\Sigma(\text{Forces along the plane}) = 0$$

$$T_1 + \mu_1 \times R_{N_1} + \mu_2 \times R_{N_2} - 100 \sin \alpha = 0$$

$$\therefore T_1 = 100 \sin \alpha - \mu_1 \times R_{N_1} - \mu_2 \times R_{N_2}$$

$$\begin{aligned}
&= 100 \times \frac{3}{5} - 0.3 \times (0.8W + 80) - 0.3 \times (0.8W) \\
&\quad \left(\because \sin \alpha = \frac{3}{5}; R_{N_1} = 0.8W + 80; R_{N_2} = 0.8W \right) \\
&= 60 - 0.24W - 24 - 0.24W \\
&= 36 - 0.48W \quad \dots(iii)
\end{aligned}$$

Substituting the values of T_1 and T_2 in equation (i), we get

$$36 - 0.48W = 1.874 (0.84W)$$

$$\therefore 36 = 0.48W + 1.574W = 2.054W$$

$$\therefore W = \frac{36}{2.054} = 17.52 \text{ N. Ans.}$$

Problem 6.22. An open-belt drive connects two pulleys 120 cm and 50 cm diameters, on parallel shafts 4 m apart. The maximum tension in the belt is 1855.3 N. The co-efficient of friction is 0.3. The driver pulley of diameter 120 cm runs at 200 r.p.m. Calculate :

(i) the power transmitted, and

(ii) torque on each of the two shafts.

Sol. Given :

Dia. of larger pulley, $d_1 = 120 \text{ cm} = 1.20 \text{ m}$

$\therefore$ Radius of larger pulley, $r_1 = \frac{120}{2} = 60 \text{ cm} = 0.6 \text{ m}$

Dia. of smaller pulley, $d_2 = 50 \text{ cm} = 0.50 \text{ m}$

$\therefore$ Radius of smaller pulley, $r_2 = 25 \text{ cm} = 0.25 \text{ m}$

Distance between the shafts, $x = 4 \text{ m}$

Max. tension, $T_1 = 1855.3 \text{ N}$

Co-efficient of friction, $\mu = 0.3$

Speed of driver pulley, $N_1 = 200 \text{ r.p.m.}$

We know that velocity of belt is given by,

$$v = \frac{\pi d_1 N_1}{60} = \frac{\pi \times 1.2 \times 200}{60} = 12.56 \text{ m/s.}$$

Let us now calculate the angle of contact (θ). For an open belt drive, angle of contact is given by equation (6.7) as

$$\theta = 180 - 2\alpha \quad \dots(i)$$

where α is given by equation (6.8) as

$$\sin \alpha = \frac{r_1 - r_2}{x} = \frac{0.6 - 0.25}{4} = 0.0875$$

$$\therefore \alpha = \sin^{-1} 0.0875 = 5.02^\circ$$

Substituting this value of α in equation (i), we get

$$\theta = 180 - 2 \times 5.02 = 169.96^\circ$$

$$= 169.96 \times \frac{\pi}{180} \text{ radians} = 2.967 \text{ radians.}$$

Let T_2 = Tension on the slack side of the belt.

Now using equation (6.6), $\frac{T_1}{T_2} = e^{\mu \times \theta}$

or
$$\frac{1855.3}{T_2} = e^{0.3 \times 2.967} = e^{0.8901} = 2.435$$

$$\therefore T_2 = \frac{1855.3}{2.435} = 761.8 \text{ N}$$

(i) *Power transmitted.* Using equation (6.11),

$$\begin{aligned} \text{Power} &= \frac{(T_1 - T_2) \times v}{1000} \\ &= \frac{(1855.3 - 761.8) \times 12.56}{1000} = \mathbf{13.73 \text{ kW. Ans.}} \end{aligned}$$

(ii) *Torque on each of the two shafts.* Torque exerted on the driving shaft is given by equation (6.13).

$$\therefore \text{Torque} = (T_1 - T_2) \times r_1 = (1855.3 - 761.8) \times 0.6 = \mathbf{656.1 \text{ Nm}}$$

Torque exerted on the driven shaft is given by equation (6.14)

$$\begin{aligned} \therefore \text{Torque} &= (T_1 - T_2) \times r_2 \\ &= (1855.3 - 761.8) \times 0.25 = \mathbf{273.4 \text{ Nm. Ans.}} \end{aligned}$$

6.12. BRAKES AND SCREWS FRICTION

A *brake* is a device used either to bring to rest a body which is in motion or to hold a body in a state of rest or of uniform motion against the action of external forces or couples. Actually the brake offers the frictional resistance to the moving body and this frictional resistance retards the motion and the body comes to rest. In this process, the kinetic energy of the body is absorbed by brakes.

A simple arrangement for applying a braking force is shown in Fig. 6.37. The face of a brake has a special friction material which has a high value of co-efficient of friction.

A single block or shoe brake consists of a block or shoe which is pressed against a rotating drum as shown in Fig. 6.37. The block is rigidly fixed to the lever. The force is applied at one end of the lever and the other end of the lever is pivoted on a fixed fulcrum O . As the force is applied to the lever, the block is pressed against the rotating drum. The friction between the block and the drum causes a tangential force to act on the drum, which tends to prevent its rotation.

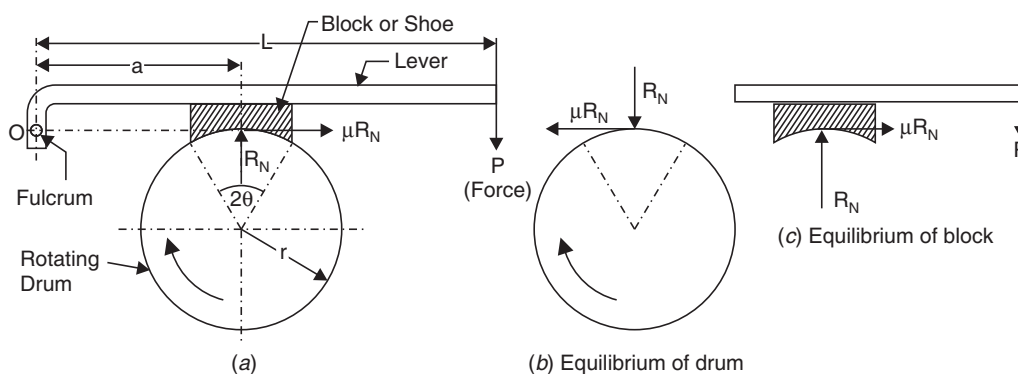


Fig. 6.37

The block is made of a softer material than that of the drum so that the block can be replaced easily on wearing.

Let P = Force applied at the lower end

r = Radius of the drum

μ = Co-efficient of friction

R_N = Normal reaction on the block

2θ = Angle made by contact surface of the block at the centre of the drum

F^* = Frictional force acting on block = μR_N

T_B = Braking torque.

When force P is applied at the lever end, the block is pressed against the rotating drum. The block exerts a radial force on the drum (*i.e.*, this force passes through the centre of the drum). The drum will exert a normal reaction (R_N) on the block. Hence the radial force on the drum will be equal to the normal reaction (R_N) on the block.

Assuming that the normal reaction R_N and the frictional force F^* ($= \mu R_N$) act at the mid-point of the block, we have

Braking torque on the drum = Frictional force $\times$ radius

$$\begin{aligned} \text{or } T_B &= F^* \times r \\ &= \mu R_N \times r \quad (\because F^* = \mu \times R_N) \quad \dots(6.15) \end{aligned}$$

The braking torque can be calculated if the value of R_N is known in equation (6.15). The value of R_N is obtained by considering the *equilibrium of the block*.

In Fig. 6.37, the drum is rotating clockwise. Hence the frictional force on the drum will be acting in the opposite direction [*i.e.*, in the anti-clockwise direction as shown in Fig. 6.37 (b)]. The frictional force on the block will be opposite to the direction of the frictional force on the drum. Hence the frictional force on the block will be in the clockwise direction as shown in Fig. 6.37 (c) (*i.e.*, in the same direction in which drum is rotating). Let the line of action of this frictional force (μR_N) passes through the fulcrum O of the lever. The forces acting on the block are :

(i) R_N (Normal reaction), (ii) μR_N (Frictional force), (iii) P (Applied force). Taking moments of all forces about the pivot O , we have

$$R_N \times a = P \times L \quad \text{(The frictional force } \mu R_N \text{ passes through } O, \text{ hence its moment is zero)}$$

$$\therefore R_N = \frac{P \times L}{a}$$

Substituting this value of R_N in equation (6.15), we get the braking torque as,

$$T_B = \mu \times \frac{P \times L}{a} \times r \quad \dots(6.16)$$

Equation (6.16) gives the value of braking torque when the line of action of the frictional force passes through the fulcrum O of the lever.

It is not necessary that the line of action of the frictional force ($\mu \times R_N$) should pass through the fulcrum O of the lever. The line of action of the frictional force may be at a distance b below or above the fulcrum O .

Let us consider these two cases :

Case 1. When the line of action of the frictional force (μR_N) is at a distance ' b ' below the fulcrum O and the drum rotates clockwise as shown in Fig. 6.38.

The forces acting on the block are : (i) R_N acting upwards, (ii) μR_N frictional force on block acting in the same direction in which drum is rotating and (iii) P (acting downwards.)

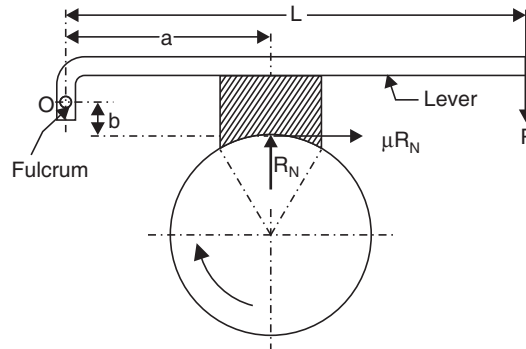


Fig. 6.38

Taking moments about the fulcrum O , we get

$$R_N \times a + \mu R_N \times b = P \times L$$

or

$$R_N (a + \mu b) = P \times L$$

$\therefore$

$$R_N = \frac{P \times L}{(a + \mu b)}$$

Substituting this value of R_N in equation (6.15), we get braking torque (T_B) as,

$$\begin{aligned} T_B &= \mu \times \frac{P \times L}{(a + \mu b)} \times r \\ &= \frac{\mu \times P \times L \times r}{(a + \mu b)} \end{aligned} \quad \dots(6.17)$$

Problem 6.23. The brake drum of a single block brake is rotating at 500 r.p.m. in the clockwise direction. The diameter of the drum is 400 mm and the single block brake is of the type as shown in Fig. 6.38. The force required at the end of the lever to apply the brake is 300 N. If angle of contact is 30° and $L = 1$ m, $a = 300$ mm and $b = 25$ mm then determine the braking torque. The co-efficient of friction is equal to 0.3.

Sol. Given (Refer to Fig. 6.38)

Speed,	$N = 500$ r.p.m.
Dia. of drum	$= 400$ mm $= 0.4$ m
$\therefore$ Radius of drum,	$r = \frac{400}{2} = 200$ mm $= 0.2$ m
Force at the end of lever,	$P = 300$ N
Angle of contact,	$2\theta = 30^\circ$
Length of lever from fulcrum,	$L = 1$ m
Distance of centre of the block from fulcrum,	$a = 300$ mm $= 0.3$ m
Perpendicular distance between line of action of frictional force and fulcrum,	$b = 25$ mm $= 0.025$ m
Rotation of drum	$=$ clockwise.
Co-efficient of friction,	$\mu = 0.3$

Taking the moments of all forces (R_N , μR_N and P) about fulcrum, we get

$$R_N \times a + \mu R_N \times b = P \times L$$

$$\text{or } R_N \times 0.3 + 0.3 \times R_N \times 0.025 = 300 \times 1$$

$$\text{or } R_N(0.3 + 0.3 \times 0.025) = 300$$

$$\begin{aligned} \text{or } R_N &= \frac{300}{0.3 + 0.3 \times 0.025} \\ &= \frac{300}{0.3075} = 975.6 \text{ N} \end{aligned}$$

Braking torque (T_B) is given by equation (6.15) as

$$\begin{aligned} T_B &= \mu R_N \times r \\ &= 0.3 \times 975.6 \times 0.2 = \mathbf{58.536 \text{ Nm.}} \quad \text{Ans.} \end{aligned}$$

Problem 6.24. Fig. 6.39 shows a simple band brake which is applied to a shaft carrying a flywheel (i.e., rotating drum) of mass 300 kg and of radius of gyration 350 mm. The flywheel rotates at 200 r.p.m. The brake drum diameter is 260 mm and co-efficient of friction is 0.20. The angle of lap of the band on the drum is 210° . If the braking torque is 39 Nm, find the force applied at the lever end.

Sol. Given :

For a simple band brake, one end of the band should be connected to the fulcrum whereas the other end of the band may be connected to the lever either towards the same side in which force P is acting or towards the opposite side in which P is acting. Here the other end is in opposite direction.

The other given data is :

mass, $m = 300 \text{ kg}$; radius of gyration, $k = 350 \text{ mm} = 0.35 \text{ m}$;

$N = 200 \text{ r.p.m.}$; $d = 260 \text{ mm}$; $r = 130 \text{ mm} = 0.13 \text{ m}$;

$\mu = 0.20$; $\theta = 210^\circ$ or $210 \times \frac{\pi}{180} \text{ rad} = 3.666 \text{ rad}$.

braking torque, $T_B = 39 \text{ Nm}$

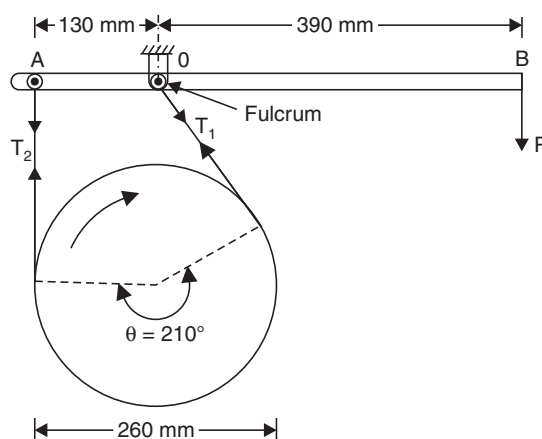


Fig. 6.39

Force applied at the end of the lever

Let P = Force applied at the lever end.

The braking torque is given by,

$$T_B = (T_1 - T_2) \times r$$

$$\text{or } 39 = (T_1 - T_2) \times 0.13$$

$$\therefore (T_1 - T_2) = \frac{39}{0.13} = 300 \text{ N} \quad \dots(i)$$

Let us now find the tensions T_1 and T_2 . We know that

$$\frac{T_1}{T_2} = e^{\mu \times \theta}$$

$$\text{or } 2.3 \log \frac{T_1}{T_2} = \mu \times \theta = 0.2 \times 3.666 = 0.7322$$

$$\therefore \log \frac{T_1}{T_2} = \frac{0.7322}{2.3} = 0.3188$$

$$\text{or } \frac{T_1}{T_2} = \text{Antilog of } 0.3188 = 2.08$$

$$\text{or } T_1 = 2.08 T_2 \quad \dots(ii)$$

Substituting the value of T_1 in equation (i), we get

$$2.08 T_2 - T_2 = 300$$

$$\text{or } 1.08 T_2 = 300$$

$$\therefore T_2 = \frac{300}{1.08} = 277.77 \text{ N}$$

Substituting this value of T_2 in equation (ii), we get

$$T_1 = 2.08 \times 277.77 = 577.76 \text{ N}$$

To find the value of P , take the moments of all forces (*i.e.*, T_1 , T_2 and P) about the fulcrum O .

$$\therefore P \times 390 = T_2 \times 130 \quad (\because T_1 \text{ passes through } O)$$

$$\therefore P = \frac{T_2 \times 130}{390}$$

$$= \frac{277.77 \times 130}{390} = 92.59 \text{ N. Ans.}$$

6.13. FRICTION IN A SQUARE THREADED SCREW

A screw-jack is a device used for lifting heavy weights or loads with the help of a small effort applied at its handle.

Fig. 6.40 shows the simple screw-jack, which consists of a nut, a screw with square threads and a handle fitted to the head of the screw. The nut also forms the body of the jack.

The load to be lifted is placed on the head of the screw. At the end of the handle, fitted to the screw head, an effort P is applied in the horizontal direction to lift the load W . The screw-jack works on the same principle on which an inclined plane works.

Let W = Weight placed on the screw head,
 P = Effort applied at the end of the handle,
 L = Length of handle,
 p = Pitch of the screw,
 d = Mean diameter of the screw,
 α = Angle of the screw or helix angle,
 ϕ = Angle of friction,
 μ = Co-efficient of friction between screw and nut = $\tan \phi$.

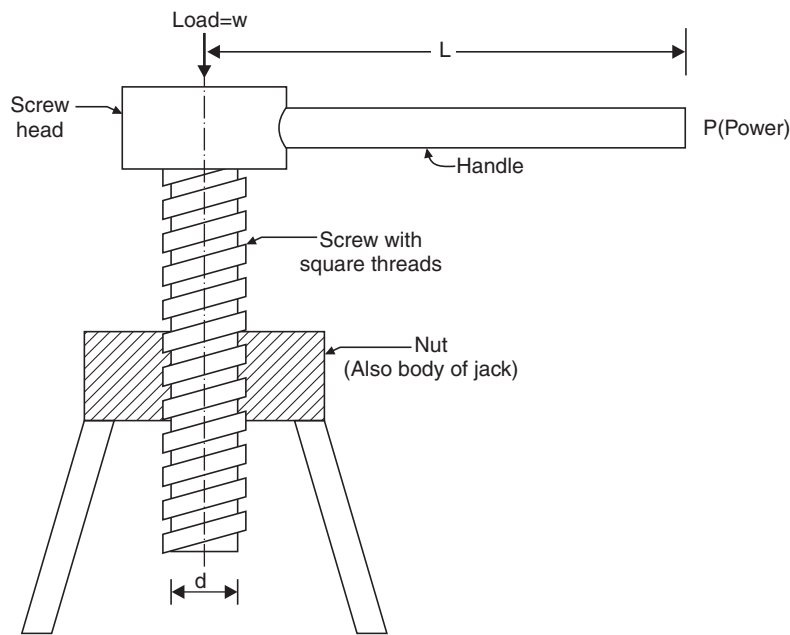


Fig. 6.40. Simple screw-jack

When the handle is rotated through one complete turn, the screw is also rotated through one turn. Then the load is lifted by a height p (pitch of screw).

The development of one complete turn of a screw thread is shown in Fig. 6.41 (a). This is similar to the inclined plane. The distance AB will be equal to the circumference (πd) and distance BC will be equal to the pitch (p) of the screw. From the Fig. 6.41 (a), we have

$$\tan \alpha = \frac{BC}{AC} = \frac{p}{\pi d} \quad \dots(6.18)$$

Let P' = Effort applied horizontally at the mean radius of the screw-jack to lift the load W ,

$$r = \text{Mean radius of the screw-jack} = \frac{d}{2}.$$

Now this case becomes similar to that of lifting a load W up an inclined plane by a horizontal force P' as shown in Fig. 6.41 (b).

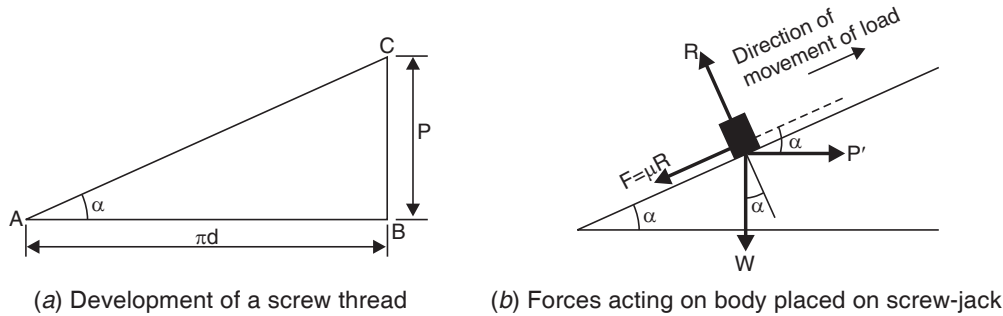


Fig. 6.41

Let R = Normal reaction

F = Force of friction = μR .

As the load W is lifted upwards, the force of friction will be acting downwards. All the forces acting on the body are shown in Fig. 6.41 (b).

Resolving forces along the inclined plane,

$$F + W \sin \alpha = P' \cos \alpha$$

or

$$\mu R + W \sin \alpha = P' \cos \alpha$$

$$(\because F = \mu R) \quad \dots(i)$$

Resolving forces normal to the inclined plane

$$R = W \cos \alpha + P' \sin \alpha$$

Substituting the value of R in equation (i), we get

$$\mu(W \cos \alpha + P' \sin \alpha) + W \sin \alpha = P' \cos \alpha$$

But $\mu = \tan \phi = \frac{\sin \phi}{\cos \phi}$

$$\therefore \frac{\sin \phi}{\cos \phi} (W \cos \alpha + P' \sin \alpha) + W \sin \alpha = P' \cos \alpha$$

or

$$W \frac{\sin \phi \cos \alpha}{\cos \phi} + P' \frac{\sin \phi \sin \alpha}{\cos \phi} + W \sin \alpha = P' \cos \alpha$$

Multiplying by $\cos \phi$, we get

$$W \sin \phi \cos \alpha + P' \sin \phi \sin \alpha + W \sin \alpha \cos \phi = P' \cos \alpha \cos \phi$$

or

$$W [\sin \phi \cos \alpha + \sin \alpha \cos \phi] = P' [\cos \alpha \cos \phi - \sin \alpha \sin \phi]$$

or

$$W \sin (\alpha + \phi) = P' \cos (\alpha + \phi)$$

$$\therefore P' = W \frac{\sin (\alpha + \phi)}{\cos (\alpha + \phi)} = W \tan (\alpha + \phi) \quad \dots(6.19)$$

Now P' is the effort applied at the mean radius of the screw-jack. But in case of screw-jack, effort is actually applied at the end of the handle as shown in Fig. 6.40. The effort applied at the end of the handle is P . Moment of P' about the axis of the screw

$$= P' \times \text{Distance of } P' \text{ from the axis of the screw}$$

$$= P' \times \text{Mean radius of the screw-jack}$$

$$= P' \times \frac{d}{2}$$

Moment of P about the axis of the screw

$$= P \times \text{Distance of } P \text{ from axis}$$

$$= P \times L.$$

Equating the two moments, we get $P' \times \frac{d}{2} = P \times L$

$$\therefore P = P' \times \frac{d}{2L} = \frac{d}{2L} \times P' \quad \dots(6.20)$$

Substituting the value of P' from equation (6.19) into equation (6.20), we get

$$P = \frac{d}{2L} \times W \tan (\alpha + \phi) \quad \dots(6.21)$$

Equation (6.21) gives the relation between the effort required at the end of the handle and the load lifted.

Torque required to work the jack

$$= T = P \times L = \frac{d}{2} W \tan (\alpha + \phi).$$

Now

$$\begin{aligned} P &= \frac{d}{2L} W \tan (\alpha + \phi) \\ &= \frac{Wd}{2L} \frac{\tan \alpha + \tan \phi}{1 - \tan \alpha \tan \phi} \quad \left(\because \tan (\alpha + \phi) = \frac{\tan \alpha + \tan \phi}{1 - \tan \alpha \tan \phi} \right) \\ &= \frac{Wd}{2L} \frac{\frac{p}{\pi d} + \mu}{1 - \frac{p}{\pi d} \mu} \quad \left(\because \tan \alpha = \frac{p}{\pi d}, \tan \phi = \mu \right) \\ &= \frac{Wd}{2L} \left(\frac{p + \mu \pi d}{\pi d - p \mu} \right) \quad \dots(6.22) \end{aligned}$$

Equation (6.22) gives the value of P in terms of co-efficient of friction and pitch of the screw.

(i) Effort Required at the end of the handle of the screw-jack to lower the load W .

The screw-jack is also used for lowering the heavy load. When the load is lowered by the screw-jack, the force by friction ($F = \mu R$) will act upwards. Fig. 6.42 shows all the forces acting on the body.

Resolving forces along the inclined plane

$$F + P' \cos \alpha = W \sin \alpha$$

$$\text{or} \quad \mu R + P' \cos \alpha = W \sin \alpha \quad \dots(i)$$

Resolving forces normal to the plane

$$R = W \cos \alpha + P' \sin \alpha$$

Substituting the value of R in equation (i), we get

$$\mu(W \cos \alpha + P' \sin \alpha) + P' \cos \alpha = W \sin \alpha$$

$$\text{or} \quad \mu W \cos \alpha + \mu P' \sin \alpha + P' \cos \alpha = W \sin \alpha$$

$$\text{or} \quad \mu P' \sin \alpha + P' \cos \alpha = W \sin \alpha - \mu W \cos \alpha$$

$$\text{or} \quad P' [\mu \sin \alpha + \cos \alpha] = W [\sin \alpha - \mu \cos \alpha]$$

$$\text{But} \quad \mu = \tan \phi = \frac{\sin \phi}{\cos \phi}$$

$$\therefore P' \left[\frac{\sin \phi}{\cos \phi} \sin \alpha + \cos \alpha \right] = W \left[\sin \alpha - \frac{\sin \phi}{\cos \phi} \cos \alpha \right]$$

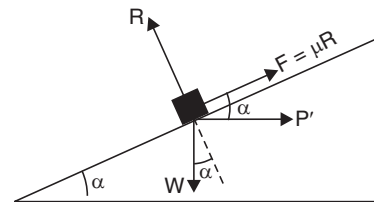


Fig. 6.42. Body moving down

Multiplying by $\cos \phi$, we get

$$P' [\sin \phi \sin \alpha + \cos \alpha \cos \phi] = W [\sin \alpha \cos \phi - \sin \phi \cos \alpha]$$

or $P' [\cos (\phi - \alpha)] = W [\sin (\phi - \alpha)]$

$$\therefore P' = W \frac{\sin (\phi - \alpha)}{\cos (\phi - \alpha)} = W \tan (\phi - \alpha) \quad \dots[6.23 (a)]$$

$$\text{If } \alpha > \phi, \text{ then } P' = W \tan (\alpha - \phi) \quad \dots[6.23 (b)]$$

But P' is the effort applied at the mean radius of the screw-jack. But in actual case, effort is applied at the handle of the jack. Let the effort applied at the handle is P as shown in Fig. 6.40. Equating the moments of P and P' about the axis of the jack, we get

$$P \times L = P' \times \frac{d}{2}$$

$$\therefore P = \frac{d}{2L} \times P'$$

$$= \frac{d}{2L} \times W \tan (\phi - \alpha) \quad \dots[6.23 (c)] \quad [\because P' = W \tan (\phi - \alpha)]$$

Equation [6.23 (c)] gives the relation between the effort required at the end of the handle to lower the load (W).

(ii) **Expression for P in terms of co-efficient of friction and pitch of the screw.**

From equation [6.23 (c)]

$$P = \frac{Wd}{2L} \tan (\phi - \alpha) = \frac{Wd}{2L} \frac{\tan \phi - \tan \alpha}{1 + \tan \phi \tan \alpha}$$

$$= \frac{Wd}{2L} \frac{\mu - \frac{p}{\pi d}}{1 + \mu \frac{p}{\pi d}} \quad \left(\because \tan \phi = \mu, \tan \alpha = \frac{p}{\pi d} \right)$$

$$= \frac{Wd}{2L} \frac{\mu \pi d - p}{\pi d + \mu p} \quad \dots(6.24)$$

Problem 6.25. (a) Find the effort required to apply at the end of a handle, fitted to the screw head of a screw-jack to lift a load of 1500 N. The length of the handle is 70 cm. The mean diameter and the pitch of the screw-jack are 6 cm and 0.9 cm respectively. The co-efficient of friction is given as 0.095.

(b) If instead of raising the load of 1500 N, the same load is lowered, determine the effort required to apply at the end of the handle.

Sol. Given :

Load,	$W = 1500 \text{ N}$
Length of handle,	$L = 70 \text{ cm} = 0.70 \text{ m}$
Mean dia. of screw-jack,	$d = 6 \text{ cm} = 0.06 \text{ m}$
Pitch of the screw-jack,	$p = 0.9 \text{ cm} = 0.009 \text{ m}$
Co-efficient of friction,	$\mu = 0.095$.

(a) **Effort (P) required at the end of the handle for raising the load (W).**

Let P = Effort required

$$\text{Using equation (6.22), we have } P = \frac{Wd}{2L} \left(\frac{p + \mu \pi d}{\pi d - p \mu} \right)$$

$$\begin{aligned}
 &= \frac{1500 \times 0.06}{2 \times 0.70} \left(\frac{0.009 + 0.095 \times \pi \times 0.6}{\pi \times 0.06 - 0.009 \times 0.095} \right) \text{ N} \\
 &= 64.2857 \left(\frac{0.0269}{0.1885 - 0.000855} \right) = \mathbf{9.21 \text{ N. Ans.}}
 \end{aligned}$$

(b) The effort (P) required at the end of the handle for lowering the load W is given by equation (6.24) as

$$\begin{aligned}
 P &= \frac{Wd (\mu\pi d - p)}{2L (\pi d + \mu p)} \\
 &= \frac{1500 \times 0.06}{2 \times 0.70} \left(\frac{0.095 \times \pi \times 0.06 - 0.009}{\pi \times 0.6 + 0.095 \times 0.009} \right) \\
 &= 64.2857 \left(\frac{0.0089}{0.1885 + 0.000855} \right) = \mathbf{3.02 \text{ N. Ans.}}
 \end{aligned}$$

Problem 6.26. A screw-jack is used to lift a load of 3 kN. The screw of the screw-jack is square threaded with two threads to 1.2 cm. If the co-efficient of friction between the nut and screw is 0.09 and the outer diameter of the screw is 6 cm, find the force required at the end of the handle of length 60 cm to lift the load.

Sol. Given :

Load lifted, $W = 3 \text{ kN} = 3 \times 1000 = 3000 \text{ N}$

No. of square thread in 1.2 cm = 2

$$\begin{aligned}
 \therefore \text{ Pitch of the screw, } p &= \frac{1.2}{\text{No. of threads}} = \frac{1.2}{2} \\
 &= 0.6 \text{ cm} = 0.006 \text{ m.}
 \end{aligned}$$

$$\text{Thickness of the threads, } t = \frac{1}{2} \times p = \frac{1}{2} \times 0.6 = 0.3 \text{ cm}$$

Outer diameter of screw, $D_1 = 6 \text{ cm}$

$$\begin{aligned}
 \text{Diameter at the base of screw, } D_2 &= D_1 - 2 \times t = 6 - 2 \times 0.3 \\
 &= 6 - 0.6 = 5.4 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ Mean diameter of screw, } d &= \frac{D_1 + D_2}{2} = \frac{6.0 + 5.40}{2} \\
 &= \frac{11.40}{2} = 5.70 \text{ cm} = 0.057 \text{ m}
 \end{aligned}$$

Co-efficient of friction, $\mu = 0.09$

Length of handle, $L = 60 \text{ cm} = 0.60 \text{ m.}$

The force (P) required at the end of the handle for raising the load (W) is given by equation (6.22) as

$$\begin{aligned}
 P &= \frac{Wd \left(\frac{p + \mu\pi d}{\pi d - p\mu} \right)}{2L} \\
 &= \frac{3000 \times 0.057}{3 \times 0.60} \left(\frac{0.006 + 0.09 \times \pi \times 0.057}{\pi \times 0.057 - 0.006 \times 0.09} \right) \\
 &= 142.50 \times \left(\frac{0.0221}{0.179 - 0.00054} \right) = 17.6468 \text{ N} \\
 &= \mathbf{17.65 \text{ N. Ans.}}
 \end{aligned}$$

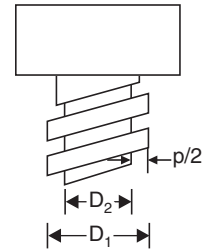


Fig. 6.43

Problem 6.27. A screw-jack is used to lift a load of 5 kN. The jack has screw with square threads having two threads per 12 mm length. If the co-efficient of friction between the nut and the screw is 0.08 and outer dia. of the screw is 60 mm, find the force required at the end of the 600 mm long lever to lift the load. (U.P. Tech. University, 2000–2001)

Sol. Given :

Load, $W = 5 \text{ kN} = 5000 \text{ N}$; No. of square threads in 12 mm = 2

$$\therefore \text{Pitch of screw, } p = \frac{12 \text{ mm}}{\text{No. of threads}} = \frac{12 \text{ mm}}{2} = 6 \text{ mm} = 0.006 \text{ m}$$

$$\text{Thickness of threads, } t = \frac{p}{2} = 0.003 \text{ m}$$

$$\text{Outer dia. of screw} = 60 \text{ mm} = 0.06 \text{ m}$$

$$\begin{aligned} \text{Inner dia. of screw} &= \text{Outer dia.} - 2t \\ &= 0.06 - 2 \times 0.003 = 0.054 \text{ m} \end{aligned}$$

$$\therefore \text{Mean dia. of screw, } d = \frac{\text{Outer dia.} + \text{Inner dia.}}{2} = \frac{0.06 + 0.054}{2} = 0.057 \text{ m}$$

$$\text{Co-efficient of friction, } \mu = 0.08, L = 600 \text{ mm} = 0.6 \text{ m}$$

The force (P) required at the end of the handle to lift the load is given by equation (6.21) as

$$\begin{aligned} P &= \frac{d}{2L} \times W \tan (\alpha + \phi) \\ &= \frac{d}{2L} W \times \frac{\tan \alpha + \tan \phi}{1 - \tan \alpha \tan \phi} \end{aligned} \quad \dots(i)$$

$$\text{where } \tan \alpha = \frac{p}{\pi d} = \frac{0.006}{\pi \times 0.057} = 0.0335$$

$$\tan \phi = \mu = 0.08$$

$$\begin{aligned} \therefore P &= \frac{0.057}{2 \times 0.6} \times 5000 \times \left(\frac{0.0335 + 0.08}{1 - 0.0335 \times 0.08} \right) \\ &= \frac{237.5 \times 0.1135}{0.099723} = 27.03 \text{ N. Ans.} \end{aligned}$$

6.13.1. Efficiency of a Screw-jack for Raising a Load W

Let P = Actual effort required at the end of the handle of the screw-jack to lift the load W

W = Load lifted

P_{ideal} = Ideal effort required (or when there is no friction) at the end of the handle of the screw-jack to lift the load W

d = Mean diameter of the screw

L = Length of the handle

ϕ = Angle of friction

α = Angle of screw or helix angle

μ = Co-efficient of friction = $\tan \phi$.

The actual effort (P) required at the end of the handle of the screw-jack to lift the load (W) is given by equation (6.21) as

$$P = \frac{d}{2L} W \tan (\alpha + \phi) \quad \dots(i)$$

The ideal effort (P_{ideal}) at the end of the handle of the screw-jack to lift the load (W) will be obtained by making friction zero. Friction will be zero, when the co-efficient of friction $\mu = \tan \phi = 0$. This means angle of friction ϕ is zero. If the value of $\phi = 0$ is substituted in equation (i), the value of P becomes as P_{ideal} .

$$\therefore P_{ideal} = \frac{d}{2L} W \tan \alpha \quad \dots(ii)$$

The efficiency of a machine (here of screw-jack) in terms of ideal effort and actual effort is given as

$$\eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{P_{ideal}}{P} \quad \dots(iii)$$

Substituting the values of P_{ideal} from equation (ii) and the value of P from (i), in equation (iii), we have

$$\eta = \frac{\frac{d}{2L} W \tan \alpha}{\frac{d}{2L} W \tan (\alpha + \phi)} = \frac{\tan \alpha}{\tan (\alpha + \phi)} \quad \dots(6.25)$$

HIGHLIGHTS

1. Force of friction always acts in the direction opposite to the direction of motion.
2. The maximum value of frictional force acting on a body, when the body is on the point of motion, is called limiting force of friction. It is denoted by F .
3. The force of friction, acting on a body when the body is moving, is called dynamic friction.
4. The ratio of the limiting force of friction (F) to the normal reaction (R) between two bodies is known as co-efficient of friction. It is denoted by μ . Mathematically, $\mu = \frac{F}{R}$.
5. The angle made by the resultant of the normal reaction (R) and the limiting force of friction (F) with the normal reaction is known as angle of friction. It is denoted by ϕ .
6. The relation between angle of friction (ϕ) and co-efficient of friction (μ) is expressed as $\tan \phi = \mu$.
7. The force of friction is always equal to μR , where R is normal reaction.
8. If a ladder is leaning against a smooth vertical wall, the force of friction between ladder and vertical wall will be zero.
9. The problems on ladder friction are solved by taking algebraic sum of horizontal and vertical components of the forces to be zero and also taking the moments of all forces about any point to be zero.
10. If a body is placed on a rough inclined plane and the angle of inclination of the plane is gradually increased, till the body just starts sliding down the plane. The angle of the inclined plane, at which the body just begins to slide down the plane, is called angle of repose.
11. Angle of repose is equal to angle of friction.
12. If the inclination of the plane, with the horizontal is less than angle of friction, the body placed on the inclined plane will be always in equilibrium without any external force.
13. The minimum force required to drag a body of weight W placed on a rough horizontal plane, when the force is applied at an angle θ with the horizontal is equal to $W \sin \theta$. And the angle θ will be equal to angle of friction for the least force.

14. The force required to drag a body of weight W , placed on a rough inclined plane of inclination α to the horizontal, when force is applied to the body in such a way that the force makes an angle θ with the inclined plane is given by

$$P = W \frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)} \dots \text{when the body is on the point of moving up the plane.}$$

$$= W \frac{\sin(\alpha - \phi)}{\cos(\theta + \phi)} \dots \text{when the body is on the point of moving down the plane.}$$

where α = Inclination of the inclined plane with horizontal.

θ = Angle made by the force with the inclined plane.

ϕ = Angle of friction.

15. If the force for the S. No. 14 is applied to the body horizontally, then the force required to drag the body is given by
 $P = W \tan(\alpha + \phi) \dots$ when the body is on the point of moving up the plane.
 $= W \tan(\alpha - \phi) \dots$ when the body is on the point of moving down the plane.
16. Wedge is a piece of metal or wood which is usually of a triangular or trapezoidal in cross-section. It is used either lifting loads or used for slight adjustments in the position of a body *i.e.*, for tightening fits or keys for shafts.
17. The problems on wedges are solved by : (i) resolving the forces on the wedge and block horizontally and vertically or (ii) by applying Lami's Theorem.

EXERCISE 6

(A) Theoretical Questions

- Define the following terms : Friction, limiting force of friction, co-efficient of friction and angle of friction.
- Explain the difference between co-efficient of friction and angle of friction.
- Explain the following :
 (i) Laws of static friction, and
 (ii) Limiting angle of friction. *(U.P. Tech. University, May 2009)*
- Prove that the angle of friction is equal to the angle of the inclined plane, when a solid body of weight W placed on the inclined plane, is about to slide down.
- What do you mean by 'angle of repose' ? Prove that angle of repose is equal to the angle of friction.
- A body of weight W is placed on an inclined rough plane. The inclination of the plane with the horizontal is less than the angle of friction. The body will
 (a) be in equilibrium (b) move downwards
 (c) move upwards (d) none of the above. [Ans. (a)]
- If in the above question, the inclination of the plane with the horizontal is more than the angle of friction, the body will
 (a) be in equilibrium (b) move downwards
 (c) move upwards (d) none of the above. [Ans. (b)]
- A body of weight W is placed on a rough inclined plane having inclination α to the horizontal. The force P is applied horizontally to drag the body. If the body is on the point of motion up the plane, prove that P is given by

$$P = W \tan(\alpha + \phi)$$

where

ϕ = Angle of friction.

9. In the above question, if the body is on the point of motion down the plane, prove that the force P is given by

$$P = W \tan (\alpha - \phi).$$

10. Define static and kinetic friction and state the laws of solid friction.
 11. Derive an expression for the least force required to drag a body on a rough horizontal plane.

(B) Numerical Problems

1. A body of weight 90 N is placed on a rough horizontal plane. Determine the co-efficient of friction if a horizontal force of 63 N just causes the body to slide over the horizontal plane. [Ans. 0.7]
2. A body of weight 150 N is placed on a rough horizontal plane. If the co-efficient of friction between the body and the horizontal plane is 0.4, determine the horizontal force required to just slide the body on the plane. [Ans. 60 N]
3. The force required to pull a body of weight 40 N on a rough horizontal plane is 15 N. Determine the co-efficient of friction if the force is applied at an angle of 20° with the horizontal. [Ans. 0.404]
4. A body of weight 60 N is placed on a rough horizontal plane. To just move the body on the horizontal plane, a push of 18 N inclined at 20° to the horizontal plane is required. Find the co-efficient of friction. [Ans. 0.255]
5. A pull of 60 N inclined at 25° to the horizontal plane, is required just to move a body placed on a rough horizontal plane. But the push required to move the body is 75 N. If the push is inclined at 25° to the horizontal, find the weight of the body and co-efficient of friction. [Ans. 253.83 N, .238]
6. Find the least force required to pull a body of weight W placed on a rough horizontal plane, when the force is applied at an angle θ with the horizontal. [Ans. $W \sin \theta$]
7. A body of weight 450 N is pulled up an inclined plane, by a force of 300 N. The inclination of the plane is 30° to the horizontal and the force is applied parallel to the plane. Determine the co-efficient of friction. [Ans. .192]
8. A body of weight 400 N is pulled up along an inclined plane having inclination 30° to the horizontal at a steady speed. If the co-efficient of friction between the body and the plane is 0.3 and force is applied parallel to the inclined plane, find the force required. Find also the work done on the body if the distance travelled by the body is 10 m along the plane. [Ans. 303.92 N, 3039.2 Nm]
9. An effort of 180 N is required just to move a certain body up an inclined plane of angle 15° , the force being parallel to the plane. If the angle of inclination of the plane is made 20° , the effort required, again applied parallel to the plane, is found to be 210 N. Find the weight of the body and co-efficient of friction. [Ans. 386.68 N, 0.214]
10. Block A weighing 20 N is a rectangular prism resting on a rough inclined plane as shown in Fig. 6.20. The block is tied up by a horizontal string which has a tension of 6 N. Find : (i) the frictional force on the block, (ii) the normal reaction of the inclined plane, and (iii) the co-efficient of friction between the surface of contact. [Ans. (i) 9.898 N, (ii) 18.382 N, (iii) 0.538]
11. A cord connects two bodies of weights 500 N and 1000 N. The two bodies are placed on an inclined plane and cord is parallel to inclined plane. The co-efficients of friction for the weight of 500 N is 0.20 and that of 1000 N is 0.4. Determine the inclination of the plane to the horizontal and tension in the cord when the motion is about to take place, down the inclined plane. The body weight 500 N is below the body weighing 1000 N. [Ans. 18.417° , 63.086 N]

UNIT II

BASIC STRUCTURAL ANALYSIS

Chapters

7. Plane Truss
8. Beams (Shear Force and Bending Moment)

7.1. DIFFERENCE BETWEEN TRUSS AND FRAME

A structure made up of several bars (or members) riveted or welded together is known as truss. The isometric view of a structure made of trusses is shown in Fig. 7.1 (a) and front view is shown in Fig. 7.1 (b). If the members of the structure are hinged or pin-joined, then the structure is known as a Frame. Hence the difference between truss and frame is that incase of truss members are riveted or welded whereas incase of frame the members are hinged or pin-joined. If the frame is composed of such members which are just sufficient to keep the frame in equilibrium, when the frame is supporting an external load, then the frame is known as *perfect*

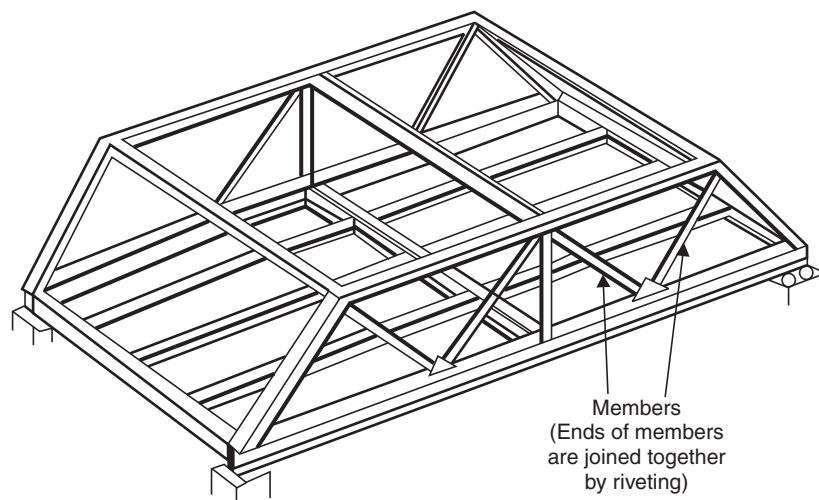


Fig. 7.1 (a). Isometric view of a structure made of truss.

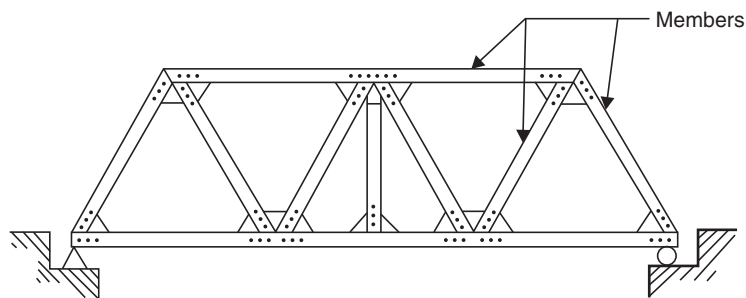


Fig. 7.1 (b). Front view of a truss.

frame. Though in actual practice the members are welded or riveted together at their joints, yet for calculation purposes the joints are assumed to be hinged or pin-jointed. In this chapter, we shall discuss how to determine the axial forces in the members of a perfect frame, when it is subjected to some external load.

7.2. PERFECT AND IMPERFECT TRUSS

The different types of frames are :

(i) Perfect frame, and

(ii) Imperfect frame.

Imperfect frame may be a deficient frame or a redundant frame.

7.2.1. Perfect Frame. The frame which is composed of such members, which are just sufficient to keep the frame in equilibrium, when the frame is supporting an external load, is known as perfect frame. The simplest perfect frame is a triangle as shown in Fig. 7.1 (c) which consists three members and three joints. The three members are : AB , BC and AC whereas the three joints are A , B and C . This frame can be easily analysed by the condition of equilibrium.

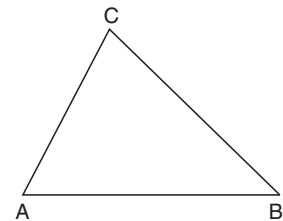


Fig. 7.1 (c)

Let the two members CD and BD and a joint D are added to the triangular frame ABC . Now, we get a frame $ABCD$ as shown in Fig. 7.2 (a). This frame can also be analysed by the conditions of equilibrium. This frame is also known as perfect frame.

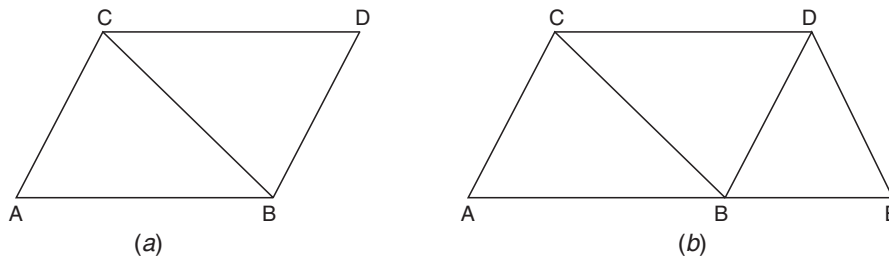


Fig. 7.2

Suppose we add a set of two members and a joint again, we get a perfect frame as shown in Fig. 7.2 (b). Hence for a perfect frame, the number of joints and number of members are given by,

$$n = 2j - 3$$

where n = Number of members, and

j = Number of joints.

7.2.2. Imperfect Frame. A frame in which number of members and number of joints are not given by

$$n = 2j - 3$$

is known, as imperfect frame. This means that number of members in an imperfect frame will be either more or less than $(2j - 3)$.

(i) If the number of members in a frame are less than $(2j - 3)$, then the frame is known as *deficient frame*.

(ii) If the number of members in a frame are more than $(2j - 3)$, then the frame is known as *redundant frame*.

7.3. ASSUMPTIONS AND ANALYSIS OF PLANE TRUSS

The assumptions made in finding out the axial forces in a frame are :

- (i) The frame is a perfect frame
- (ii) The frame carries load at the joints
- (iii) All the members are pin-jointed.

7.3.1. Reactions of Supports of a Frame. The frames are generally supported

- (i) on a roller support or
- (ii) on a hinged support.

If the frame is supported on a roller support, then the line of action of the reaction will be at right angles to the roller base as shown in Figs. 7.3 and 7.4.

If the frame is supported on a hinged support, then the line of action of the reaction will depend upon the load system on the frame.

The reactions at the supports of a frame are determined by the conditions of equilibrium. The external load on the frame and the reactions at the supports must form a system of equilibrium.

7.3.2. Analysis of a Frame. Analysis of a frame consists of :

- (i) Determinations of the reactions at the supports and
- (ii) Determination of the axial forces in the members of the frame.

The reactions are determined by the condition that the applied load system and the induced reactions at the supports form a system in equilibrium.

The forces in the members of the frame are determined by the condition that every joint should be in equilibrium and so, the forces acting at every joint should form a system in equilibrium.

A frame is analysed by the following methods :

- (i) Method of joints,
- (ii) Method of sections, and
- (iii) Graphical method.

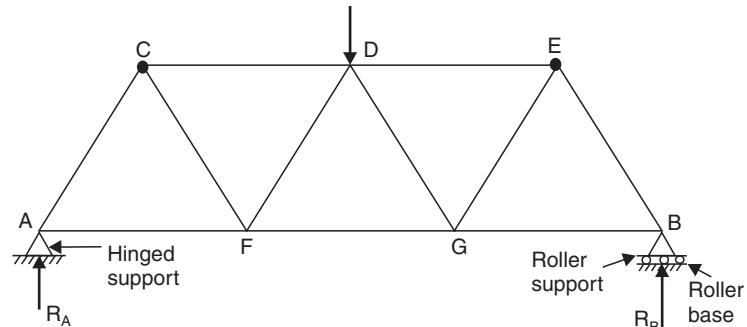


Fig. 7.3

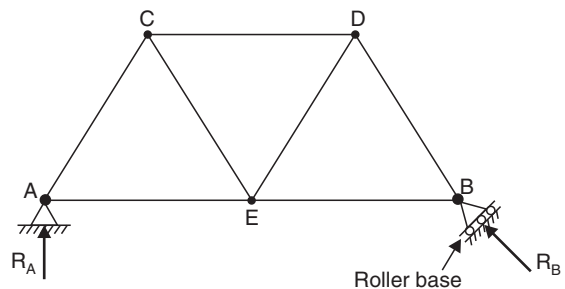


Fig. 7.4

7.4. METHOD OF JOINTS

In this method, after determining the reactions at the supports, the equilibrium of every joint is considered. This means the sum of all the vertical forces as well as the horizontal forces acting on a joint is equated to zero. The joint should be selected in such a way that at any time there are only two members, in which the forces are unknown. The force in the member will be compressive if the member *pushes the joint* to which it is connected whereas the force in the member will be tensile if the member *pulls the joint* to which it is connected.

Problem 7.1. Find the forces in the members AB, AC and BC of the truss shown in Fig. 7.5.

Sol. First determine the reactions R_B and R_C . The line of action of load of 20 kN acting at A is vertical. This load is at a distance of $AB \times \cos 60^\circ$ from the point B. Now let us find the distance AB.

The triangle ABC is a right-angled triangle with angle $BAC = 90^\circ$. Hence AB will be equal to $BC \times \cos 60^\circ$.

$$\therefore AB = 5 \times \cos 60^\circ = 5 \times \frac{1}{2} = 2.5 \text{ m}$$

Now the distance of line of action of 20 kN from B is

$$AB \times \cos 60^\circ \text{ or } 2.5 \times \frac{1}{2} = 1.25 \text{ m.}$$

Taking the moments about B, we get

$$R_C \times 5 = 20 \times 1.25 = 25$$

$$\therefore R_C = \frac{25}{5} = 5 \text{ kN}$$

and $R_B = \text{Total load} - R_C = 20 - 5 = 15 \text{ kN}$

Now let us consider the equilibrium of the various joints.

Joint B

Let F_1 = Force in member AB

F_2 = Force in member BC

Let the force F_1 is acting towards the joint B and the force F_2 is acting away* from the joint B as shown in Fig. 7.6. (The reaction R_B is acting vertically up. The force F_2 is horizontal. The reaction R_B will be balanced by the vertical component of F_1 . The vertical component of F_1 must act downwards to balance R_B . Hence F_1 must act towards the joint B so that its vertical component is downward. Now the horizontal component of F_1 is towards the joint B. Hence force F_2 must act away from the joint to balance the horizontal component of F_1).

Resolving the forces acting on the joint B, vertically

$$F_1 \sin 60^\circ = 15$$

$$\therefore F_1 = \frac{15}{\sin 60^\circ} = \frac{15}{0.866} = 17.32 \text{ kN (Compressive)}$$

As F_1 is pushing the joint B, hence this force will be compressive. Now resolving the forces horizontally, we get

$$F_2 = F_1 \cos 60^\circ = 17.32 \times \frac{1}{2} = 8.66 \text{ kN (tensile)}$$

As F_2 is pulling the joint B, hence this force will be tensile.

Joint C

Let F_3 = Force in the member AC

F_2 = Force in the member BC

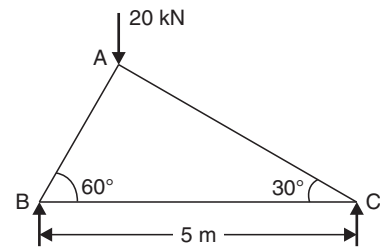


Fig. 7.5

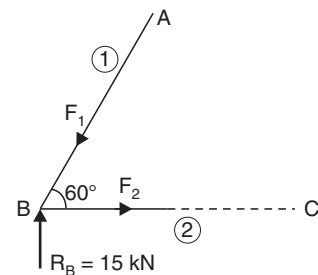


Fig. 7.6

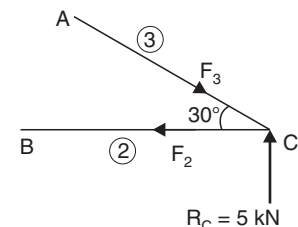


Fig. 7.7

*The direction of F_2 can also be taken towards the joint B. Actually when we consider the equilibrium of the joint B, if the magnitude of F_1 and F_2 comes out to be positive then the assumed direction of F_1 and F_2 are correct. But if any one of them is having a negative magnitude then the assumed direction of that force is wrong. Correct direction then will be the reverse of the assumed direction.

The force F_2 has already been calculated in magnitude and direction. We have seen that force F_2 is tensile and hence it will pull the joint C . Hence it must act away from the joint C as shown in Fig. 7.7.

Resolving forces vertically, we get

$$F_3 \sin 30^\circ = 5 \text{ kN}$$

$$\therefore F_3 = \frac{5}{\sin 30^\circ} = 10 \text{ kN (Compressive)}$$

As the force F_3 is pushing the joint C , hence it will be compressive.

Problem 7.2. A truss of span 7.5 m carries a point load of 1 kN at joint D as shown in Fig. 7.8. Find the reactions and forces in the members of the truss.

Sol. Let us first determine the reactions R_A and R_B

Taking moments about A , we get

$$R_B \times 7.5 = 5 \times 1$$

$$\therefore R_B = \frac{5}{7.5} = \frac{2}{3} = 0.667 \text{ kN}$$

$$\begin{aligned} \therefore R_A &= \text{Total load} - R_B \\ &= 1 - 0.667 = 0.333 \text{ kN} \end{aligned}$$

Now consider the equilibrium of the various joints.

Joint A

Let F_1 = Force in member AC
 F_2 = Force in member AD .

Let the force F_1 is acting towards the joint A and F_2 is acting away from the joint A as shown in Fig. 7.9.

Resolving the forces vertically, we get

$$F_1 \sin 30^\circ = R_A$$

$$\begin{aligned} \text{or } F_1 &= \frac{R_A}{\sin 30^\circ} = \frac{0.333}{0.5} \\ &= 0.666 \text{ kN (Compressive)} \end{aligned}$$

Resolving the forces horizontally, we get

$$\begin{aligned} F_2 &= F_1 \times \cos 30^\circ \\ &= 0.666 \times 0.866 = 0.5767 \text{ kN (Tensile)} \end{aligned}$$

Joint B

Let F_4 = Force in member BC
 F_5 = Force in member BD

Let the direction of F_4 and F_5 are assumed as shown in Fig. 7.10.

Resolving the forces vertically, we get

$$F_4 \sin 30^\circ = R_B = 0.667$$

$$\text{or } F_4 = \frac{0.667}{\sin 30^\circ} = 1.334 \text{ kN (Compressive)}$$

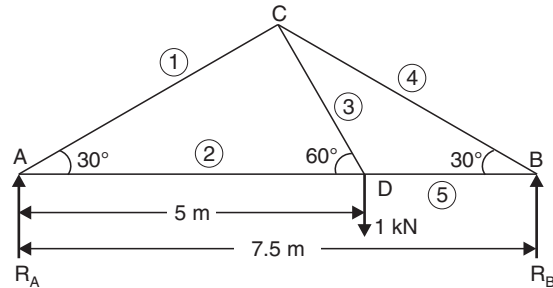


Fig. 7.8

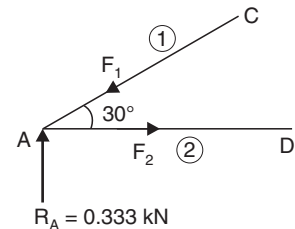


Fig. 7.9

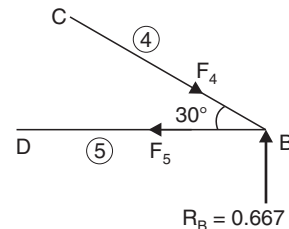


Fig. 7.10

Resolving the forces horizontally, we get

$$F_5 = F_4 \cos 30^\circ = 1.334 \times 0.866 = 1.155 \text{ kN (Tensile)}$$

Joint D

Let F_3 = Force in member CD . The forces F_2 and F_5 have been already calculated in magnitude and direction. The forces F_2 and F_5 are tensile and hence they will be pulling the joint D as shown in Fig. 7.11. Let the direction* of F_3 is assumed as shown in Fig. 7.11.

Resolving the forces vertically, we get

$$F_3 \sin 60^\circ = 1$$

$$\begin{aligned} \therefore F_3 &= \frac{1}{\sin 60^\circ} = \frac{1}{0.866} \\ &= 1.1547 \text{ kN (Tensile)} \end{aligned}$$

Hence the forces in the members are :

$$F_1 = 0.666 \text{ kN (Compressive)}$$

$$F_2 = 0.5767 \text{ kN (Tensile)}$$

$$F_3 = 1.1547 \text{ kN (Tensile)}$$

$$F_4 = 1.334 \text{ kN (Compressive)}$$

$$F_5 = 1.155 \text{ kN (Tensile)}. \quad \text{Ans.}$$

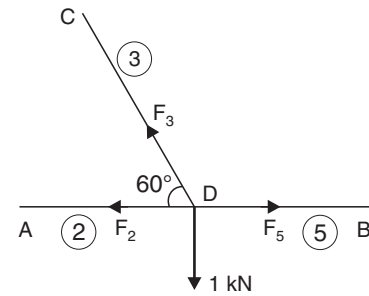


Fig. 7.11

Problem 7.3. A truss of span 5 m is loaded as shown in Fig. 7.12. Find the reactions and forces in the members of the truss.

Sol. Let us first determine the reactions R_A and R_B .

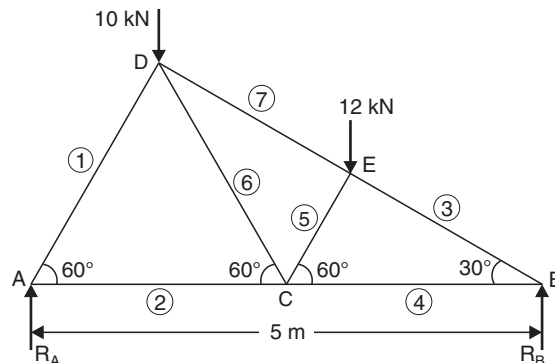


Fig. 7.12

Triangle ABD is a right-angled triangle having angle $ADB = 90^\circ$.

$$\therefore AD = AB \cos 60^\circ = 5 \times 0.5 = 2.5 \text{ m}$$

The distance of the line of action of the vertical load 10 kN from point A will be $AD \cos 60^\circ$

$$\text{or } 2.5 \times 0.5 = 1.25 \text{ m.}$$

From triangle ACD , we have $AC = AD = 2.5 \text{ m}$

$$\therefore BC = 5 - 2.5 = 2.5 \text{ m}$$

*The horizontal force F_5 is more than F_2 . Hence the horizontal component of F_3 must be in the direction of F_2 . This is only possible if F_3 is acting away from D .

In right-angled triangle CEB , we have

$$BE = BC \cos 30^\circ = 2.5 \times \frac{\sqrt{3}}{2}$$

The distance of the line of action of the vertical load of 12 kN from point B will be $BE \times \cos 30^\circ$

or
$$BE \times \frac{\sqrt{3}}{2} = \left(2.5 \times \frac{\sqrt{3}}{2}\right) \times \frac{\sqrt{3}}{2} = 1.875 \text{ m.}$$

$\therefore$ The distance of the line of action of the load of 12 kN from point A will be $(5 - 1.875) = 3.125 \text{ m.}$

Now taking the moments about A , we get

$$R_B \times 5 = 10 \times 1.25 + 12 \times 3.125 = 50$$

$$\therefore R_B = \frac{50}{5} = 10 \text{ kN}$$

$$\therefore R_A = \text{Total load} - R_B = (10 + 12) - 10 = 12 \text{ kN}$$

Now consider the equilibrium of the various joints.

Joint A

Let F_1 = Force in member AD , and

F_2 = Force in member AC .

Let the directions of F_1 and F_2 are assumed as shown in Fig. 7.13.

Resolving the forces vertically,

$$F_1 \sin 60^\circ = 12$$

$$\therefore F_1 = \frac{12}{\sin 60^\circ} = 13.856 \text{ kN (Compressive)}$$

Resolving the forces horizontally,

$$F_2 = F_1 \cos 60^\circ = 13.856 \times 0.5 = 6.928 \text{ kN (Tensile)}$$

Now consider the joint B .

Joint B

Let F_3 = Force in member BE , and

F_4 = Force in member BC .

Let the directions of F_3 and F_4 are assumed as shown in Fig. 7.14.

Resolving the forces vertically, we get

$$F_3 \sin 30^\circ = 10$$

$$\therefore F_3 = \frac{10}{\sin 30^\circ} = 20 \text{ kN (Compressive)}$$

Now resolving the forces horizontally, we get

$$F_4 = F_3 \cos 30^\circ = 20 \times 0.866 = 17.32 \text{ kN (Tensile)}$$

Now consider the joint C .

Joint C

Let F_5 = Force in member CE

F_6 = Force in member CD

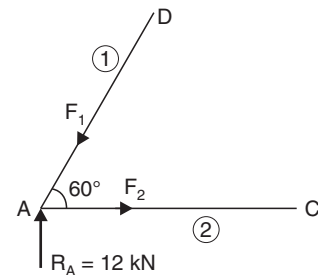


Fig. 7.13

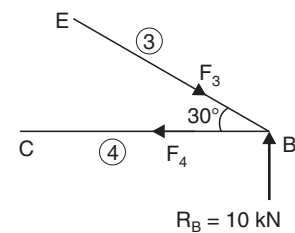


Fig. 7.14

Let the directions of F_5 and F_6 are assumed as shown in Fig. 7.15.

The forces F_2 and F_4 are already known in magnitude and directions. They are tensile and hence will be pulling the joint C as shown in Fig. 7.15.

Resolving forces vertically, we get

$$F_6 \sin 60^\circ + F_5 \sin 60^\circ = 0$$

$$\text{or} \quad F_6 = -F_5 \quad \dots(i)$$

Resolving forces horizontally, we get

$$F_2 - F_6 \cos 60^\circ = F_4 - F_5 \cos 60^\circ$$

$$\text{or} \quad 6.928 - \frac{F_6}{2} = 17.32 - \frac{F_5}{2}$$

$$\text{or} \quad \frac{-F_6 + F_5}{2} = 17.32 - 6.928 = 10.392$$

$$\text{or} \quad -F_6 + F_5 = 10.392 \times 2 = 20.784$$

$$\text{or} \quad F_5 + F_5 = 20.784$$

$$\therefore F_5 = \frac{20.784}{2} = 10.392 \text{ kN}$$

$$\text{and} \quad F_6 = -F_5 = -10.392 \text{ kN}$$

The magnitude of F_6 is -ve, hence the assumed direction of F_6 is wrong. The correct direction F_6 will be as shown in Fig. 7.15 (a).

$$\therefore F_5 = 10.392 \quad (\text{Compressive})$$

$$\text{and} \quad F_6 = 10.392 \quad (\text{Tensile})$$

Now consider the joint E .

Joint E

Let F_7 = Force in member ED

Let F_7 is acting as shown in Fig. 7.16.

The forces F_3 and F_5 are known in magnitude and directions. They are compressive hence they will be pushing the joint E as shown in Fig. 7.16.

Resolving the forces along BED , we get

$$F_7 + 12 \cos 60^\circ = F_3$$

$$\text{or} \quad F_7 = F_3 - 12 \times 0.5 = 20 - 6 = 14 \text{ kN (Compressive)}$$

As F_7 is positive hence the assumed direction of F_7 is correct.

Problem 7.4. A truss of span 9 m is loaded as shown in Fig. 7.17. Find the reactions and forces in the member of the truss.

Sol. Let us first calculate the reactions R_A and R_B .

Taking moments about A , we get

$$R_B \times 9 = 9 \times 3 + 12 \times 6 = 27 + 72 = 99$$

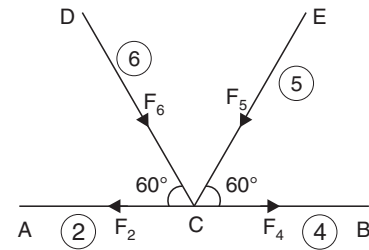


Fig. 7.15

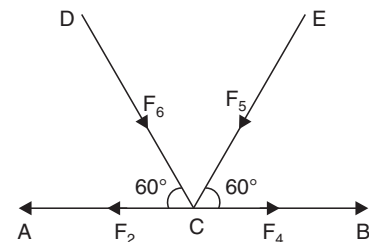


Fig. 7.15 (a)

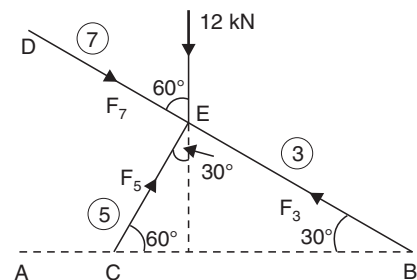


Fig. 7.16

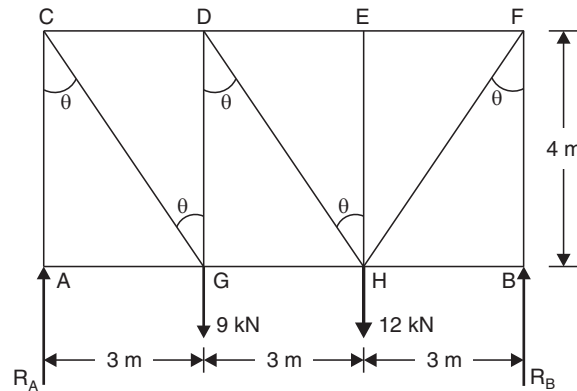


Fig. 7.17

$$\therefore R_B = \frac{99}{9} = 11 \text{ kN}$$

and

$$R_A = \text{Total load} - R_B = (9 + 12) - 11 = 10 \text{ kN}$$

In this problem, there are some members in which force is zero.

These members are obtained directly as given below :

“If three forces act at a joint and two of them are along the same straight line, then for the equilibrium of the joint, the third force should be equal to zero.”

1. Three forces are acting at the point A (i.e., R_A , F_{AC} and F_{AG}), two of which (i.e., R_A , R_{AC}) are along the same straight line. Hence the third force (i.e., R_{AG}) is zero.

2. Similarly, three forces are acting at the joint B (i.e., R_B , F_{BF} and F_{BH}), two of which (i.e., R_B and F_{BH}) are along the same straight line. Hence the third force F_{BF} should be zero.

3. At the joint E also, three forces (i.e., F_{ED} , F_{EF} and F_{EH}) are acting, two of which (i.e., F_{ED} and F_{EF}) are along the same straight line. Hence the third force F_{EH} must be zero.

Now the equilibrium of various joints can be considered.

Joint A [See Fig. 7.17 (a)]

$$F_{AG} = \text{Force in member AG} = 0$$

$$\begin{aligned} F_{AC} &= \text{Force in member AC} \\ &= R_A = 10 \text{ kN (Compressive)} \end{aligned}$$

Now consider the equilibrium of joint C.

Joint C [See Fig. 7.17 (b)]

Let F_{CD} = Force in member CD

$$F_{CG} = \text{Force in member CG}$$

$$F_{AC} = 10 \text{ kN (Compressive)}$$

Let the directions of F_{CG} and F_{CD} are assumed as shown in Fig. 7.17 (b).

Resolving the forces vertically, we get

$$F_{CG} \cos \theta = 10$$

$$\therefore F_{CG} = \frac{10}{\cos \theta}$$

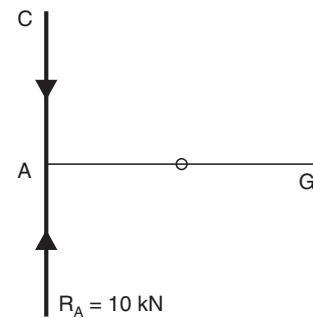


Fig. 7.17 (a)

But $\cos \theta = \frac{AC}{CG} = \frac{4}{5} \quad (\because CG = \sqrt{3^2 + 4^2} = 5)$

$\therefore F_{CG} = \frac{10}{(4/5)} = 10 \times \frac{5}{4} = 12.5 \text{ kN} \quad (\text{Tensile})$

Resolving forces horizontally, we get

$$\begin{aligned} F_{CD} &= F_{CG} \sin \theta \\ &= 12.5 \times \frac{3}{5} = 7.5 \text{ kN} \quad (\text{Compressive}) \end{aligned}$$

Now consider the equilibrium of joint G .

Joint G

The force in member CG is 12.5 kN (Tensile).

Hence at the joint G , this force will be pulling the joint G as shown in Fig. 7.17 (c).

Resolving the forces vertically, we get

$$\begin{aligned} 12.5 \cos \theta + F_{GD} &= 9 \\ \therefore F_{GD} &= 9 - 12.5 \cos \theta \\ &= 9 - 12.5 \times \frac{4}{5} \quad \left(\because \cos \theta = \frac{4}{5} \right) \\ &= 9 - 10 = -1 \text{ kN}. \end{aligned}$$

As the magnitude of F_{GD} is negative, hence its assumed direction is wrong. The correct direction will be as shown in Fig. 7.17 (d).

Then, $F_{GD} = 1 \text{ kN}$ (Compressive)

Resolving the forces horizontally, we get

$$\begin{aligned} 12.5 \sin \theta &= F_{GH} \\ \text{or } F_{GH} &= 12.5 \times \frac{3}{5} \quad \left(\because \sin \theta = \frac{3}{5} \right) \\ &= 7.5 \text{ kN (Tensile)} \end{aligned}$$

Now consider the equilibrium of joint D .

Joint D

The forces in the members CD and GD have been already calculated. They are 7.5 kN and 1 kN respectively. Both are compressive.

Let F_{DH} = Force in member DH , and

F_{DE} = Force in member DE .

Resolving the forces vertically, we get

$$\begin{aligned} F_{DH} \cos \theta &= 1 \text{ kN} \\ \therefore F_{DH} &= \frac{1}{\cos \theta} = \frac{1}{(4/5)} \\ &= \frac{5}{4} = 1.25 \text{ kN (Tensile)} \end{aligned}$$

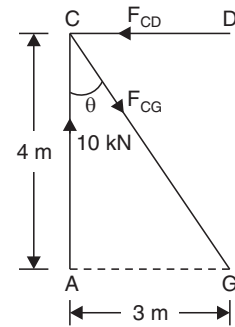


Fig. 7.17 (b)

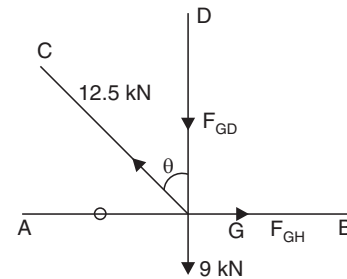


Fig. 7.17 (c)

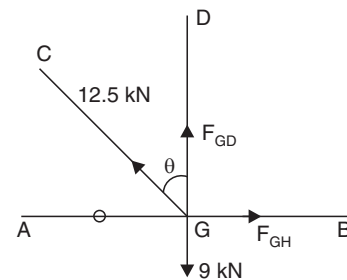


Fig. 7.17 (d)

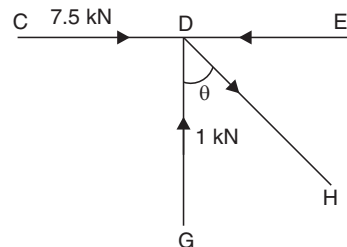


Fig. 7.17 (e)

Resolving the forces horizontally, we get

$$7.5 + F_{DH} \sin \theta = F_{DE}$$

$$\begin{aligned} \text{or} \quad F_{DE} &= 7.5 + 1.25 \times \frac{3}{5} & \left(\because \sin \theta = \frac{3}{5} \right) \\ &= 7.5 + 0.75 = 8.25 \text{ kN (Compressive)} \end{aligned}$$

Now consider the equilibrium of joint E .

Joint E

As shown in Fig. 7.17 (f), at joint E three forces are acting. The forces *i.e.*, F_{DE} and F_{EF} are in the same straight line.

Hence force F_{EH} must be zero.

$$\begin{aligned} \text{Force in } EF, \text{ i.e., } F_{EF} &= F_{DE} \\ &= 8.25 \text{ kN (Compressive)} \end{aligned}$$

Now consider the joint H .

Joint H

It is already shown that forces in the members EH and BH are zero.

Also the forces in the member GH is 7.5 kN tensile and in the member DH is 1.25 kN tensile.

Let F_{HF} is the force in the member HF .

Resolving forces vertically, we get

$$1.25 \cos \theta + F_{HF} \cos \theta = 12$$

$$\begin{aligned} \text{or} \quad 1.25 \times \frac{4}{5} + F_{HF} \times \frac{4}{5} &= 12 & \left(\because \cos \theta = \frac{4}{5} \right) \\ 1.0 + 0.8 F_{HF} &= 12 \end{aligned}$$

$$\begin{aligned} \text{or} \quad F_{HF} &= \frac{12 - 1.0}{0.8} = \frac{11}{0.8} \\ &= 13.75 \text{ (Tensile)} \end{aligned}$$

Now consider the joint B .

Joint B

See Fig. 7.17 (h).

$$\begin{aligned} \text{The force in member } BF & \\ &= 11 \text{ kN (Compressive)} \end{aligned}$$

Now the forces in each member are known.

They are shown in Fig. 7.18. Also these forces are shown below.

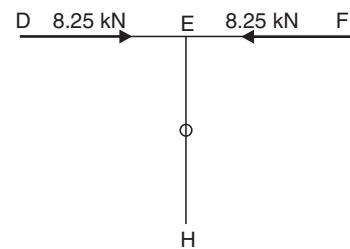


Fig. 7.17 (f)

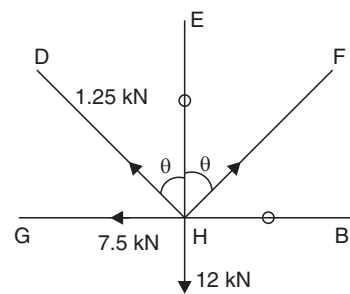


Fig. 7.17 (g)

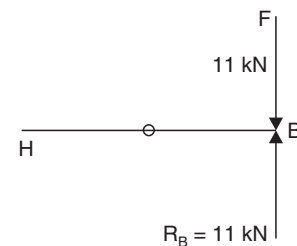


Fig. 7.17 (h)

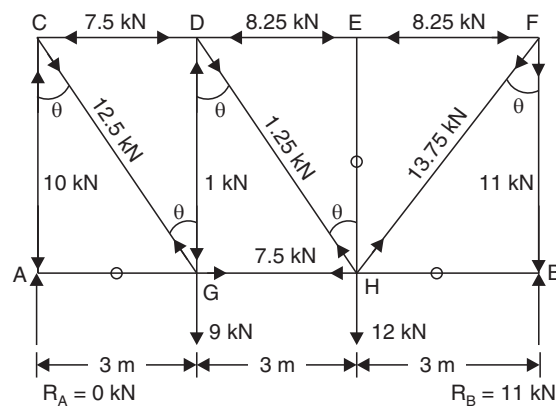


Fig. 7.18

Member	Force in member
AC	10 kN (Comp.)
AG	0
CG	12.5 kN (Tens.)
CD	7.5 kN (Comp.)
DG	10 kN (Comp.)
DE	8.25 kN (Comp.)
DH	1.25 kN (Tens.)
GH	7.5 kN (Tens.)
EH	0
EF	8.25 kN (Comp.)
HB	0
HF	13.75 kN (Tens.)
BF	11 kN (Comp.)

Problem 7.5. A plane truss is loaded and supported as shown in Fig. 7.19. Determine the nature and magnitude of the forces in the members 1, 2 and 3.

Sol. First calculate the reactions R_A and R_B

Taking moments about A, we get

$$R_B \times 4 = 1 \times 1000$$

$$\therefore R_B = \frac{1000}{4} = 250 \text{ N}$$

$$\therefore R_A = 1000 - 250 = 750 \text{ N}$$

From figure, we know that

$$\tan \theta = \frac{CH}{AH} = \frac{2.25}{4} = 0.75$$

$$\cos \theta = \frac{AH}{AC} = \frac{3}{3.75} = 0.8$$

$$\left(\because AC = \sqrt{3^2 + 2.25^2} = 3.75 \right)$$

and

$$\sin \theta = \frac{CH}{AC} = \frac{2.25}{3.75} = 0.6$$

Resolving the forces horizontally, we get

$$1250 \cos \theta + F_1 \cos \theta = F_{DF} \times \cos \theta$$

$$\text{or } 1250 + F_1 = F_{DF} \text{ or } F_1 - F_{DF} = -1250 \quad \dots(ii)$$

Adding (i) and (ii), we get

$$2F_1 = 416.66 - 1250 = -833.34$$

$$\therefore F_1 = -\frac{833.34}{2} = -416.67 \text{ N}$$

Substituting the value of F_1 in equation (i), we get

$$-416.67 + F_{DF} = 416.66$$

$$\text{or } F_{DF} = 416.66 + 416.67 = 833.33 \text{ N (Comp.)}$$

The magnitude of F_1 is negative. Hence its assumed direction is wrong. The correct direction of F_1 is shown in Fig. 7.19 (c).

$$\therefore F_1 = 416.67 \text{ N (Compressive). Ans.}$$

To find the forces F_2 and F_3 , consider the joint F .

Joint F

The forces in the members DF and EF are already known. They are :

$$F_{DF} = 833.33 \text{ N (Compressive)}$$

$$F_{EF} = 1000 \text{ N (Tensile).}$$

These forces are acting at the joint F as shown in Fig. 7.19 (d).

Let F_2 = Force in member FG , and

F_3 = Force in member FH .

Resolving forces vertically, we get

$$833.33 \sin \theta = F_2$$

$$\text{or } F_2 = 833.33 \times 0.6 \quad (\because \sin \theta = 0.6)$$

$$= 499.998 \text{ N} \approx 500 \text{ N (Tensile). Ans.}$$

Resolving forces horizontally, we get

$$F_3 + 833.33 \cos \theta = 1000$$

$$\text{or } F_3 = 1000 - 833.33 \times 0.8 \quad (\because \cos \theta = 0.8)$$

$$= 333.336 \text{ N (Tensile). Ans.}$$

7.4.1. Method of Joints Applied to Cantilever Trusses. In case of cantilever trusses, it is not necessary to determine the support reactions. The forces in the members of cantilever truss can be obtained by starting the calculations from the free end of the cantilever.

Problem 7.6. Determine the forces in all the members of a cantilever truss shown in Fig. 7.20.

Sol. Here the calculations can be started from end C . Hence consider the equilibrium of the joint C .

Joint C

Let F_{CD} = Force in member CD , and

F_{CA} = Force in member CA .

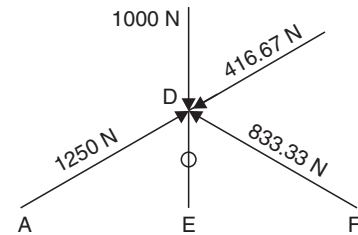


Fig. 7.19 (c)

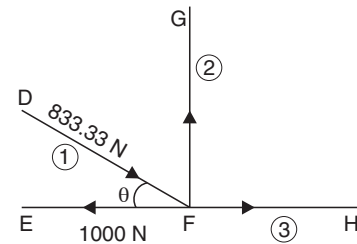


Fig. 7.19 (d)

Their assumed directions are shown in Fig. 7.20.

Resolving the force vertically, we get

$$F_{CD} \times \sin 60^\circ = 1000$$

$$\therefore F_{CD} = \frac{1000}{\sin 60^\circ} = \frac{1000}{0.866} = 1154.7 \text{ N (Tensile)}$$

Resolving the forces horizontally, we get

$$\begin{aligned} F_{CA} &= F_{CD} \times \cos 60^\circ \\ &= 1154.7 \times 0.5 \\ &= 577.35 \text{ N (Compressive)} \end{aligned}$$

Now consider the equilibrium of the joint *D*.

Joint D

[See Fig. 7.20 (a)]

The force $F_{CD} = 1154.7 \text{ N}$ (tensile) is already calculated.

Let F_{AD} = Force in member *AD*, and

F_{BD} = Force in member *BD*.

Their assumed directions are shown in Fig. 7.20 (a).

Resolving the forces vertically, we get

$$F_{AD} \cos 30^\circ = 1154.7 \cos 30^\circ$$

$$\therefore F_{AD} = \frac{1154.7 \cos 30^\circ}{\cos 30^\circ} = 1154.7 \text{ N (Compressive)}$$

Resolving the forces horizontally, we get

$$\begin{aligned} F_{BD} &= F_{AD} \sin 30^\circ + F_{DC} \sin 30^\circ \\ &= 1154.7 \times 0.5 + 1154.7 \times 0.5 \\ &= 1154.7 \text{ N (Tensile)} \end{aligned}$$

Now the forces are shown in a tabular form below :

Member	Force in the member	Nature of force
<i>AC</i>	577.35 N	Compressive
<i>CD</i>	1154.7 N	Tensile
<i>AD</i>	1154.7 N	Compressive
<i>BD</i>	1154.7 N	Tensile

Problem 7.7. Determine the forces in all the members of a cantilever truss shown in Fig. 7.21.

Sol. Start the calculations from joint *C*.

From triangle *ACE*, we have

$$\tan \theta = \frac{AE}{AC} = \frac{3}{4}$$

$$\text{Also } EC = \sqrt{3^2 + 4^2} = 5$$

$$\therefore \cos \theta = \frac{AC}{CE} = \frac{4}{5} = 0.8$$

$$\sin \theta = \frac{AE}{CE} = \frac{3}{5} = 0.6.$$

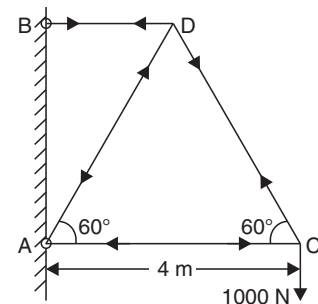


Fig. 7.20

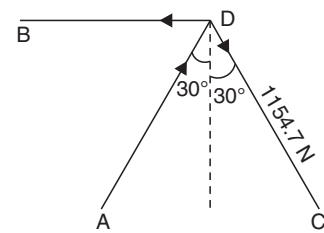


Fig. 7.20 (a)

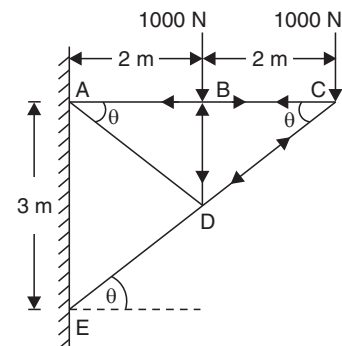


Fig. 7.21

Joint C

The direction of forces at the joint C are shown in Fig. 7.21.

Resolving the forces vertically, we get

$$F_{CD} \sin \theta = 1000$$

$$\therefore F_{CD} = \frac{1000}{\sin \theta} = \frac{1000}{0.6} = 1666.66 \text{ N (Compressive)}$$

Resolving the forces horizontally, we get

$$F_{CB} = F_{CD} \times \cos \theta = 1666.66 \times 0.8 = 1333.33 \text{ N (Tensile)}$$

Now consider the equilibrium of joint B .

Joint B

Resolving vertically, we get

$$F_{BD} = 1000 \text{ N (Compressive)}$$

$$F_{BA} = F_{CB} = 1333.33 \text{ (Tensile)}$$

Now consider the joint D .

Joint D

The forces in member CD and BD have already been calculated. They are 1666.66 N and 1000 N respectively as shown in Fig. 7.21 (a).

Let F_{DA} = Force in member DA , and

F_{DE} = Force in member DE .

Resolving forces vertically, we get

$$1000 + 1666.66 \sin \theta = F_{AD} \sin \theta + F_{ED} \sin \theta$$

$$\text{or } 1000 + 1666.66 \times 0.6 = F_{AD} \times 0.6 + F_{ED} \times 0.6$$

$$\text{or } F_{AD} + F_{ED} = \frac{1000}{0.6} + 1666.66 = 3333.32 \quad \dots(i)$$

Resolving forces horizontally, we get

$$1666.66 \cos \theta + F_{AD} \cos \theta = F_{ED} \cos \theta$$

$$\text{or } 1666.66 + F_{AD} = F_{ED} \quad \text{or } F_{ED} - F_{AD} = 1666.66 \quad \dots(ii)$$

Adding equations (i) and (ii), we get

$$2F_{ED} = 3333.32 + 1666.66 = 4999.98$$

$$\therefore F_{ED} = \frac{4999.98}{2} = 2499.99 \approx 2500 \text{ N (Compressive)}$$

Substituting this value in equation (i), we get

$$F_{AD} + 2500 = 3333.32$$

$$\therefore F_{AD} = 3333.32 - 2500 = 833.32 \text{ N (Tensile)}$$

Now the forces are shown in a tabular form below :

Member	Force in the member	Nature of force
AB	1333.33 N	Tensile
BC	1333.33 N	Tensile
CD	1666.66 N	Compressive
DE	2500 N	Compressive
AD	833.32 N	Tensile
BD	1000 N	Compressive

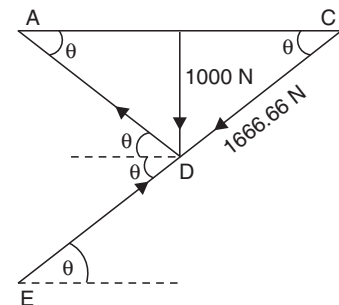


Fig. 7.21 (a)

7.4.2. Method of Joints Applied to Trusses Carrying Horizontal Loads. If a truss carries horizontal loads (with or without vertical loads), hinged at one end and supported on rollers at the other end, then the support reaction at the roller supported end will be normal, whereas the support reactions at the hinged end will consist of: (i) horizontal reaction and (ii) vertical reaction.

The horizontal reaction will be obtained by adding algebraically all the horizontal loads; whereas the vertical reaction will be obtained by subtracting the roller support reaction from the total vertical loads. Now the forces in the members of the truss can be determined.

Problem 7.8. Determine the forces in the truss shown in Fig. 7.22 which carries a horizontal load of 12 kN and a vertical load of 18 kN.

Sol. The truss is supported on rollers at B and hence the reaction at B must be normal to the roller base i.e., the reaction at B, in this case, should be vertical.

At the end A, the truss is hinged and hence the support reactions at the hinged end A will consist of a horizontal reaction H_A and a vertical reaction R_A .

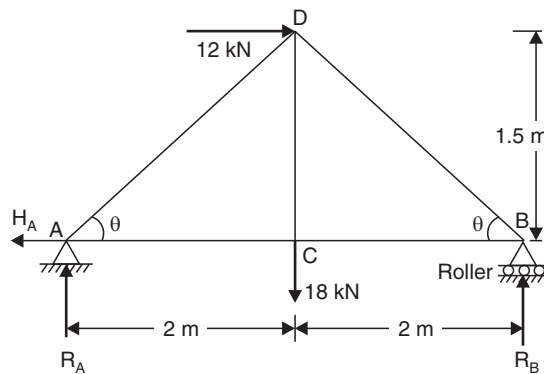


Fig. 7.22

Taking moments of all forces at A, we get

$$R_B \times 4 = 18 \times 2 + 12 \times 1.5 = 36 + 18 = 54$$

$$\therefore R_B = \frac{54}{4} = 13.5 \text{ kN } (\uparrow)$$

$$\therefore R_A = \text{Total vertical load} - R_B = 18 - 13.5 = 4.5 \text{ kN } (\uparrow)$$

and $H_A = \text{Sum of all horizontal loads} = 12 \text{ kN } (\leftarrow)$

Now the forces in the members can be calculated.

In triangle BCD, $BD = \sqrt{BC^2 + CD^2} = \sqrt{2^2 + 1.5^2} = 2.5 \text{ m}$

$$\therefore \cos \theta = \frac{BC}{BD} = \frac{2}{2.5} = 0.8$$

$$\sin \theta = \frac{CD}{BD} = \frac{1.5}{2.5} = 0.6$$

Let us first consider the equilibrium of joint A.

Joint A

The reactions R_A and H_A are known in magnitude and direction. Let the directions of the forces in the members AC and AD are as shown in Fig. 7.22 (a).

Resolving the forces vertically, we get

$$F_{AD} \sin \theta = R_A$$

or
$$F_{AD} = \frac{R_A}{\sin \theta} = \frac{4.5}{0.6} = 7.5 \text{ kN (Compressive)}$$

Resolving the forces horizontally, we get

$$\begin{aligned} F_{AC} &= H_A + F_{AD} \cos \theta \\ &= 12 + 7.5 \times 0.8 = 18 \text{ kN (Tensile)} \end{aligned}$$

Now consider the joint C.

Joint C

At the joint C, the force O in member CA and vertical load 18 kN are known in magnitude and directions. For equilibrium of the joint C.

$$F_{BC} = F_{CA} = 18 \text{ kN (Tensile)}$$

$$F_{CD} = 18 \text{ kN (Tensile)}$$

Now consider the joint B.

Joint B

At the joint B, R_B and force F_{BC} are known in magnitude and direction.

Let F_{BD} is the force in member BD .

Resolving the forces vertically, we get

$$F_{BD} \sin \theta = R_B$$

$$\begin{aligned} \therefore F_{BD} &= \frac{R_B}{\sin \theta} = \frac{13.5}{0.6} \\ &= 22.5 \text{ kN (Compressive)} \end{aligned}$$

Now the forces are shown in a tabular form below :

Member	Force in the member	Nature of force
AC	18 kN	Tensile
AD	7.5 kN	Compressive
CD	18 kN	Tensile
CB	18 kN	Tensile
BD	22.5 kN	Compressive

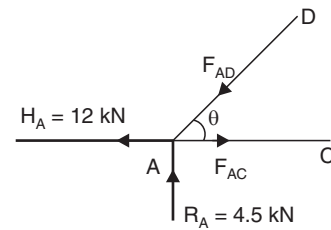


Fig. 7.22 (a)

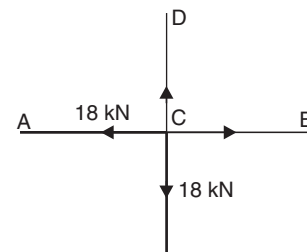


Fig. 7.22 (b)

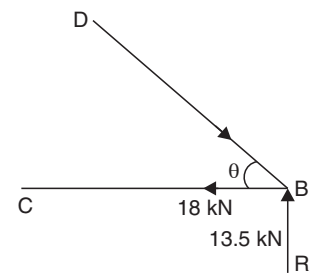


Fig. 7.22 (c)

Problem 7.9. Determine the forces in the truss shown in Fig. 7.23 which is subjected to horizontal and vertical loads. Mention the nature of forces in each case.

Sol. The truss is supported on rollers at B and hence R_B will be vertical. The truss is hinged at A and hence the support reactions at A will consist of a horizontal reaction H_A and a vertical reaction R_A .

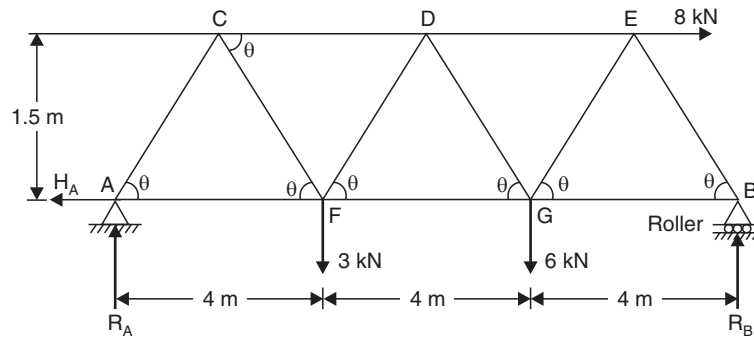


Fig. 7.23

Taking moment about A, we get

$$R_B \times 12 = 8 \times 1.5 + 3 \times 4 + 6 \times 8$$

$$= 72$$

$$\therefore R_B = \frac{72}{12} = 6 \text{ kN } (\uparrow)$$

and

$$R_A = \text{Total vertical loads} - R_B$$

$$= (3 + 6) - 6$$

$$= 3 \text{ kN } (\uparrow)$$

and

$$H_A = \text{Sum of all horizontal loads}$$

$$= 8 \text{ kN } (\leftarrow)$$

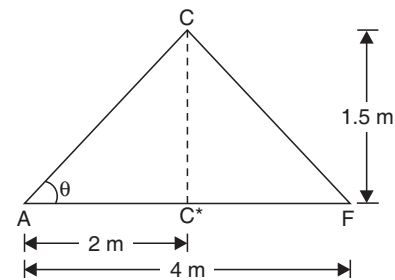


Fig. 7.23 (a)

In the triangle ACC^* , $AC = \sqrt{AC^{*2} + CC^{*2}} = \sqrt{2^2 + 1.5^2} = 2.5$

$$\therefore \cos \theta = \frac{AC^*}{AC} = \frac{2}{2.5} = 0.8 \quad \text{and} \quad \sin \theta = \frac{CC^*}{AC} = \frac{1.5}{2.5} = 0.6$$

Now the forces in the members can be calculated. Consider the joint A.

Joint A

The reactions R_A and H_A are known in magnitude and direction. Let the directions of the forces F_{CA} and F_{FA} are as shown in Fig. 7.23 (b).

Resolving the forces vertically, we get

$$F_{CA} \times \sin \theta = 3 \text{ kN}$$

$$\therefore F_{CA} = \frac{3}{\sin \theta} = \frac{3}{0.6} = 5 \text{ kN} \quad (\text{Compressive})$$

Resolving the forces horizontally, we get

$$F_{FA} = F_{CA} \cos \theta + H_A$$

$$= 5 \times 0.8 + 8 = 12 \text{ kN} \quad (\text{Tensile})$$

Now consider joint C.

Joint C

The force F_{CA} is known in magnitude and direction. The assumed directions of the forces F_{CD} and F_{CF} are shown in Fig. 7.23 (c).

Resolving forces vertically, we get

$$F_{CA} \sin \theta = F_{CF} \sin \theta$$

$$\therefore F_{CA} = F_{CF} = 5 \text{ kN}$$

$$\therefore F_{CF} = 5 \text{ kN (Tensile)}$$

$$(\because F_{CA} = 5 \text{ kN})$$

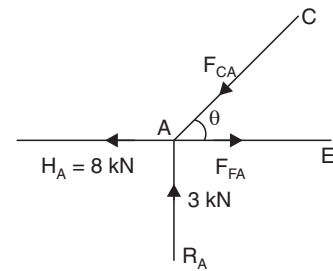


Fig. 7.23 (b)

Resolving forces horizontally, we get

$$\begin{aligned} F_{CD} &= F_{CA} \cos \theta + F_{CF} \cos \theta \\ &= 5 \times 0.8 + 5 \times 0.8 = 8 \text{ kN (Compressive)} \end{aligned}$$

Now consider the joint F .

Joint F

The forces F_{FA} and F_{FC} are known in magnitude and directions. The assumed directions of the forces F_{DF} and F_{GF} are shown in Fig. 7.23 (d).

Resolving the forces vertically, we get

$$\begin{aligned} 5 \times \sin \theta + F_{DF} \sin \theta &= 3 \\ \text{or } F_{DF} &= -\frac{5 \sin \theta + 3}{\sin \theta} \\ &= -5 + \frac{3}{\sin \theta} = -5 + \frac{3}{0.6} = -5 + 5 = 0 \end{aligned}$$

Resolving the forces horizontally, we get

$$\begin{aligned} 12 + 5 \cos \theta &= F_{GF} + F_{DF} \cos \theta \\ \text{or } 12 + 5 \times 0.8 &= F_{GF} + 0 \quad \text{or } 12 + 4 = F_{GF} \\ \therefore F_{GF} &= 12 + 4 = 16 \text{ kN (Tensile)} \end{aligned}$$

Now consider the joint D .

Joint D

The forces F_{DC} and F_{FD} are known in magnitude and direction. The assumed directions of F_{DG} and F_{DE} are shown in Fig. 7.23 (e).

Resolving the forces vertically, we get

$$\begin{aligned} F_{DG} \sin \theta &= F_{DF} \times \sin \theta = 0 \\ \therefore F_{DG} &= 0 \end{aligned}$$

Resolving forces horizontally, we get

$$\begin{aligned} F_{DE} &= F_{CD} = 8 \text{ kN} \\ \therefore F_{DE} &= 8 \text{ kN (Compressive)} \end{aligned}$$

Now consider the joint G .

Joint G

The forces F_{DG} and F_{FG} are known in magnitude and direction. The assumed directions of F_{GE} and F_{GB} are shown in Fig. 7.23 (f).

Resolving the forces vertically, we get

$$\begin{aligned} F_{GE} \sin \theta &= F_{DG} \sin \theta + 6 = 6 \\ \text{or } F_{GE} &= \frac{6}{\sin \theta} = \frac{6}{0.6} = 10 \text{ kN (Tensile)} \end{aligned}$$

Resolving forces horizontally,

$$\begin{aligned} F_{GB} &= 16 - F_{GE} \cos \theta \\ &= 16 - 10 \times 0.8 = 8 \text{ kN (Tensile)} \end{aligned}$$

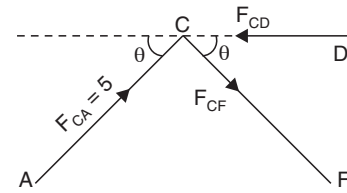


Fig. 7.23 (c)

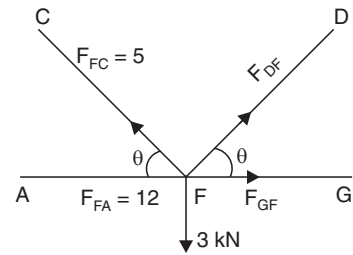


Fig. 7.23 (d)

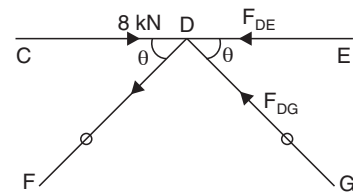


Fig. 7.23 (e)

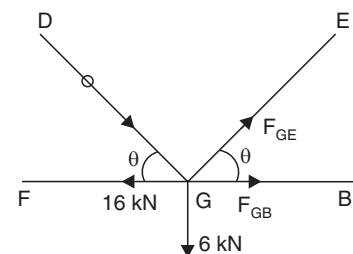


Fig. 7.23 (f)

Now consider the joint E .

Joint E

The forces F_{GE} and F_{DE} are known in magnitude and directions. Let F_{BE} is acting in a direction as shown in Fig. 7.23 (g).

Resolving forces vertically, we get

$$F_{GE} \sin \theta = F_{BE} \sin \theta$$

$$\therefore F_{BE} = F_{GE} = 10$$

$$\therefore F_{BE} = 10 \text{ kN} \quad (\text{Compressive})$$

If we have calculated the forces in member BE and BG , considering joint B , we would have got the same results.

Now the forces in each member are shown in Fig. 7.23 (h).

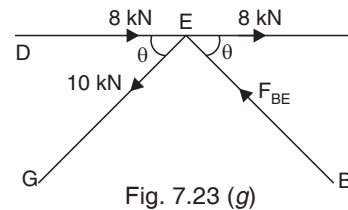


Fig. 7.23 (g)

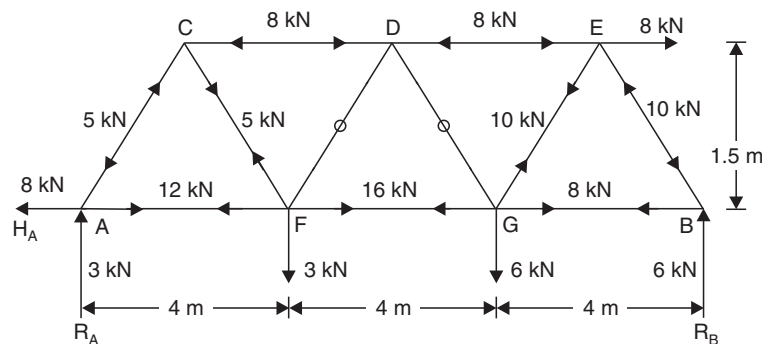


Fig. 7.23 (h)

7.4.3. Method of Joints Applied to Trusses Carrying Inclined Loads. If a truss carries inclined loads, hinged at one end and supported on rollers at the other end, then the support reaction at the roller supported end will be normal, whereas the support reactions at the hinged end will consists of :

- (i) Horizontal reaction and (ii) Vertical reaction.

The inclined loads are resolved into horizontal and vertical components.

The horizontal reaction will be obtained by adding algebraically all the horizontal components of the inclined loads ; whereas the vertical reaction will be obtained by subtracting the roller support reaction from the total vertical components of the inclined loads. Now the forces in the members of truss can be determined.

Problem 7.10. Determine the forces in the truss shown in Fig. 7.24 which is subjected to inclined loads.

Sol. The truss is supported on roller at B and hence R_B will be vertical.

The truss is hinged at A and hence the support reactions at A will consists of a horizontal reaction H_A and a vertical reaction R_A .

Now length $AC = 4 \times \cos 30^\circ = 4 \times 0.866 = 3.464 \text{ m}$

and length $AD = 2 \times AC = 2 \times 3.464 = 6.928 \text{ m}$

Now taking moments about A , we get

$$\begin{aligned} R_B \times 12 &= 2 \times AC + 1 \times AD + 1 \times AE \\ &= 2 \times 3.464 + 1 \times 6.928 + 1 \times 4 = 17.856 \end{aligned}$$

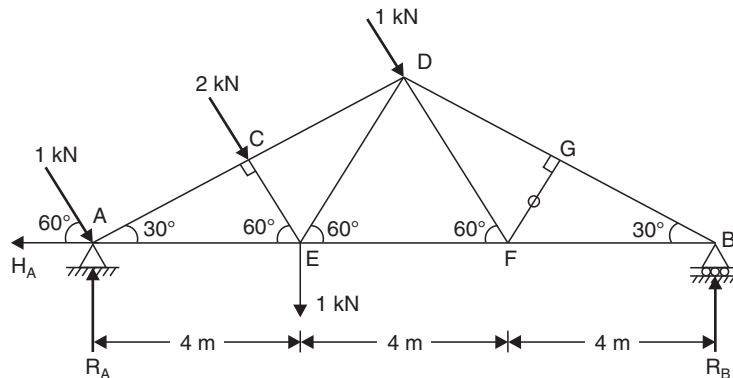


Fig. 7.24

$$\therefore R_B = \frac{17.856}{12} = 1.49 \text{ kN}$$

Total vertical components of inclined loads

$$= (1 + 2 + 1) \times \sin 60^\circ + 1.0$$

$$= 4 \times 0.866 + 1.0 = 4.464 \text{ kN}$$

Total horizontal components of inclined loads

$$= (1 + 2 + 1) \cos 60^\circ = 4 \times 0.5 = 2 \text{ kN}$$

$$\text{Now } R_A = \text{Vertical components of inclined loads} - R_B$$

$$= 4.464 - 1.49 = 2.974 \text{ kN } (\uparrow)$$

and

$$H_A = \text{Sum of all horizontal components} = 2 \text{ kN}$$

Now the forces in the members can be calculated.

Consider the equilibrium of joint A.

Joint A

Let F_{AE} = Force in member AE

and F_{AC} = Force in member AC

Their directions are assumed as shown in Fig. 7.24 (a).

Resolving the forces vertically, we get

$$F_{AC} \times \sin 30^\circ + 1 \times \sin 60^\circ = 2.974$$

$$\text{or } F_{AC} \times 0.5 + 0.866 = 2.974$$

$$\therefore F_{AC} = \frac{2.974 - 0.866}{0.5}$$

$$= 4.216 \text{ kN (Compressive)}$$

Resolving the forces horizontally, we get

$$F_{AE} = 2 + F_{AC} \cos 30^\circ - 1 \times \cos 60^\circ$$

$$= 2 + 4.216 \times 0.866 - 0.5$$

$$= 5.15 \text{ kN (Tensile)}$$

Now consider the joint C.

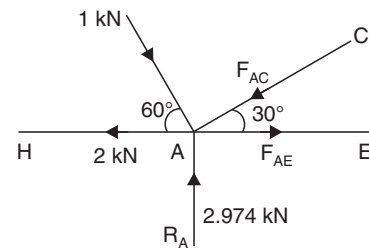


Fig. 7.24 (a)

Joint C

From Fig. 7.24 (b), we have

$$F_{CD} = F_{AC} = 4.216 \quad (\text{Compressive})$$

and $F_{CE} = 2 \text{ kN} \quad (\text{Compressive})$

Now consider joint E.

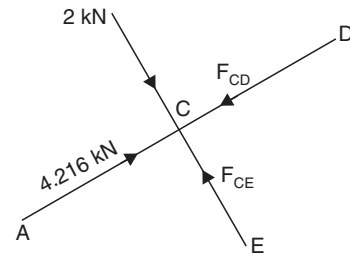


Fig. 7.24 (b)

Joint E [See Fig. 7.24 (c)]

Resolving forces vertically, we get

$$1 + 2 \times \sin 60^\circ = F_{ED} \times \sin 60^\circ$$

or $F_{ED} = 2 + \frac{1}{\sin 60^\circ} = 3.155 \text{ (Tensile)}$

Resolving forces horizontally, we get

$$5.15 - 2 \times \cos 60^\circ - F_{ED} \cos 60^\circ - F_{EF} = 0$$

or $5.15 - 2 \times \frac{1}{2} - 3.15 \times \frac{1}{2} - F_{EF} = 0$

$$F_{EF} = 5.15 - 1 - 1.57 = 2.58 \text{ kN} \quad (\text{Tensile})$$

At the joint G, two forces, *i.e.*, F_{BG} and F_{DG} are in the same straight line and hence the third force, *i.e.*, F_{GF} should be zero.

$$\therefore F_{GF} = 0$$

Now consider the joint F.

Joint F [See Fig. 7.24 (d)]

Resolving forces vertically, we get

$$F_{DF} \times \sin 60^\circ = 0$$

$$\therefore F_{DF} = 0$$

Resolving horizontally, we get

$$F_{FB} = F_{EF} = 2.58 \text{ kN}$$

$$\therefore F_{FB} = 2.58 \text{ kN} \quad (\text{Compressive})$$

Now consider the joint B.

Joint B

Resolving vertically, we get

$$F_{BG} \times \sin 30^\circ = 1.49$$

$$\therefore F_{BG} = \frac{1.49}{0.5} = 2.98 \text{ kN} \quad (\text{Compressive})$$

Joint G

$$F_{GD} = F_{BG} = 2.98 \text{ kN} \quad (\text{Compressive})$$

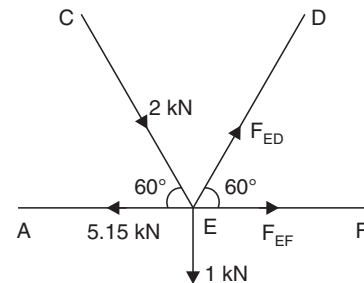


Fig. 7.24 (c)

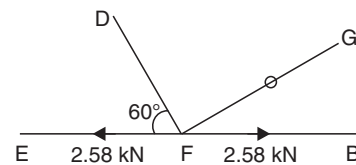


Fig. 7.24 (d)

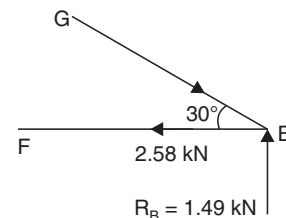


Fig. 7.24 (e)

The forces are shown in a tabular form as

Member	Force in the member	Nature of force
AC	4.216 kN	Compressive
AE	5.15 kN	Tensile
CE	2 kN	Compressive
CD	4.216 kN	Compressive
ED	3.155 kN	Tensile
EF	2.58 kN	Tensile
DF	0	Nil
DG	2.98 kN	Compressive
GB	2.98 kN	Compressive
FB	2.58 kN	Compressive
FG	0	Nil

7.5. METHOD OF SECTIONS

When the forces in a few members of a truss are to be determined, then the method of section is mostly used. This method is very quick as it does not involve the solution of other joints of the truss.

In this method, a section line is passed through the members, in which forces are to be determined as shown in Fig. 7.25. The section line should be drawn in such a way that it does not cut more than three members in which the forces are unknown. The part of the truss, on any one side of the section line, is treated as a free body in equilibrium under the action of external forces on that part and forces in the members cut by the section line. The unknown forces in the members are then determined by using equations of equilibrium as

$$\Sigma F_x = 0, \Sigma F_y = 0 \text{ and } \Sigma M = 0.$$

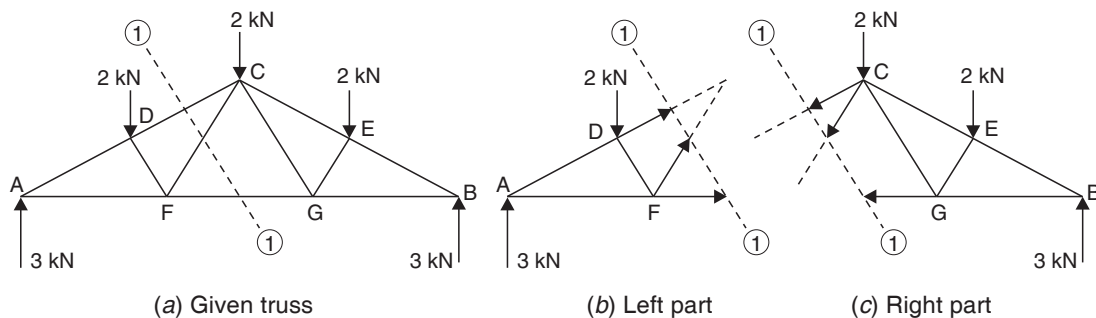


Fig. 7.25

If the magnitude of the forces, in the members cut by a section line, is positive then the assumed direction is correct. If magnitude of a force is negative, then reverse the direction of that force.

Problem 7.11. Find the forces in the members AB and AC of the truss shown in Fig. 7.26 using method of section. (U.P. Tech. University, 2002–2003)

Sol. First determine the reaction R_B and R_C .

The distance of line of action of 20 kN from point B is

$$AB \times \cos 60^\circ \text{ or } 2.5 \times \frac{1}{2} = 1.25 \text{ m}$$

Taking moments about point B , we get

$$R_C \times 5 = 20 \times 1.25$$

$$\therefore R_C = \frac{20 \times 1.25}{5} = 5 \text{ kN}$$

and

$$R_B = 20 - 5 = 15 \text{ kN}$$

Now draw a section line (1-1), cutting the members AB and BC in which forces are to be determined. Now consider the equilibrium of the left part of the truss. This part is shown in Fig. 7.27.

Let the directions of F_{BA} and F_{BC} are assumed as shown in Fig. 7.27.

Now taking the moments of all the forces acting on the left part about point C , we get

$$15 \times 5 + (F_{BA} \times AC)^* = 0$$

($\because$ The perpendicular distance between the line of action of F_{BA} and point C is equal to AC)

$$\text{or } 75 + F_{BA} \times 5 \times \cos 30^\circ = 0$$

$$\text{or } F_{BA} = \frac{-75}{5 \times \cos 30^\circ} = -17.32 \text{ kN}$$

The negative sign shows that F_{BA} is acting in the opposite direction (*i.e.*, towards point B). Hence force F_{BA} will be a compressive force.

$$\therefore F_{BA} = 17.32 \text{ kN (Compressive). Ans.}$$

Again taking the moments of all the forces acting on the left part about point A , we get

$$15 \times \text{Perpendicular distance between the line of action of } 15 \text{ kN and point } C \\ = F_{BC} \times \text{Perpendicular distance between } F_{BC} \text{ and point } A$$

$$15 \times 2.5 \times \cos 60^\circ = F_{BC} \times 2.5 \times \sin 60^\circ$$

$$\therefore F_{BC} = \frac{15 \times 2.5 \times \cos 60^\circ}{2.5 \times \sin 60^\circ} = \frac{15 \times 0.5}{0.866} \\ = 8.66 \text{ kN (Tensile). Ans.}$$

These forces are same as obtained in Problem 7.1.

Problem 7.12. A truss of span 5 m is loaded as shown in Fig. 7.28. Find the reactions and forces in the members marked 4, 5 and 7 using method of section.

Sol. Let us first determine the reactions R_A and R_B .

Triangle ABD is a right-angled triangle having angle

$$ADB = 90^\circ$$

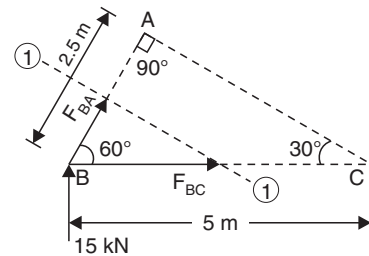


Fig. 7.26

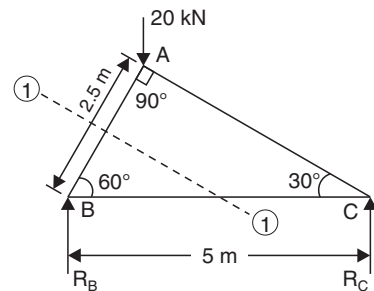


Fig. 7.27

$$(\because AC = BC \times \cos 30^\circ)$$

*The moment of the force F_{BA} about point C , is also taken by resolving the force F_{BA} into vertical and horizontal components at point B . The moment of the horizontal component about C is zero, whereas the moment of vertical component will be $(F_{BA} \times \sin 60^\circ) \times 5 = F_{BA} \times 5 \times \sin 60^\circ$ or $F_{BA} \times 5 \times \cos 30^\circ$.

$$(\because \sin 60^\circ = \cos 30^\circ)$$

$$\therefore AD = AB \cos 60^\circ = 5 \times 0.5 = 2.5 \text{ m}$$

The distance of line of action of the vertical load of 10 kN from point A will be $AD \cos 60^\circ$ or $2.5 \times 0.5 = 1.25 \text{ m}$.

From triangle ACD , we have

$$AC = AD = 2.5 \text{ m}$$

$$\therefore BC = 5 - 2.5 = 2.5 \text{ m}$$

In right-angled triangle CEB , we have

$$BE = BC \cos 30^\circ = 2.5 \times \frac{\sqrt{3}}{2}$$

$\therefore$ The distance of line of action of vertical load 12 kN

from point B will be $BE \cos 30^\circ$ or $BE \times \frac{\sqrt{3}}{2}$

$$= \left(2.5 \times \frac{\sqrt{3}}{2} \right) \times \frac{\sqrt{3}}{2} = 1.875 \text{ m}$$

$\therefore$ The distance of the line of action of the load of 12 kN from point A will be $(5 - 1.875) = 3.125 \text{ m}$

Now taking the moments about A, we get

$$R_B \times 5 = 10 \times 1.25 + 12 \times 3.125 = 50$$

$$\therefore R_B = \frac{50}{5} = 10 \text{ kN and } R_A = (10 + 12) - 10 = 12 \text{ kN}.$$

Now draw a section line (1-1), cutting the members 4, 5 and 7 in which forces are to be determined. Consider the equilibrium of the right part of the truss (because it is smaller than the left part).

This part is shown in Fig. 7.29. Let F_4 , F_5 and F_7 are the forces in members 4, 5 and 7. Let their directions are assumed as shown in Fig. 7.29.

Now taking the moments of all the forces acting on the right part about point E, we get

$$R_B \times BE \cos 30^\circ = F_4 \times (BE \times \sin 30^\circ)$$

$$\text{or } 10 \times \left(2.5 \times \frac{\sqrt{3}}{2} \right) \times \frac{\sqrt{3}}{2} = F_4 \times 2.5 \times \frac{\sqrt{3}}{2} \times 0.5$$

$$\text{or } 10 \times \frac{\sqrt{3}}{2} = F_4 \times 0.5$$

$$\therefore F_4 = 10 \times \frac{\sqrt{3}}{2} \times \frac{1}{0.5} = 17.32 \text{ kN (Tensile)}.$$

Now taking the moments of all the forces about point B acting on the right part, we get

$$12 \times BE \cos 30^\circ + F_5 \times BE = 0$$

$$\text{or } 12 \times \cos 30^\circ + F_5 = 0$$

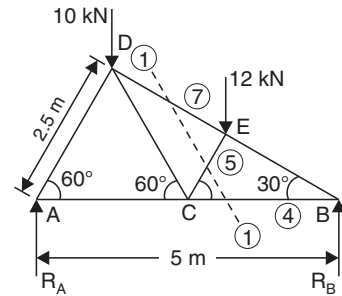


Fig. 7.28

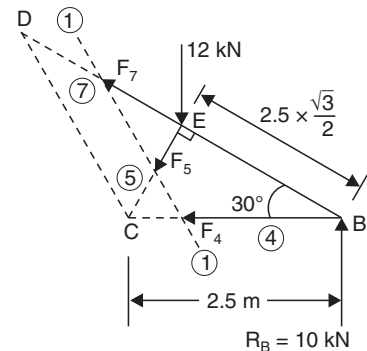


Fig. 7.29

$$\therefore F_5 = -12 \times \cos 30^\circ = -10.392 \text{ kN}$$

–ve sign indicates that F_5 is compressive.

$$\therefore F_5 = \mathbf{10.392 \text{ kN (Compressive)}}. \text{ Ans.}$$

Now taking the moments about point C of all the forces acting on the right parts, we get

$$12 \times (2.5 - BE \cos 30^\circ) = F_7 \times CE + R_B \times BC$$

$$\text{or } 12 \times \left(2.5 - 2.5 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \right) = F_7 \times 2.5 \times \sin 30^\circ + 10 \times 2.5$$

$$\text{or } 12 \times (2.5 - 1.875) = F_7 \times 1.25 + 25 \text{ or } 7.5 = 1.25F_7 + 25$$

$$\text{or } F_7 = \frac{7.5 - 25}{1.25} = -14 \text{ kN}$$

Negative sign shows that F_7 is compressive.

$$\therefore F_7 = \mathbf{14 \text{ kN (Compressive)}}. \text{ Ans.}$$

These forces are same as obtained in Problem 7.3.

Problem 7.13. A truss of span 9 m is loaded as shown in Fig. 7.30. Find the reactions and forces in the members marked 1, 2 and 3.

Sol. Let us first calculate the reactions R_A and R_B .

Taking moments about A , we get

$$R_B \times 9 = 9 \times 3 + 12 \times 6 = 27 + 72 = 99$$

$$\therefore R_B = \frac{99}{9} = 11 \text{ kN}$$

$$\text{and } R_A = (9 + 12) - 11 = 10 \text{ kN}$$

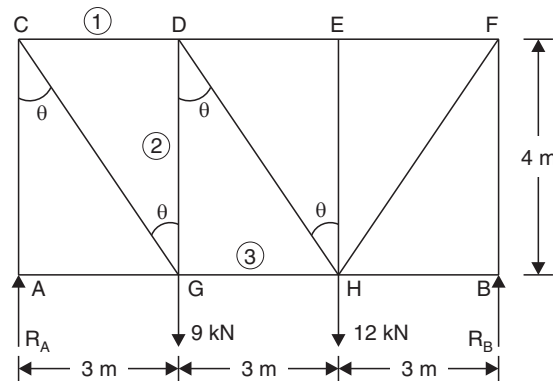


Fig. 7.30

Now draw a section line (1-1), cutting the members 1, 2 and 3 in which forces are to be determined. Consider the equilibrium of the left part of the truss (because it is smaller than the right part). This part is shown in Fig. 7.30 (a). Let F_1 , F_2 and F_3 are the forces of members 1, 2 and 3 respectively. Let their directions are assumed as shown in Fig. 7.30 (a).

Taking moments of all the forces acting on the left part about point D , we get

$$10 \times 3 = F_2 \times 4$$

$$\therefore F_3 = \frac{10 \times 3}{4} = 7.5 \text{ kN (Tensile). Ans.}$$

Now taking the moments of all the forces acting on the left part about point G , we get

$$10 \times 3 + F_1 \times 4 = 0$$

$$\therefore F_1 = \frac{-30}{4} = -7.5 \text{ kN}$$

Negative sign shows that force F_1 is compressive.

$$\therefore F_1 = 7.5 \text{ kN (Compressive). Ans.}$$

Now taking the moments about the point C , we get

$$F_2 \times 3 - 9 \times 3 + F_3 \times 4 = 0$$

$$\text{or } F_2 \times 3 - 27 + 7.5 \times 4 = 0 \quad (\because F_3 = 7.5)$$

$$\text{or } F_2 = \frac{27 - 7.5 \times 4}{3} = \frac{-3}{3} = -1.0 \text{ kN}$$

Negative sign shows that force F_2 is compressive.

$$\therefore F_2 = 1.0 \text{ kN (Compressive). Ans.}$$

Problem 7.14. A truss of 12 m span is loaded as shown in Fig. 7.31. Determine the forces in the members DG , DF and EF , using method of section.

Sol. The truss is supported on rollers at B and hence R_B will be vertical. The truss is hinged at A and hence the support sections at A will consists of a horizontal section H_A and a vertical section R_A .

$$\begin{aligned} \text{In triangle } AEC, \quad AC &= AE \times \cos 30^\circ \\ &= 4 \times 0.866 = 3.464 \text{ m} \end{aligned}$$

$$\text{Now length, } AD = 2 \times AC = 2 \times 3.464 = 6.928 \text{ m}$$

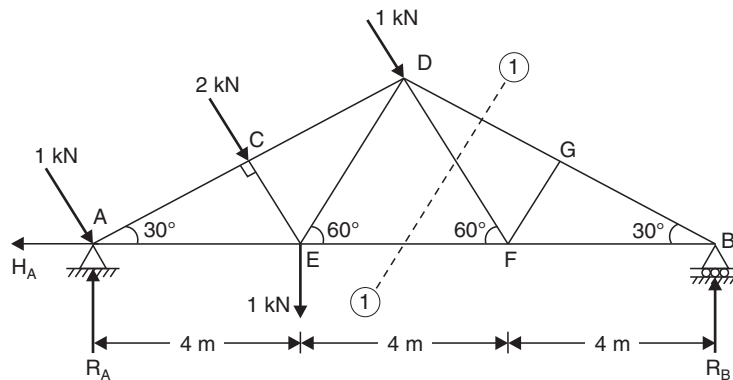


Fig. 7.31

Now taking the moments about A , we get

$$\begin{aligned} R_B \times 12 &= 2 \times AC + 1 \times AD + 1 \times AE \\ &= 2 \times 3.464 + 1 \times 6.928 + 1 \times 4 = 17.856 \end{aligned}$$

$$\therefore R_B = \frac{17.856}{12} = 1.49 \text{ kN}$$

Now draw the section line (1-1), passing through members DG , DF and EF in which the forces are to be determined. Consider the equilibrium of the right part of the truss. This part is shown in Fig. 7.31 (a). Let F_{DG} , F_{FD} and F_{FE} are the forces in members DG , FD and EF respectively. Let their directions are assumed as shown in Fig. 7.31 (a). Taking moments of all forces acting on right part about point F , we get

$$R_B \times 4 + F_{DG} \times FG = 0$$

or $1.49 \times 4 + F_{DG} \times (4 \times \sin 30^\circ) = 0$
 $(\because FG = 4 \times \sin 30^\circ)$

or $F_{DG} = \frac{-1.49 \times 4}{4 \times \sin 30^\circ} = -2.98 \text{ kN}$

–ve sign shows that the force F_{DG} is compressive.

$\therefore F_{DG} = 2.98 \text{ kN}$ (Compressive). **Ans.**

Now taking the moments about point D , we get

$$R_B \times BD \cos 30^\circ = F_{FE} \times BD \times \sin 30^\circ$$

or $R_B \times \cos 30^\circ = F_{FE} \times \sin 30^\circ$

$\therefore F_{FE} = \frac{1.49 \times \cos 30^\circ}{\sin 30^\circ} = \frac{1.49 \times 0.866}{0.5}$
 $= 2.58 \text{ kN}$ (Tensile). **Ans.**

Now taking the moments of all forces acting on the right part about B , we get

$$F_{FD} \times \perp \text{ distance between } F_{FD} \text{ and } B = 0$$

$\therefore F_{FD} = 0$. **Ans.** ($\because \perp$ distance between F_{FD} and B is not zero)

Problem 7.15. Determine the support reactions and nature and magnitude of forces in the members of truss shown in Fig. 7.32.

(U.P. Tech. University, 2001–2002)

Sol. Let us start from joint A where forces in two members are unknown.

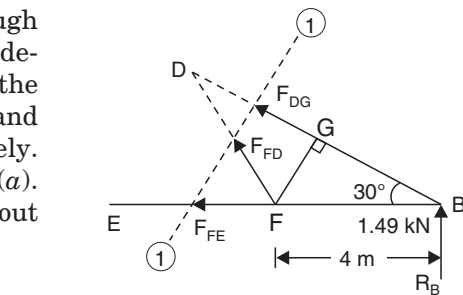


Fig. 7.31 (a)

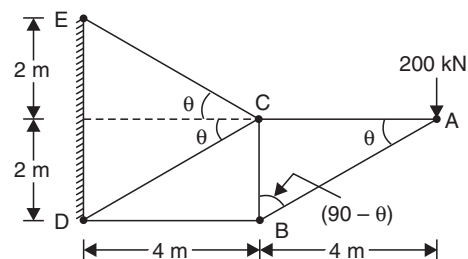


Fig. 7.32

Joint A

In triangle ABC , $\tan \theta = \frac{BC}{CA} = \frac{2}{4}$

$\therefore \cos \theta = \frac{AC}{AB} = \frac{4}{\sqrt{2^2 + 4^2}} = \frac{4}{\sqrt{20}}$ and $\sin \theta = \frac{2}{\sqrt{20}}$

Refer to Fig. 7.32 (a). The forces are shown at joint A . Resolving forces vertically, we get

$$F_{AB} \sin \theta = 200$$

$\therefore F_{AB} = \frac{200}{\sin \theta} = \frac{200}{\frac{2}{\sqrt{20}}} = \frac{200 \times \sqrt{20}}{2} = 447.2 \text{ kN}$. **(Ans.)**

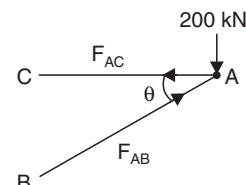


Fig. 7.32 (a)

Resolving forces horizontally, we get

$$\begin{aligned} F_{AC} &= F_{AB} \cos \theta \\ &= (100 \times \sqrt{20}) \times \frac{4}{\sqrt{20}} = 400 \text{ kN (Tensile)}. \end{aligned}$$

Joint B [Refer to Fig. 7.32 (b)]

$$\angle ABC = 90 - \theta$$

Resolving forces vertically,

$$\begin{aligned} F_{BC} &= F_{AB} \cos (90 - \theta) \\ &= F_{AB} \sin \theta \\ &= (100 \times \sqrt{20}) \times \frac{2}{\sqrt{20}} \\ &\left(\because F_{AB} = 100 \times \sqrt{20} \text{ and } \sin \theta = \frac{2}{\sqrt{20}} \right) \end{aligned}$$

$$\therefore F_{BC} = 200 \text{ kN (Tensile)}$$

Resolving forces horizontally, we get

$$\begin{aligned} F_{BD} &= F_{AB} \sin (90 - \theta) = F_{AB} \cos \theta \\ &= (100 \times \sqrt{20}) \times \frac{4}{\sqrt{20}} = 400 \text{ kN (Compressive)} \end{aligned}$$

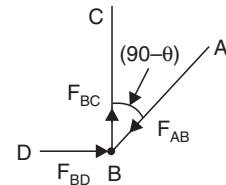


Fig. 7.32 (b)

Joint C

Refer to Fig. 7.32 (c).

Resolving forces horizontally,

$$\begin{aligned} F_{CE} \cos \theta + F_{CD} \cos \theta &= F_{AC} \\ \text{or } F_{CE} \times \frac{4}{\sqrt{20}} + F_{CD} \times \frac{4}{\sqrt{20}} &= 400 \end{aligned}$$

$$\begin{aligned} \therefore F_{CE} + F_{CD} &= 400 \times \frac{\sqrt{20}}{4} \\ &= 100 \times \sqrt{20} \quad \dots(i) \end{aligned}$$

Resolving forces vertically, we get

$$\begin{aligned} F_{CE} \sin \theta - F_{CD} \sin \theta - F_{BC} &= 0 \\ \text{or } (F_{CE} - F_{CD}) \sin \theta &= F_{BC} = 200 \quad (\because F_{BC} = 200 \text{ kN}) \\ \therefore F_{CE} - F_{CD} &= \frac{200}{\sin \theta} = \frac{200}{\left(\frac{2}{\sqrt{20}} \right)} \\ &= 100 \times \sqrt{20} \quad \dots(ii) \end{aligned}$$

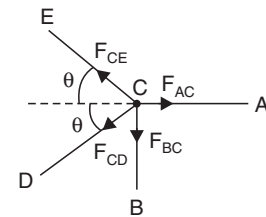


Fig. 7.32 (c)

Adding equations (i) and (ii),

$$\begin{aligned} 2F_{CE} &= 200 \times \sqrt{20} \\ \text{or } F_{CE} &= 100 \times \sqrt{20} \text{ (Tensile)} \end{aligned}$$

Substituting this value in equation (i), we get

$$F_{CD} = 100 \times \sqrt{20} - 100 \times \sqrt{20} = 0$$

Support Reactions

To find the support reactions, consider joint D and E .

Joint D

The force $F_{BD} = 400$ kN

whereas $F_{DC} = 0$. Hence at joint D , there will be only horizontal reaction R_{DH} , which will balance force F_{BD} .

$$\therefore R_{DH} = F_{BD} = 400 \text{ kN.}$$

Joint E

At joint E , the force $F_{EC} = 100 \times \sqrt{20}$ kN. To balance this force, there will be horizontal reaction and vertical reaction at E .

Let R_{EV} = Vertical component of reaction at E

R_{EH} = Horizontal component of reaction at E

Resolving forces horizontally, we get

$$R_{EH} = F_{EC} \cos \theta = (100 \times \sqrt{20}) \times \frac{4}{\sqrt{20}} = 400 \text{ kN. Ans.}$$

Resolving forces vertically, we get

$$R_{EV} = F_{EC} \sin \theta = (100 \times \sqrt{20}) \times \frac{2}{\sqrt{20}} = 200 \text{ kN. Ans.}$$

Now the nature and magnitude of forces in the members are :

$AB \rightarrow 447.2$ kN (Compressive)

$BC \rightarrow 200$ kN (Tensile)

$AC \rightarrow 400$ kN (Tensile)

$BD \rightarrow 400$ kN (Compressive)

$CD \rightarrow 0$

$CE \rightarrow 447.2$ kN (Tensile)

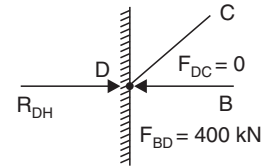


Fig. 7.32 (d)

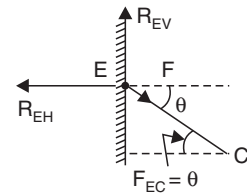


Fig. 7.32 (e)

7.6. ZERO FORCE MEMBERS

The members of a truss in which force is zero, are known as zero force members.

These members are obtained directly as given below :

“If three forces act at a joint and two of them are along the same straight line, then for the equilibrium of the joint, the third force should be equal to zero.”

Refer to Fig. 7.32 (f).

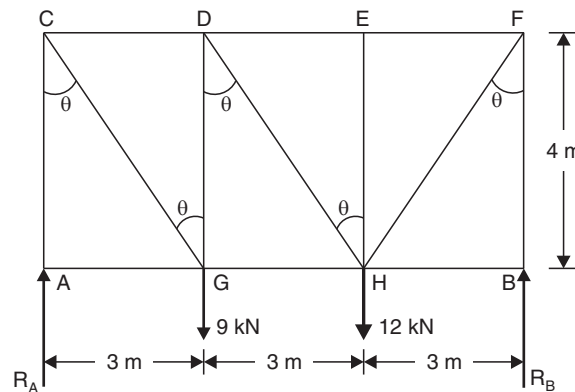


Fig. 7.32 (f)

1. Three forces are acting at the point A (i.e., R_A , F_{AC} and F_{AG}), two of which (i.e., R_A , R_{AC}) are along the same straight line. Hence the third force (i.e., R_{AG}) is zero.

2. Similarly, three forces are acting at the joint B (i.e., R_B , F_{BF} and F_{BH}), two of which (i.e., R_B and F_{BH}) are along the same straight line. Hence the third force F_{BF} should be zero.

3. At the joint E also, three forces (i.e., F_{ED} , F_{EF} and F_{EH}) are acting, two of which (i.e., F_{ED} and F_{EF}) are along the same straight line. Hence the third force F_{EH} must be zero.

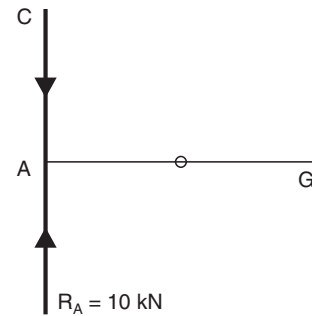


Fig. 7.32 (g)

HIGHLIGHTS

1. The relation between number of joints (j) and number of members (n) in a perfect frame is given by $n = 2j - 3$.
2. Deficient frame is a frame in which number of members are less than $(2j - 3)$ whereas a redundant frame is a frame in which number of members are more than $(2j - 3)$.
3. The reaction on a roller support is at right angles to the roller base.
4. The forces in the members of a frame are determined by :
 - (i) Method of joints
 - (ii) Method of sections, and
 - (iii) Graphical method.
5. The force in a member will be compressive if the member pushes the joint to which it is connected whereas the force in the member will be tensile if the member pulls the joint to which it is connected.
6. While determining forces in a member by method of joints, the joint should be selected in such a way that at any time there are only two members, in which the forces are unknown.
7. If three forces act at a joint and two of them are along the same straight line then third force would be zero.
8. If a truss (or frame) carries horizontal loads, then the support reaction at the hinged end will consists of (i) horizontal reaction and (ii) vertical reaction.
9. If a truss carries inclined loads, then the support reaction at the hinged end will consists of : (i) horizontal reaction and (ii) vertical reaction. They will be given as :
 Horizontal reaction = Horizontal components of inclined loads
 Vertical reaction = Total vertical components of inclined loads – Roller support reaction.
10. Method of section is mostly used, when the forces in a few members of a truss are to be determined.
11. The following steps are necessary for obtaining a graphical solution of a frame :
 - (i) Making a space diagram,
 - (ii) Constructing a vector diagram, and
 - (iii) Preparing a force table.
12. The various members of a frame are named according to Bow's notation.

EXERCISE 7

(A) Theoretical Questions

1. Define and explain the terms : Perfect frame, imperfect frame, deficient frame and a redundant frame. (U.P. Tech. University, 2002–2003)
2. (a) What is a frame ? State the difference between a perfect frame and an imperfect frame.
 (b) What are the assumptions made in finding out the forces in a frame ?

3. What are the different methods of analysing (or finding out the forces) a perfect frame ? Which one is used where and why ?
4. How will you find the forces in the members of a truss by method of joints when
 - (i) the truss is supported on rollers at one end and hinged at other end and carries vertical loads.
 - (ii) the truss is acting as a cantilever and carries vertical loads.
 - (iii) the truss is supported on rollers at one end and hinged at other end and carries horizontal and vertical loads.
 - (iv) the truss is supported on rollers at one end and hinged at other end and carries inclined loads ?
5. (a) What is the advantage of method of section over method of joints ? How will you use method of section in finding forces in the members of a truss ?
 (b) Explain with simple sketches the terms (i) method of sections and (ii) method of joints, as applied to trusses.
6. How will you find the forces in the members of a joint by graphical method ? What are the advantages or disadvantages of graphical method over method of joints and method of section ?
7. What is the procedure of drawing a vector diagram for a frame ? How will you find out (i) magnitude of a force, and (ii) nature of a force from the vector diagram ?
8. How will you find the reactions of a cantilever by graphical method ?
9. What are the assumptions made in the analysis of a simple truss ?
10. Explain what you understand by perfect frame, deficient frame and redundant frame ?

(B) Numerical Problems

1. Find the forces in the members AB , AC and BC of the truss shown in Fig. 7.33.

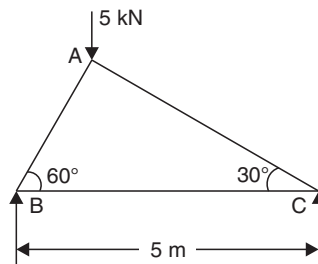


Fig. 7.33

[Ans. $AB = 4.33 \text{ kN (Comp.)}$
 $AC = 2.5 \text{ kN (Comp.)}$
 $BC = 2.165 \text{ kN (Tens.)}$]

2. A truss of span 7.5 m carries a point load of 500 N at joint D as shown in Fig. 7.34. Find the reactions and forces in the members of the truss.

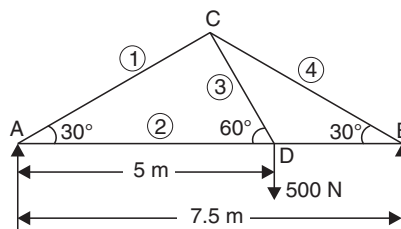
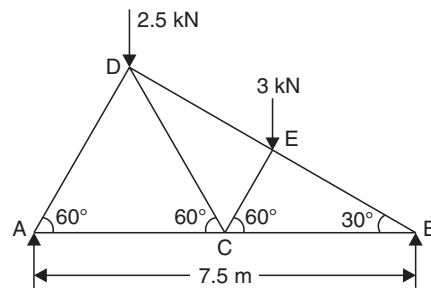


Fig. 7.34

[Ans. $R_A = 166.5 \text{ N}$
 $R_B = 333.5 \text{ N}$
 $F_1 = 333 \text{ N (Comp.)}$
 $F_2 = 288.5 \text{ N (Tens.)}$
 $F_3 = 577.5 \text{ N (Tens.)}$
 $F_4 = 667 \text{ N (Comp.)}$
 $F_5 = 577.5 \text{ N (Tens.)}$]

3. A truss of span 7.5 m is loaded as shown in Fig. 7.35. Find the reactions and forces in the members of the truss.



[Ans. $AD = 3.464$ kN (Comp.)
 $AC = 1.732$ kN (Tens.)
 $CD = 2.598$ kN (Tens.)
 $CE = 2.598$ kN (Comp.)
 $DE = 3.50$ kN (Comp.)
 $BE = 5$ kN (Comp.)
 $BC = 4.33$ kN (Tens.)]

Fig. 7.35

- 3A. A truss is shown in Fig. 7.36. Find the forces in all the members of the truss and indicate it is in tension or compression. (U.P. Tech. University 2000–2001)

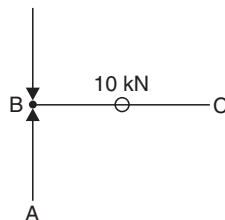
[Hint. In the problem, length of members are not given. Assume $AD = DE = 3$ m and $\angle DAC = \angle DEC = 60^\circ$ as from figure it appears that $AD = DE$ and $\angle DAC = \angle DEC$

$$M_A = 0, 10 \times 3 + 15 \times 3 + 20 \times 6 - 6 \times R_E = 0,$$

$$\text{or } R_E = \frac{30 + 45 + 120}{6} = 32.5 \text{ kN}$$

and $R_A = 10 + 15 + 20 + 10 - R_E = 55 - 32.5 = 22.5$
 Start from joint B where forces in two members are unknown

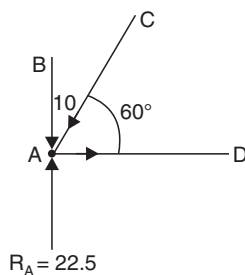
Joint B



$$F_{BA} = 10 \text{ kN (Comp.)}$$

$$F_{BC} = 0$$

Joint A



$$\Sigma V = 0$$

$$22.5 - 10 - F_{AC} \sin 60^\circ = 0$$

$$\therefore F_{AC} = \frac{12.5}{\sin 60^\circ} = 14.43 \text{ kN}$$

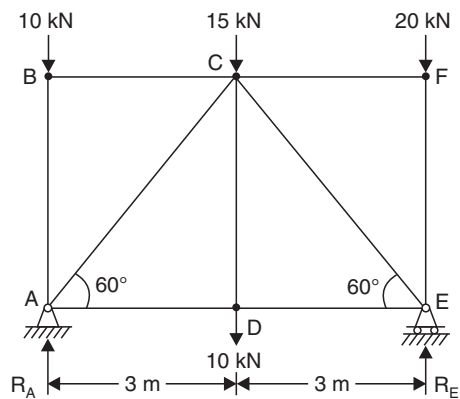
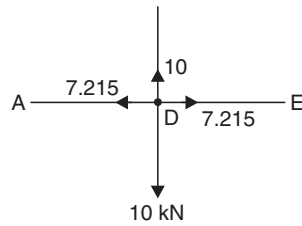
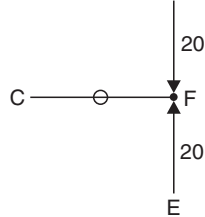


Fig. 7.36

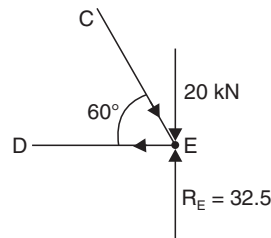
Joint D

$$\Sigma H = 0, F_{AD} = F_{AC} \cos 60^\circ = 7.215$$

Joint F

$$F_{FE} = 20 \text{ kN (Comp.)}$$

$$F_{FC} = 0$$

Joint E

$$\Sigma V = 0, 32.5 - 20 - F_{CE} \sin 60^\circ = 0$$

$$\frac{12.5}{\sin 60^\circ} = F_{CE} = 14.43 \text{ kN}$$

$$\Sigma H = 0, F_{ED} = F_{CE} \cos 60^\circ = 7.215 \text{ kN}$$

4. Determine the forces in the various members of the truss shown in Fig. 7.37.

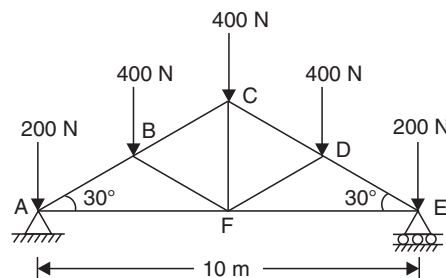


Fig. 7.37

$$[\text{Ans. } AB = 1200 \text{ N (Comp.)}]$$

$$BC = 800 \text{ N (Comp.)}$$

$$CD = 800 \text{ N (Comp.)}$$

$$DE = 1200 \text{ N (Comp.)}$$

$$EF = 600 \text{ N (Tens.)}$$

$$AF = 600 \text{ N (Tens.)}$$

$$BF = DF = 400 \text{ N (Comp.)}$$

$$FC = 400 \text{ N (Tens.)}]$$

5. A plan truss is loaded and supported as shown in Fig. 7.38. Determine the nature and magnitude of forces in the members 1, 2 and 3.

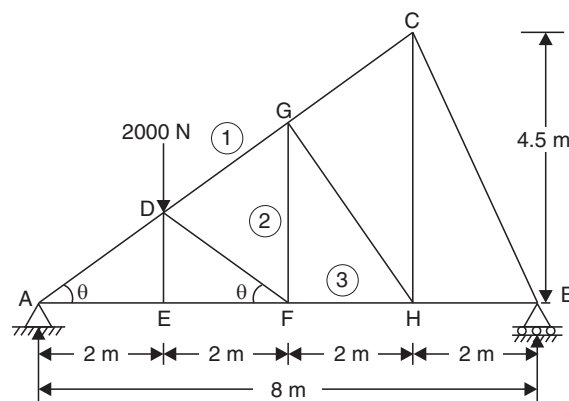


Fig. 7.38

$$[\text{Ans. } F_1 = 833.34 \text{ N (Comp.)}]$$

$$F_2 = 1000 \text{ N (Tens.)}$$

$$F_3 = 666.66 \text{ (Tens.)}]$$

6. Determine the forces in all the members of a cantilever truss shown in Fig. 7.39.

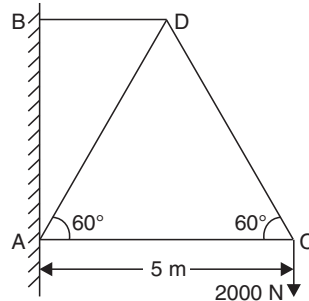


Fig. 7.39

[Ans. $AC = 1154.7 \text{ N (Comp.)}$
 $CD = 2309.4 \text{ N (Tens.)}$
 $AD = 2309.4 \text{ N (Comp.)}$
 $BD = 2309.4 \text{ N (Tens.)}$]

7. A cantilever truss is loaded as shown in Fig. 7.40. Find the force in member AB.

[Ans. $AB = 15 \text{ kN (Tens.)}$]

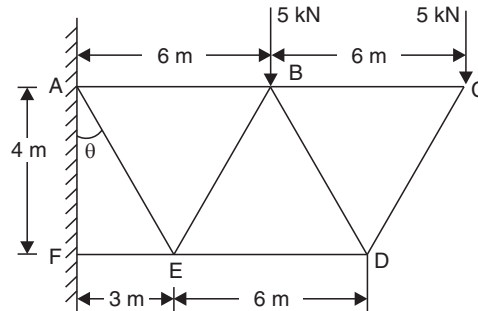


Fig. 7.40

- 7A. Find the axial forces in all the members of the truss shown in Fig. 7.41.

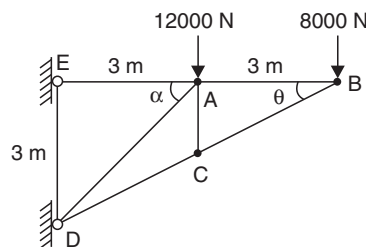


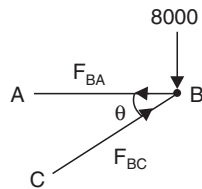
Fig. 7.41

[Hint. Start from joint B

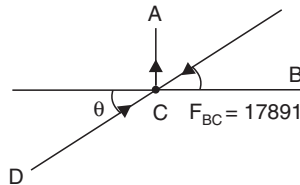
First find angles θ and α

$$\tan \theta = \frac{ED}{EB} = \frac{3}{6} = \frac{1}{2} \quad \therefore \quad \theta = \tan^{-1} 0.5 = 26.56^\circ$$

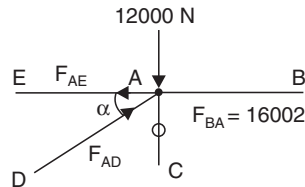
$$\tan \alpha = \frac{EA}{ED} = \frac{3}{3} = 1 \quad \therefore \quad \alpha = \tan^{-1} 1.0 = 45^\circ$$

Joint B

$$\begin{aligned}\Sigma F_y = 0, F_{BC} \sin \theta &= 8000 \\ \therefore F_{BC} &= \frac{8000}{\sin \theta} = \frac{8000}{\sin 26.56^\circ} = 17891 \text{ N (Comp.)} \\ \Sigma F_x = 0 &= F_{BC} \cos \theta = 17891 \times \cos 26.56^\circ = 16002 \text{ N (Tensile)}\end{aligned}$$

Joint C

$$\begin{aligned}\Sigma F_x = 0, F_{BC} \cos \theta &= F_{CD} \cos \theta \\ \therefore F_{CD} &= F_{BC} = 17891 \text{ N (Comp.)} \\ \Sigma F_y = 0, F_{CA} - F_{BC} \sin \theta + F_{CD} \sin \theta &= 0 \\ \therefore F_{CA} &= 0 \quad (\because F_{BC} = F_{CD})\end{aligned}$$

Joint A

$$\begin{aligned}\Sigma F_y = 0, F_{AD} \cos \alpha &= 12000 \\ \therefore F_{AD} &= \frac{12000}{\sin 45^\circ} = 16970 \\ \Sigma F_x = 0, F_{AD} \cos \alpha - F_{AE} + F_{BA} &= 0 \\ \text{or } F_{AD} \cos 45^\circ - F_{AE} + 16002 &= 0 \\ 16970 \cos 45^\circ - F_{AE} + 16002 &= 0 \\ \therefore F_{AE} &= 16002 + 16970 \cos 45^\circ \\ &= 16002 + 11999 = 28001 \text{ N (Tens.)}\end{aligned}$$

8. Determine the forces in the truss shown in Fig. 7.42 which carries a horizontal load of 16 kN and a vertical load of 24 kN.

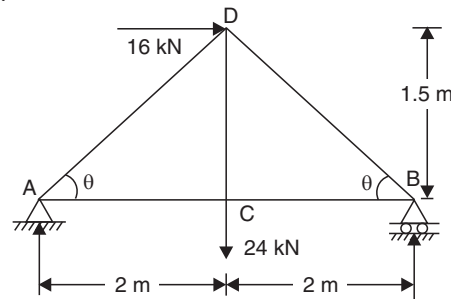


Fig. 7.42

[Ans. AC = 24 kN (Tens.)
AD = 10 kN (Comp.)
CD = 24 kN (Tens.)
CB = 24 kN (Tens.)
BD = 30 kN (Comp.)]

9. Find the forces in the member AB and AC of the truss shown in Fig. 7.33 of numerical problem 1, using method of sections.
[Ans. AB = 4.33 kN (Comp.)
AC = 2.5 kN (Comp.)]
10. Find the forces in the members marked 1, 3, 5 of truss shown in Fig. 7.34 of numerical problem 2, using method of sections.
[Ans. $F_1 = 333 \text{ N (Comp.)}$
 $F_3 = 577.5 \text{ N (Tens.)}$
 $F_5 = 577.5 \text{ N (Tens.)}$]
11. Find the forces in the members DE, CE and CB of the truss, shown in Fig. 7.35 of numerical problem 3, using method of sections.
[Ans. DE = 3.5 kN (Comp.)
CE = 2.598 kN (Comp.)
BC = 4.33 kN (Tens.)]

12. Using method of section, determine the forces in the members CD , FD and FE of the truss shown in Fig. 7.37 of numerical problem 4.
 [Ans. $CD = 800$ N (Comp.)
 $FD = 400$ N (Comp.)
 $FE = 600$ N (Tens.)]
13. Using method of section, determine the forces in the members CD , ED and EF of the truss shown in Fig. 7.43.
 [Ans. $CD = 4.216$ kN (Comp.)
 $ED = 3.155$ kN (Tens.)
 $EF = 2.58$ kN (Tens.)]

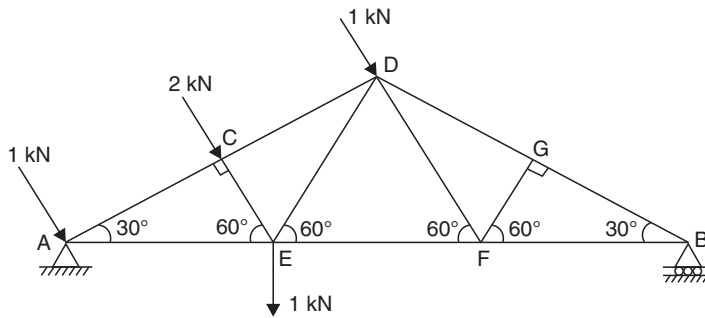


Fig. 7.43

14. A cantilever truss is loaded and supported as shown in Fig. 7.44. Find the value of load P which would produce an axial force of magnitude 3 kN in the member AC using method of section.
 (U.P. Tech. University, 2002–2003)

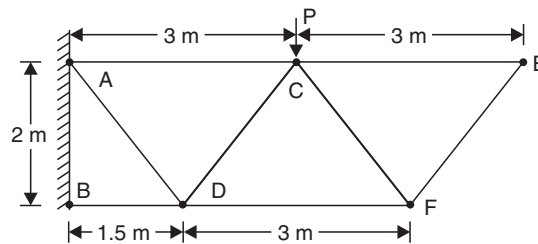


Fig. 7.44

[Hint. Force in member AC , $F_{AC} = 3$ kN]

Now pass a section ①-① as shown in Fig. 7.44 (a).

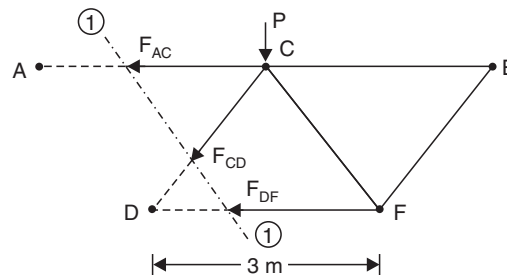


Fig. 7.44 (a)

Take moments about point D .

$$\sum M_D = 0 ; F_{AC} \times 2 - P \times 1.5 = 0$$

$$\text{But } F_{AC} = 3 \text{ kN}$$

$$\therefore 3 \times 2 - P \times 1.5 = 0$$

$$\text{or } 6 = 1.5P \text{ or } P = 4 \text{ kN. Ans.]}$$

Beams (Shear Force and Bending Moment)

8.1. TYPES OF BEAMS

The following are the important types of beams :

1. Cantilever beam,	2. Simply supported beam,
3. Overhanging beam,	4. Fixed beams, and
5. Continuous beam.	

8.1.1. Cantilever Beam. A beam which is fixed at one end and free at the other end, is known as cantilever beam. Such beam is shown in Fig. 8.1. At the fixed end, there will be fixing moment. Also at the fixed end, there can be horizontal and vertical reactions, depending upon the type of load acting on the beam.

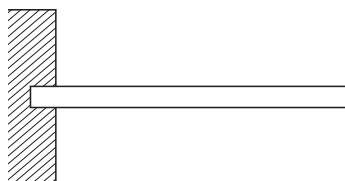


Fig. 8.1

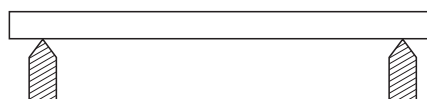


Fig. 8.2

8.1.2. Simply Supported Beam. A beam supported or resting freely on the supports at its both ends, is known as simply supported beam. Such beam is shown in Fig. 8.2.

8.1.3. Overhanging Beam. If the end portion of a beam is extended beyond the support, such beam is known as overhanging beam. Overhanging beam is shown in Fig. 8.3.

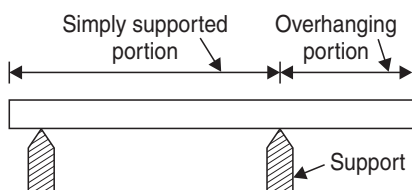


Fig. 8.3

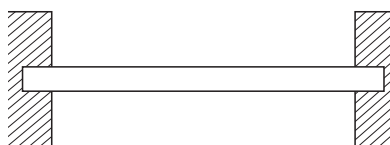


Fig. 8.4

8.1.4. Fixed Beams. A beam whose both ends are fixed or built-in walls, is known as fixed beam. Such beam is shown in Fig. 8.4. A fixed beam is also known as a *built-in* or *encastred* beam. At the fixed ends, there will be fixing moments and reactions.

8.1.5. Continuous Beam. A beam which is provided more than two supports as shown in Fig. 8.5, is known as continuous beam.

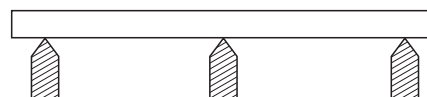


Fig. 8.5

8.2. TYPES OF LOAD

A beam is normally horizontal and the loads acting on the beams are generally vertical. The following are the important types of load acting on a beam :

1. Concentrated or point load,
2. Uniformly distributed load, and
3. Uniformly varying load.

8.2.1. Concentrated or Point Load. A concentrated load is one which is considered to act at a point, although in practice it must really be distributed over a small area. In Fig. 8.6, W shows the point load.

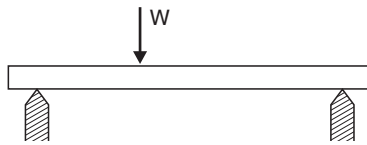


Fig. 8.6

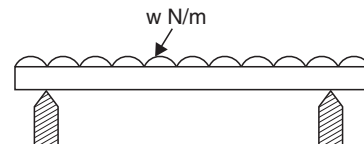


Fig. 8.6 (a)

8.2.2. Uniformly Distributed Load. A uniformly distributed load is one which is spread over a beam in such a manner that rate of loading w is uniform along the length (*i.e.*, each unit length is loaded to the same rate) as shown in Fig. 8.6 (a). The rate of loading is expressed as w N/m run. Uniformly distributed load is, represented by u.d.l.

For solving the numerical problems, the total uniformly distributed load is converted into a point load, acting at the centre of uniformly distributed load.

8.2.3. Uniformly Varying Load. A uniformly varying load is one which is spread over a beam in such a manner that rate of loading varies from point to point along the beam as shown in Fig. 8.6 (b) in which load is zero at one end and increases uniformly to the other end. Such load is known as triangular load.

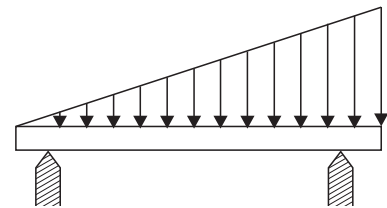


Fig. 8.6 (b)

For solving numerical problems the total load is equal to the area of the triangle and this total load is assumed to be acting at the C.G. of the triangle *i.e.*, at a distance of $\frac{2}{3}$ rd of total length of beam from left end.

8.3. STATICALLY DETERMINATE BEAMS

A beam is statically determinate if the equations of static equilibrium alone are sufficient to determine the reactions of the beam.

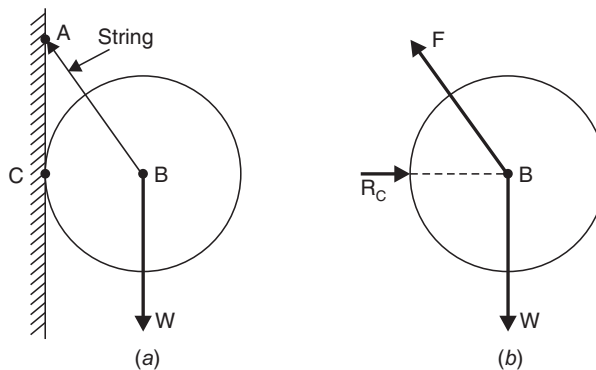


Fig. 8.7

Similarly, statically determinate problems are those, which can be solved by equations of static equilibrium alone. In Fig. 8.7 (a), a ball of weight W is supported by a string AB and is resting against a smooth vertical wall. The free body diagram is shown in Fig. 8.7 (b). There are two unknown R_C and F . They can be obtained from static equilibrium equations alone *i.e.*, $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Hence this is a statically determinate problem.

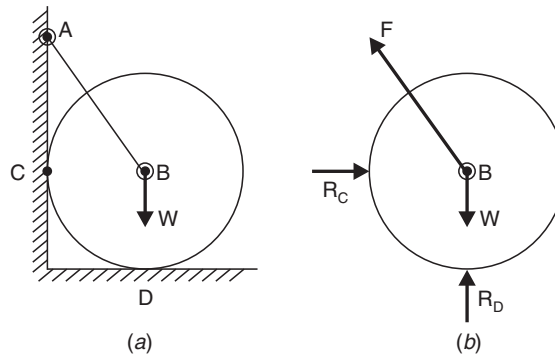


Fig. 8.8

In Fig. 8.8 (a) ball of weight W is supported by a string AB and is resting against a smooth vertical wall and also on a smooth horizontal surface. The free body diagram is shown in Fig. 8.8 (b).

There are three unknowns R_C , F and R_D . They are concurrent forces and hence equilibrium equations are $\Sigma F_x = 0$, and $\Sigma F_y = 0$. [Note that $\Sigma M = 0$ automatically for concurrent forces]. Hence from two equilibrium conditions, three unknown cannot be determined. Hence this is statically indeterminate problem.

8.4. SHEAR FORCE AND BENDING MOMENT IN BEAMS

The algebraic sum of the vertical forces at any section of a beam to the right or left of the section is known as shear force. It is briefly written as S.F. The algebraic sum of the moments of all the forces acting to the right or left of the section is known as bending moment. It is written as B.M. In this chapter, the shear force and bending moment diagrams for different types of beams (*i.e.*, cantilevers, simply supported, fixed, overhanging etc.) for different types of loads (*i.e.*, point load, uniformly distributed loads, varying loads etc.) acting on the beams, will be considered.

8.5. SHEAR FORCE AND BENDING MOMENT DIAGRAMS

A shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which shows the variation of the bending moment along the length of the beam.

8.5.1. Sign Conventions for Shear Force and Bending Moment

(i) *Shear force.* Fig. 8.9 shows a simply supported beam AB , carrying a load of 1000 N at its middle point. The reactions at the supports will be equal to 500 N. Hence $R_A = R_B = 500$ N.

Now imagine the beam to be divided into two portions by the section $X-X$. The resultant of the load and reaction to the left of $X-X$ is 500 N vertically upwards. (Note in this case, there

is no load to the left of $X-X$). And the resultant of the load and reaction to the right of $X-X$ is $(1000 \downarrow - 500 \uparrow = 500 \downarrow \text{ N})$ 500 N downwards. The resultant force acting on any one of the parts normal to the axis of the beam is called the *shear force* at the section $X-X$. Here the shear force at the section $X-X$ is 500 N.

The shear force at a section will be considered positive when the resultant of the forces to the left of the section is upwards, or to the right of the section is downwards. Similarly the shear force at a section will be considered negative if the resultant of the forces to the left of the section is downwards, or to the right of the section is upwards. Here the resultant force to the left of the section is upwards and hence the shear force will be positive.

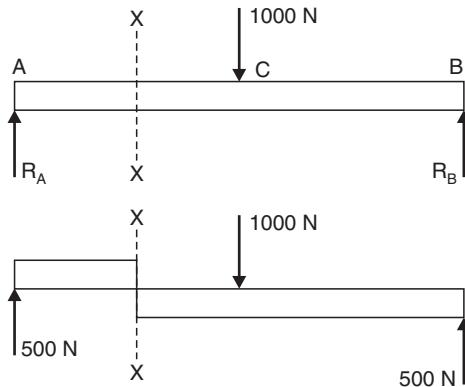


Fig. 8.9

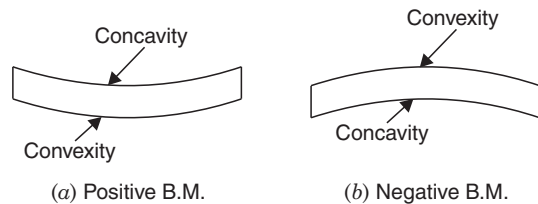


Fig. 8.10

(ii) *Bending moment.* The bending moment at a section is considered positive if the bending moment at that section is such that it tends to bend the beam to a curvature having concavity at the top as shown in Fig. 8.10 (a). Similarly the bending moment (B.M.) at a section is considered negative if the bending moment at that section is such that it tends to bend the beam to a curvature having convexity at the top as shown in Fig. 8.10 (b). The positive B.M. is often called sagging moment and negative B.M. as hogging moment.

Consider the simply supported beam AB , carrying a load of 1000 N at its middle point. Reactions R_A and R_B are equal and are having magnitude 500 N as shown in Fig. 8.11. Imagine the beam to be divided into two portions by the section $X-X$. Let the section $X-X$ is at a distance of 1 m from A .

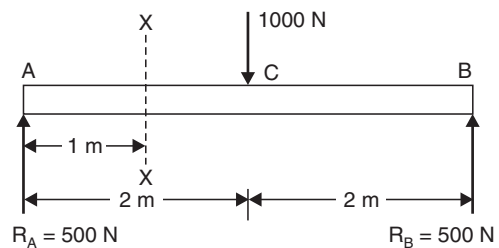


Fig. 8.11

The moments of all the forces (*i.e.*, load and reaction) to the left of $X-X$ at the section $X-X$ is $R_A \times 1 = 500 \times 1 = 500 \text{ Nm}$ (clockwise). Also the moments of all the forces (*i.e.*, load and reaction) to the right of $X-X$ at the section $X-X$ is $R_B \times 3$ (anti-clockwise) $- 1000 \times 1$ (clockwise) $= 500 \times 3 \text{ Nm} - 1000 \times 1 \text{ Nm} = 1500 - 1000 = 500 \text{ Nm}$ (anti-clockwise).

Hence the tendency of the bending moment at $X-X$ is to bend the beam so as to produce concavity at the top as shown in Fig. 8.12.

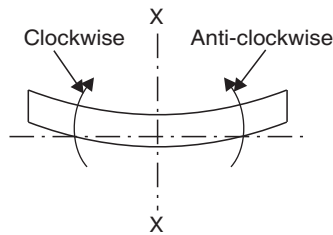


Fig. 8.12

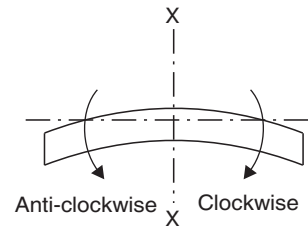


Fig. 8.13

The bending moment at a section is the algebraic sum of the moments of forces and reactions acting on one side of the section. Hence bending moment at the section $X-X$ is 500 Nm.

The bending moment will be considered positive when the moment of the forces and reaction on the left portion is clockwise, and on the right portion anti-clockwise. In Fig. 8.12, the bending moment at the section $X-X$ is positive.

Similarly the bending moment will be considered negative when the moment of the forces and reactions on the left portion is anti-clockwise, and on the right portion clockwise as shown in Fig. 8.13. In Fig. 8.13, the bending moment at the section $X-X$ is negative.

8.5.2. Important Points for Drawing Shear Force and Bending Moment Diagrams. In Art. 8.5, it is mentioned that the shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which shows the variation of the bending moment along the length of beam. In these diagrams, the shear force or bending moment are represented by ordinates whereas the length of the beam represents abscissa.

The following are the important points for drawing shear force and bending moment diagrams :

1. Consider the left or the right portion of the section.
2. Add the forces (including reaction) normal to the beam on one of the portion. If right portion of the section is chosen, a force on the right portion acting downwards is positive while a force acting upwards is negative.

If the left portion of the section is chosen, a force on the left portion acting upwards is positive while a force acting downwards is negative.

3. The positive values of shear force and bending moments are plotted above the base line, and negative values below the base line.

4. The shear force diagram will increase or decrease suddenly *i.e.*, by a vertical straight line at a section where there is a vertical point load.

5. The shear force between any two vertical loads will be constant and hence the shear force diagram between two vertical loads will be horizontal.

6. The bending moment at the two supports of a simply supported beam and at the free end of a cantilever will be zero. But at the fixed end of the cantilever, there will be bending moment (or fixing moments).

8.6. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER

A cantilever is a beam whose one end is fixed and other end is free. The calculation of shear force and bending moment are done from free end. The shear force and bending moment diagrams are drawn from free end.

8.6.1. Shear Force and Bending Moment Diagrams for a Cantilever with a Point Load at the Free End. Fig. 8.14 shows a cantilever AB of length L fixed at A and free at B and carrying a point load W at the free end B .

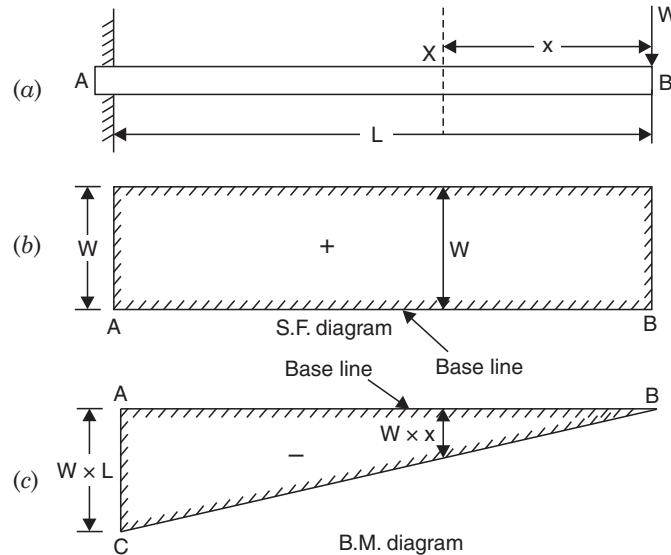


Fig. 8.14

Let F_x = Shear force at X , and
 M_x = Bending moment at X .

Take a section X at a distance x from the free end. Consider the right portion of the section.

The shear force at this section is equal to the resultant force acting on the right portion at the given section. But the resultant force acting on the right portion at the section X is W and acting in the downward direction. But a force on the right portion acting downwards is considered positive. Hence shear force at X is positive.

$$\therefore F_x = +W$$

The shear force will be constant at all sections of the cantilever between A and B as there is no other load between A and B . The shear force diagram is shown in Fig. 8.14 (b).

Bending Moment Diagram

The bending moment at the section X is given by

$$M_x = -W \times x \quad \dots(i)$$

(Bending moment will be negative as for the right portion of the section, the moment of W at X is clockwise. Also the bending of cantilever will take place in such a manner that convexity will be at the top of the beam).

From equation (i), it is clear that B.M. at any section is proportional to the distance of the section from the free end.

$$\text{At } x = 0 \text{ i.e., at } B, \quad \text{B.M.} = 0$$

$$\text{At } x = L \text{ i.e., at } A, \quad \text{B.M.} = W \times L$$

Hence B.M. follows the straight line law. The B.M. diagram is shown in Fig. 8.14 (c). At point A , take $AC = W \times L$ in the downward direction. Join point B to C .

The shear force and bending moment diagrams for several concentrated loads acting on a cantilever, will be drawn in the similar manner.

Problem 8.1. A cantilever beam of length 2 m carries the point loads as shown in Fig. 8.15. Draw the shear force and B.M. diagrams for the cantilever beam.

Sol. Given :

Refer to Fig. 8.15.

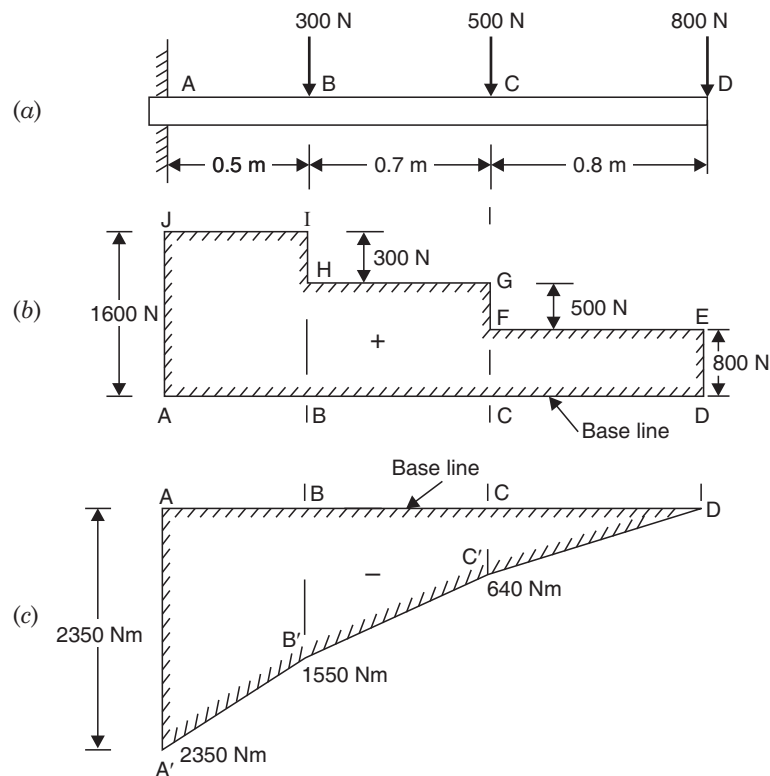


Fig. 8.15

Shear Force Diagram

The shear force at D is $+ 800$ N. This shear force remains constant between D and C . At C , due to point load, the shear force becomes $(800 + 500) = 1300$ N. Between C and B , the shear force remains 1300 N. At B again, the shear force becomes $(1300 + 300) = 1600$ N. The shear force between B and A remains constant and equal to 1600 N. Hence the shear force at different points will be as given below :

$$\begin{aligned} \text{S.F. at } D, & \quad F_D = + 800 \text{ N} \\ \text{S.F. at } C, & \quad F_C = + 800 + 500 = + 1300 \text{ N} \\ \text{S.F. at } B, & \quad F_B = + 800 + 500 + 300 = 1600 \text{ N} \\ \text{S.F. at } A, & \quad F_A = + 1600 \text{ N.} \end{aligned}$$

The shear force, diagram is shown in Fig. 8.15 (b) which is drawn as :

Draw a horizontal line AD as base line. On the base line mark the points B and C below the point loads. Take the ordinate $DE = 800$ N in the upward direction. Draw a line EF parallel to AD . The point F is vertically above C . Take vertical line $FG = 500$ N. Through G , draw a horizontal line GH in which point H is vertically above B . Draw vertical line $HI = 300$ N. From I , draw a horizontal line IJ . The point J is vertically above A . This completes the shear force diagram.

Bending Moment Diagram

The bending moment at D is zero :

(i) The bending moment at any section between C and D at a distance x and D is given by,

$$M_x = -800 \times x \text{ which follows a straight line law.}$$

At C , the value of $x = 0.8$ m.

$$\therefore \text{ B.M. at } C, \quad M_C = -800 \times 0.8 = -640 \text{ Nm.}$$

(ii) The B.M. at any section between B and C at a distance x from D is given by (At C , $x = 0.8$ and at B , $x = 0.8 + 0.7 = 1.5$ m. Hence here x varies from 0.8 to 1.5).

$$M_x = -800x - 500(x - 0.8) \quad \dots(i)$$

Bending moment between B and C also varies by a straight line law.

B.M. at B is obtained by substituting $x = 1.5$ m in equation (i),

$$\begin{aligned} \therefore M_B &= -800 \times 1.5 - 500(1.5 - 0.8) \\ &= -1200 - 350 = -1550 \text{ Nm.} \end{aligned}$$

(iii) The B.M. at any section between A and B at a distance x from D is given by (At B , $x = 1.5$ and at A , $x = 2.0$ m. Hence here x varies from 1.5 m to 2.0 m)

$$M_x = -800x - 500(x - 0.8) - 300(x - 1.5) \quad \dots(ii)$$

Bending moment between A and B varies by a straight line law.

B.M. at A is obtained by substituting $x = 2.0$ m in equation (ii),

$$\begin{aligned} \therefore M_A &= -800 \times 2 - 500(2 - 0.8) - 300(2 - 1.5) \\ &= -800 \times 2 - 500 \times 1.2 - 300 \times 0.5 \\ &= -1600 - 600 - 150 = -2350 \text{ Nm.} \end{aligned}$$

Hence the bending moments at different points will be as given below :

$$\begin{aligned} M_D &= 0 \\ M_C &= -640 \text{ Nm} \\ M_B &= -1550 \text{ Nm} \\ M_A &= -2350 \text{ Nm.} \end{aligned}$$

and

The bending moment diagram is shown in Fig. 8.15 (c) which is drawn as.

Draw a horizontal line AD as a base line and mark the points B and C on this line. Take vertical lines $CC' = 640$ Nm, $BB' = 1550$ Nm and $AA' = 2350$ Nm in the downward direction. Join points D , C' , B' and A' by straight lines. This completes the bending moment diagram.

8.6.2. Shear Force and Bending Moment Diagrams for a Cantilever with a Uniformly Distributed Load. Fig. 8.16 shows a cantilever of length L fixed at A and carrying a uniformly distributed load of w per unit length over the entire length of the cantilever.

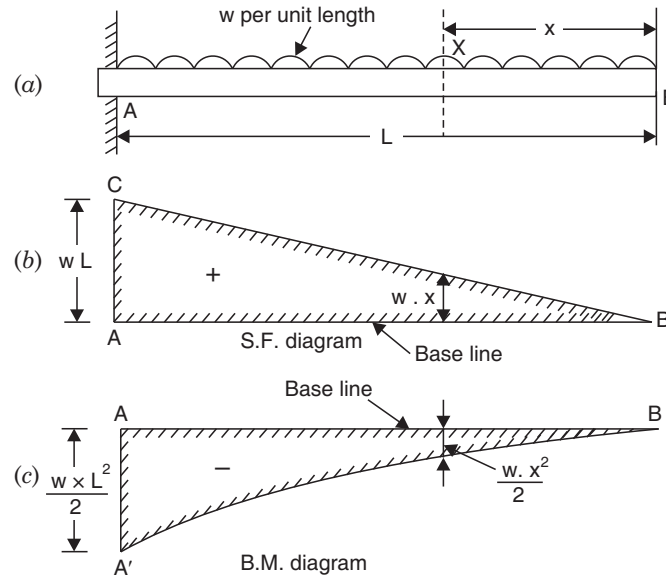


Fig. 8.16

Take a section X at a distance of x from the free end B .

Let F_x = Shear force at X , and
 M_x = Bending moment at X .

Here we have considered the right portion of the section. The shear force at the section X will be equal to the resultant force acting on the right portion of the section. But the resultant force on the right portion = $w \times \text{Length of right portion} = w.x$.

This resultant force is acting downwards. But the resultant force on the right portion acting downwards is considered positive. Hence shear force at X is positive.

$$\therefore F_x = + w.x$$

The above equation shows that the shear force follows a straight line law.

At B , $x = 0$ and hence $F_x = 0$

At A , $x = L$ and hence $F_x = w.L$

The shear force diagram is shown in Fig. 8.16 (b).

Bending Moment Diagram

It is mentioned in Art. 8.2.2 that the uniformly distributed load over a section is converted into point load acting at the C.G. of the section.

The bending moment at the section X is given by

$$\begin{aligned} M_x &= - (\text{Total load on right portion}) \\ &\quad \times \text{Distance of C.G. of right portion from } X \\ &= - (w \cdot x) \cdot \frac{x}{2} = - w \cdot x \cdot \frac{x}{2} = - w \cdot \frac{x^2}{2} \quad \dots(i) \end{aligned}$$

(The bending moment will be negative as for the right portion of the section, the moment of the load at x is clockwise. Also the bending of cantilever will take place in such a manner that convexity will be at the top of the cantilever).

From equation (i), it is clear that B.M. at any section is proportional to the square of the distance of the section from the free end. This follows a parabolic law.

At B, $x = 0$ hence $M_x = 0$

At A, $x = L$ hence $M_x = -w \cdot \frac{L^2}{2}$.

The bending moment diagram is shown in Fig. 8.16 (c).

Problem 8.2. A cantilever of length 2.0 m carries a uniformly distributed load of 1 kN/m run over a length of 1.5 m from the free end. Draw the shear force and bending moment diagrams for the cantilever.

Sol. Given :

U.D.L., $w = 1 \text{ kN/m run}$

Refer to Fig. 8.17.

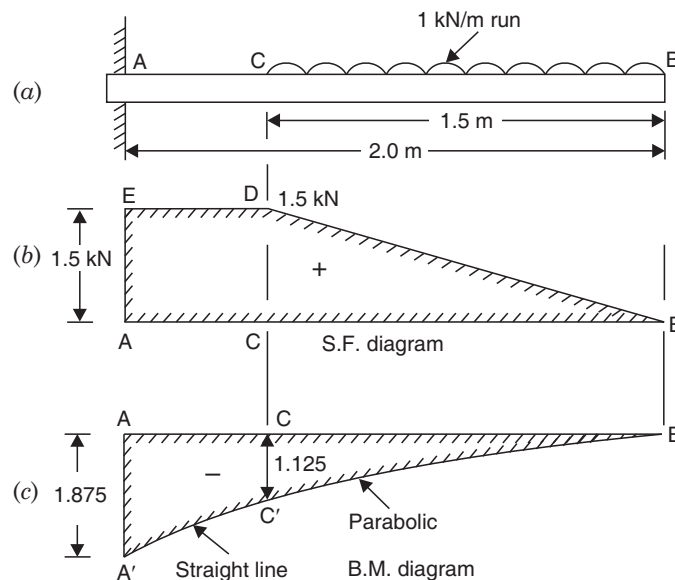


Fig. 8.17

Shear Force Diagram

Consider any section between C and B a distance of x from the free end B. The shear force at the section is given by

$$F_x = w \cdot x \quad (\text{+ve sign is due to downward force on right portion of the section})$$

$$= 1.0 \times x \quad (\because w = 1.0 \text{ kN/m run})$$

At B, $x = 0$ hence $F_x = 0$

At C, $x = 1.5$ hence $F_x = 1.0 \times 1.5 = 1.5 \text{ kN}$.

The shear force follows a straight line law between C and B. As between A and C there is no load, the shear force will remain constant. Hence shear force between A and C will be represented by a horizontal line.

The shear force diagram is shown in Fig. 8.17 (b) in which

$$F_B = 0, F_C = 1.5 \text{ kN and } F_A = F_C = 1.5 \text{ kN}.$$

Bending Moment Diagram

(i) The bending moment at any section between *C* and *B* at a distance *x* from the free end *B* is given by

$$M_x = -(w \cdot x) \cdot \frac{x}{2} = -\left(1 \cdot \frac{x^2}{2}\right) = -\frac{x^2}{2} \quad \dots(i)$$

(The bending moment will be negative as for the right portion of the section the moment of load at *x* is clockwise).

$$\text{At } B, x = 0 \text{ hence } M_B = -\frac{0^2}{2} = 0$$

$$\text{At } C, x = 1.5 \text{ hence } M_C = -\frac{1.5^2}{2} = -1.125 \text{ Nm}$$

From equation (i) it is clear that the bending moment varies according to parabolic law between *C* and *B*.

(ii) The bending moment at any section between *A* and *C* at a distance *x* from the free end *B* is obtained as : (here *x* varies from 1.5 m to 2.0 m)

$$\text{Total load due to U.D.L.} = w \times 1.5 = 1.5 \text{ kN.}$$

This load is acting at a distance of $\frac{1.5}{2} = 0.75$ m from the free end *B* or at a distance of (*x* – 0.75) from any section between *A* and *C*.

$$\therefore \text{Moment of this load at any section between } A \text{ and } C \text{ at a distance } x \text{ from free end} \\ = (\text{Load due to U.D.L.}) \times (x - 0.75)$$

$$\therefore M_x = -1.5 \times (x - 0.75) \quad \dots(ii)$$

(–ve sign is due to clockwise moment for right portion)

From equation (ii) it is clear that the bending moment follows straight line law between *A* and *C*.

$$\text{At } C, x = 1.5 \text{ m hence } M_C = -1.5 (1.5 - 0.75) = -1.125 \text{ Nm}$$

$$\text{At } A, x = 2.0 \text{ m hence } M_A = -1.5 (2 - 0.75) = -1.875 \text{ Nm.}$$

Now the bending moment diagram is drawn as shown in Fig. 8.17 (c). In this diagram line *CC'* = 1.125 Nm and *AA'* = 1.875 Nm. The points *B* and *C'* are on a parabolic curve whereas the points *A'* and *C'* are joined by a straight line.

Problem 8.3. A cantilever of length 2.0 m carries a uniformly distributed load of 2 kN/m length over the whole length and a point load of 3 kN at the free end. Draw the S.F. and B.M. diagrams for the cantilever.

Sol. Given :

$$\text{Length, } L = 2.0 \text{ m}$$

$$\text{U.D.L., } w = 2 \text{ kN/m length}$$

$$\text{Point load at free end} = 3 \text{ kN}$$

Refer to Fig. 8.18.

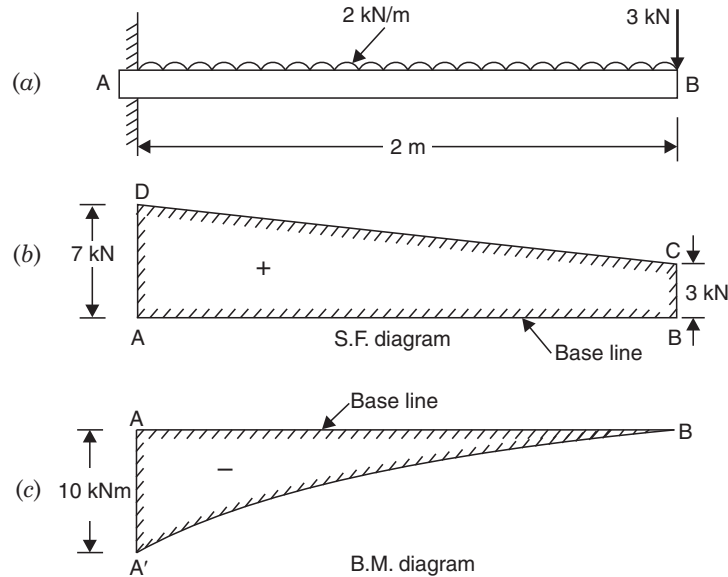


Fig. 8.18

Shear Force Diagram

The shear force at $B = 3 \text{ kN}$

Consider any section at a distance x from the free end B . The shear force at the section is given by,

$$F_x = 3.0 + w \cdot x \quad \text{(+ve sign is due to downward force on right portion of the section)}$$

$$= 3.0 + 2 \times x \quad (\because w = 2 \text{ kN/m})$$

The above equation shows that shear force follows a straight line law.

At B , $x = 0$ hence $F_B = 3.0 \text{ kN}$

At A , $x = 2 \text{ m}$ hence $F_A = 3 + 2 \times 2 = 7 \text{ kN}$.

The shear force diagram is shown in Fig. 8.18 (b) in which $F_B = BC = 3 \text{ kN}$ and $F_A = AD = 7 \text{ kN}$. The points C and D are joined by a straight line.

Bending Moment Diagram

The bending moment at any section at a distance x from the free end B is given by,

$$M_x = - \left(3x + wx \cdot \frac{x}{2} \right)$$

$$= - \left(3x + \frac{2x^2}{2} \right) \quad (\because w = 2 \text{ kN/m})$$

$$= - (3x + x^2) \quad \dots(i)$$

(The bending moment will be negative as for the right portion of the section, the moment of loads at x is clockwise).

The equation (i) shows that the B.M. varies according to the parabolic law. From equation (i), we have

$$\text{At } B, x = 0 \text{ hence } M_B = -(3 \times 0 + 0^2) = 0$$

$$\text{At } A, x = 2 \text{ m hence } M_A = -(3 \times 2 + 2^2) = -10 \text{ kNm}$$

Now the bending moment diagram is drawn as shown in Fig. 8.18 (c). In this diagram, $AA' = 10 \text{ kNm}$ and points A' and B are joined by a parabolic curve.

Problem 8.4. A cantilever of length 2 m carries a uniformly distributed load of 1.5 kN/m run over the whole length and a point load of 2 kN at a distance of 0.5 m from the free end. Draw the S.F. and B.M. diagrams for the cantilever.

Sol. Given :

Length, $L = 2 \text{ m}$

U.D.L., $w = 1.5 \text{ kN/m run}$

Point load, $W = 2 \text{ kN}$

Distance of point load from free end = 0.5 m

Refer to Fig. 8.19.

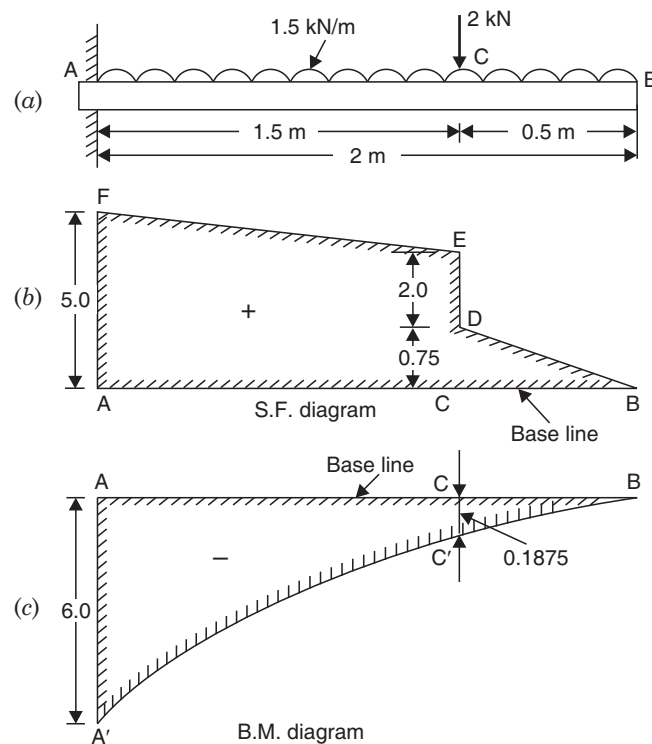


Fig. 8.19

Shear Force Diagram

(i) Consider any section between C and B at a distance x from the free end. The shear force at the section is given by,

$$F_x = + w \cdot x$$

$$= 1.5 \times x$$

(+ve sign is due to downward force on right portion)

...(i)

In equation (i), x varies from 0 to 0.5. The equation (i) shows that shear force varies by a straight line law between B and C .

$$\text{At } B, x = 0 \text{ hence } F_B = 1.5 \times 0 = 0$$

$$\text{At } C, x = 0.5 \text{ hence } F_C = 1.5 \times 0.5 = 0.75 \text{ kN}$$

(ii) Now consider any section between A and C at a distance x from free end B . The shear force at the section is given by

$$\begin{aligned} F_x &= +w \cdot x + 2 \text{ kN} && (+\text{ve sign is due to downward force on right portion of the section}) \\ &= 1.5x + 2 && \dots(ii) \end{aligned}$$

In equation (ii), x varies from 0.5 to 2.0. The equation (ii) also shows that shear force varies by a straight line law between A and C .

$$\text{At } C, x = 0.5 \text{ hence } F_C = 1.5 \times 0.5 + 2 = 2.75 \text{ kN}$$

$$\text{At } A, x = 2.0 \text{ hence } F_A = 1.5 \times 2.0 + 2 = 5.0 \text{ kN}$$

Now draw the shear force diagram as shown in Fig. 8.19 (b) in which $CD = 0.75 \text{ kN}$, $DE = 2.0 \text{ kN}$ or $CE = 2.75 \text{ kN}$ and $AF = 5.0 \text{ kN}$. The point B is joined to point D by a straight line whereas the point E is also joined to point F by a straight line.

Bending Moment Diagram

(i) The bending moment at any section between C and B at a distance x from the free end B is given by

$$\begin{aligned} M_x &= -(w \cdot x) \cdot \frac{x}{2} \\ &= -(1.5 \times x) \cdot \frac{x}{2} && (\because w = 1.5 \text{ kN/m}) \\ &= -0.75x^2 && \dots(iii) \end{aligned}$$

(The bending moment will be negative as for the right portion of the section the moment at the section is clockwise).

In equation (iii), x varies from 0 to 0.5. Equation (iii) shows that B.M. varies between C and B by a parabolic law.

$$\text{At } B, x = 0 \text{ hence } M_B = -0.75 \times 0 = 0$$

$$\text{At } C, x = 0.5 \text{ hence } M_C = -0.75 \times 0.5^2 = -0.1875 \text{ kNm.}$$

(ii) The bending moment at any section between A and C at a distance x from the free end B is given by

$$\begin{aligned} M_x &= -(w \cdot x) \cdot \frac{x}{2} - 2(x - 0.5) = -(1.5 \times x) \cdot \frac{x}{2} - 2(x - 0.5) && (\because w = 1.5 \text{ kN/m}) \\ &= -0.75x^2 - 2(x - 0.5) && \dots(iv) \end{aligned}$$

In equation (iv), x varies from 0.5 to 2.0. Equation (iv) shows that B.M. varies by a parabolic law between A and C .

$$\text{At } C, x = 0.5 \text{ hence } M_C = -0.75 \times 0.5^2 - 2(0.5 - 0.5) = -0.1875 \text{ kNm}$$

$$\text{At } A, x = 2.0 \text{ hence } M_A = -0.75 \times 2^2 - 2(2.0 - 0.5) \text{ kNm} = -3.0 - 3.0 = -6.0 \text{ kNm}$$

Now the bending moment diagram is drawn as shown in Fig. 8.19 (c). In this diagram line $CC' = 0.1875$ and $AA' = 6.0$. The points A' , C' and B are on parabolic curves.

Problem 8.4 (A). Fig. 8.19A (i) shows vertical forces 20 kN, 40 kN and uniformly load of 20 kN/m in 3 m length. Find the resultant force of the system and draw the shear force diagram and B.M. Diagram.

(U.P. Tech. University, 2001)

Sol. As the point C is fixed, there will be reaction and bending moments (fixing moments) at C.

Resultant force

$= 20 + 40 + 3 \times 20 = 120 \text{ kN}$ acting downward.

The resultant moment at C

$$\begin{aligned} &= 20 \times 3 + 40 \times 2 + (3 \times 20) \times \frac{3}{2} \\ &= 60 + 80 + 90 \\ &= 230 \text{ kNm anti-clockwise at C.} \end{aligned}$$

The reaction at C will be $R_c = 120 \text{ kN}$ acting vertically upward (opposite to resultant force) and a fixing moment $= 230 \text{ kNm}$ acting clockwise (opposite to resultant moment) as shown in Fig. 8.19A (ii).

S.F. Diagram

S.F. at A $= -20 \text{ kN}$

S.F. at B due to U.D.L.

$$= -20 - 20 \times 1 = -40 \text{ kN}$$

S.F. at B due to point load at B

$$= -40 - 40 = -80 \text{ kN}$$

S.F. at C due to U.D.L.

$$= -80 - 20 \times 2 = -120 \text{ kN}$$

S.F. at C due to point load

$$= -120 + 120 = 0$$

The S.F. Diagram is shown in Fig. 8.19A (iii).

B.M. Diagram

B.M. at A $= 0$

$$\begin{aligned} \text{B.M. at B} &= -20 \times 1 - 20 \times 1 \times \frac{1}{2} \\ &= -30 \text{ kNm} \end{aligned}$$

B.M. at C without fixing moment

$$\begin{aligned} &= -20 \times 3 - 40 \times 2 - 20 \times 3 \times \frac{3}{2} \\ &= -60 - 80 - 90 \\ &= -230 \text{ kNm} \end{aligned}$$

B.M. at C with fixing moment

$$= -230 + 230 = 0$$

The B.M. Diagram is shown in Fig. 8.19A (iv).

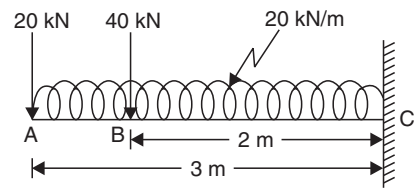


Fig. 8.19A (i)

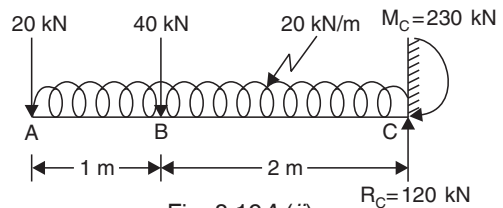


Fig. 8.19A (ii)

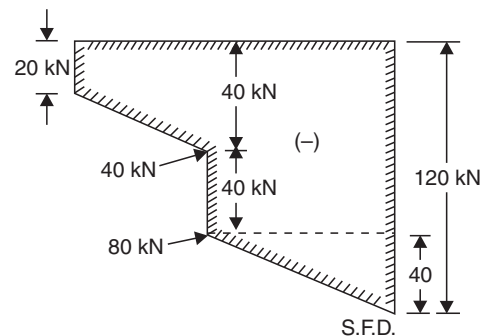


Fig. 8.19A (iii)

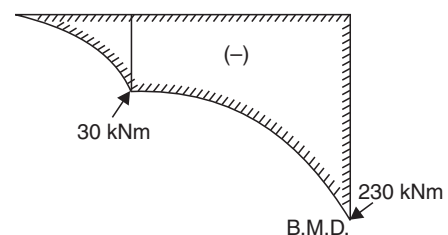


Fig. 8.19A (iv)

Problem 8.5. A cantilever 1.5 m long is loaded with a uniformly distributed load of 2 kN/m run over a length of 1.25 m from the free end. It also carries a point load of 3 kN at a distance of 0.25 m from the free end. Draw the shear force and bending moment diagrams of the cantilever.

Sol. Given :

Length, $L = 1.5$ m
 U.D.L., $w = 2$ kN/m
 Point load, $W = 3$ kN
 Refer to Fig. 8.20.

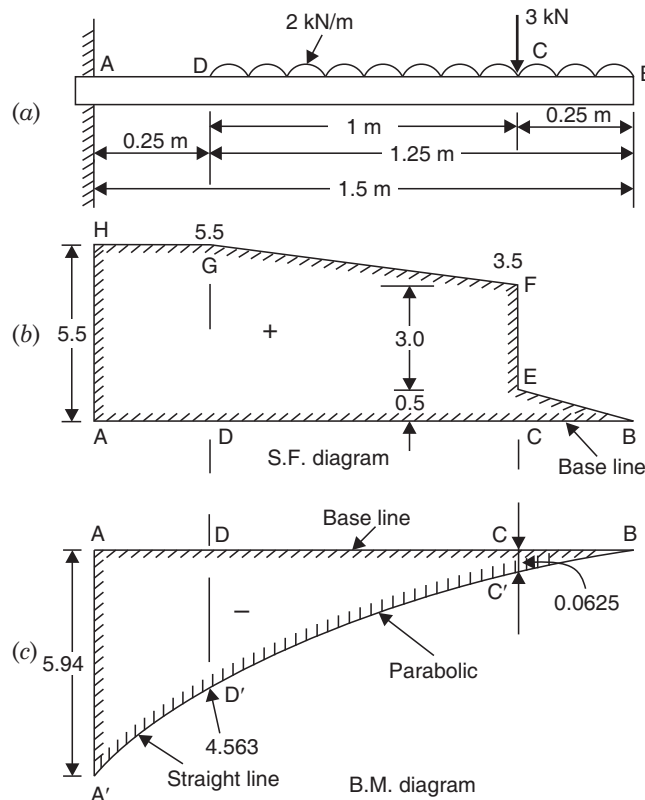


Fig. 8.20

Shear Force Diagram

The shear force at B is zero.

The shear force increases to $2 \times 0.25 = 0.5$ kN by a straight line at C . Due to point load of 3 kN, the shear force suddenly increases to $0.5 + 3 = 3.5$ kN at C .

The shear force further increases to $3.5 + 2 \times 1 = 5.5$ kN by a straight line at D . The shear force remains constant between A and D as there is no load between A and D .

Now the shear force diagram is drawn as shown in Fig. 8.20 (b). In this diagram line $CE = 0.5$ kN, $CF = 3.5$ kN, $DG = 5.5$ kN and $AH = 5.5$ kN. The point B is joined to E by a straight line. The point F is also joined to G by a straight line. Line GH is horizontal.

*Bending Moment Diagram*B.M. at $B = 0$

$$\text{B.M. at } D = -2 \times 0.25 \times \frac{0.25}{2} = -0.0625 \text{ kNm}$$

$$\text{B.M. at } D = -2 \times 1.25 \times \frac{1.25}{2} - (3 \times 1) = -4.563 \text{ kNm}$$

$$\text{B.M. at } A = -2 \times 1.25 \times \left(\frac{1.25}{2} + 0.25 \right) - 3 \times (1 + 0.25) = -5.94 \text{ kNm.}$$

The bending moment between B and C and between C and D varies by a parabolic law. But B.M. between A and D varies by a straight line law.

Now the bending moment diagram is drawn as shown in Fig. 8.20 (c). In this diagram line $CC' = 0.0625$, $DD' = 4.563$ and $AA' = 5.9$. The points B , C' and D' are on parabolic curve whereas points A' and D' are joined by a straight line.

Problem 8.6. A cantilever of length 5.0 m is loaded as shown in Fig. 8.21. Draw the S.F. and B.M. diagrams for the cantilever. (U.P. Tech. University, May 2009)

Sol. The shear force at B is 2.5 kN and remains constant between B and C .

The shear force increases by a straight line law to $2.5 + 2 \times 1 = 4.5$ kN at D . The shear force remains constant between D and E . At point E , the shear force suddenly increases to $4.5 + 3 = 7.5$ kN due to point load at E . Again the shear force remains constant between A and E . Now the shear force diagram is drawn as shown in Fig. 8.21 (b).

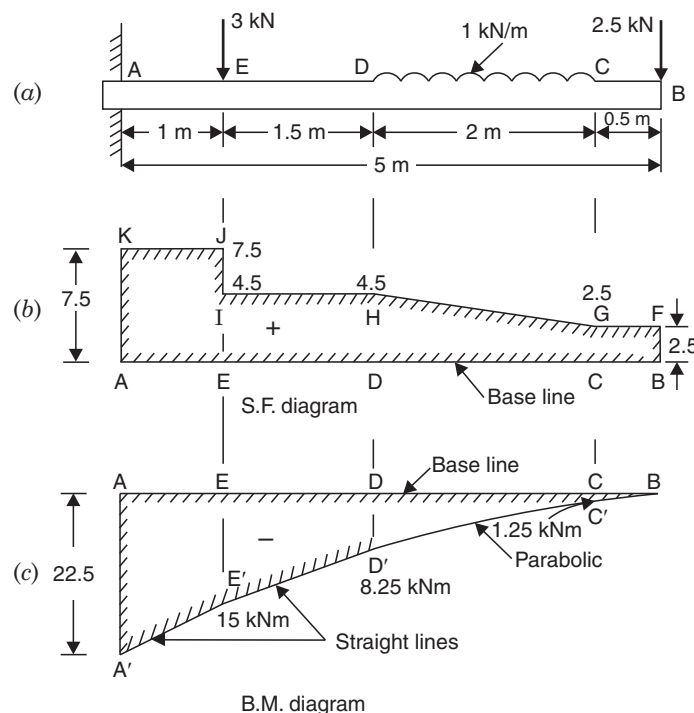


Fig. 8.21

Bending Moment Diagram

$$\text{B.M. at } B = 0$$

$$\text{B.M. at } C = -2.5 \times 0.5 = -1.25 \text{ kNm}$$

$$\text{B.M. at } D = -2.5 \times 2.5 - 2 \times 1 \times 1 = -8.25 \text{ kNm}$$

$$\text{B.M. at } E = -2.5 \times 4 - 2 \times 1 \times (1.5 + 1.0) = -10 - 5 = -15 \text{ kNm}$$

$$\begin{aligned} \text{B.M. at } A &= -2.5 \times 5 - 2 \times 1 \times (1 + 1.5 + 1.0) - 3 \times 1 \\ &= -12.5 - 7.0 - 3 = -22.5 \text{ kNm.} \end{aligned}$$

Now the bending moment diagram is drawn as shown in Fig. 8.21 (c). In this diagram, the B.M. varies according to parabolic law between points C and D only. Between other points B.M. varies according to straight line law.

8.6.3. Shear Force and Bending Moment Diagrams for a Cantilever Carrying a Gradually Varying Load. Fig. 8.22 shows a cantilever of length L fixed at A and carrying a gradually varying load from zero at the free end to w per unit length at the fixed end.

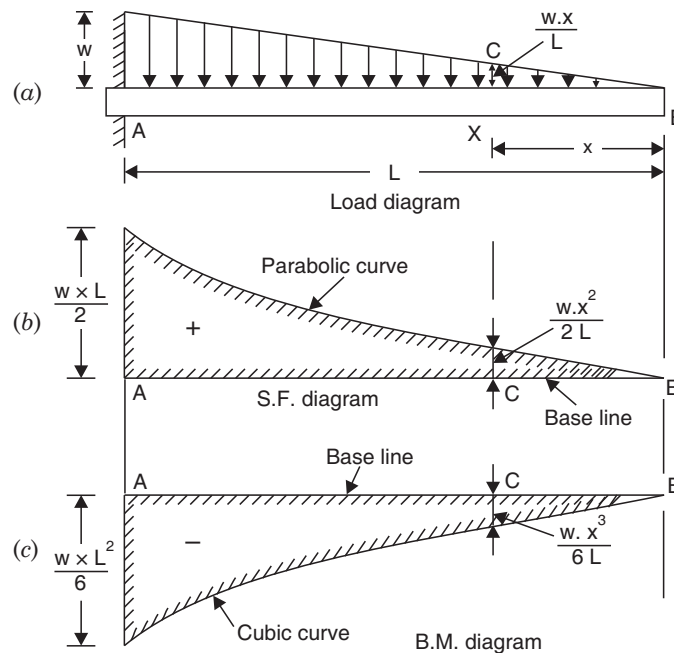


Fig. 8.22

Take a section X at a distance x from the free end B .

Let F_x = Shear force at the section X , and

M_x = Bending moment at the section X .

Let us first find the rate of loading at the section X . The rate of loading is zero at B and is w per metre run at A . This means that rate of loading for a length L is w per unit length. Hence rate of loading for a length of x will be $\frac{w}{L} \times x$ per unit length. This is shown in Fig. 8.22 (a) by CX , which is also known as load diagram. Hence $CX = \frac{w \cdot x}{L}$.

The shear force and the section X at a distance x from free end is given by,

$$\begin{aligned}
 F_x &= \text{Total load on the cantilever for a length } x \text{ from the free end } B \\
 &= \text{Area of triangle } BCX \\
 &= \frac{XB \cdot XC}{2} = \frac{x \left(\frac{w \cdot x}{L} \right)}{2} \quad \left(\because XB = x, XC = \frac{w \cdot x}{L} \right) \\
 &= \frac{w \cdot x^2}{2L} \quad \dots(i)
 \end{aligned}$$

The equation (i) shows that the S.F. varies according to the parabolic law.

$$\text{At } B, x = 0 \text{ hence } F_B = \frac{w \times 0^2}{2L} = 0$$

$$\text{At } A, x = L \text{ hence } F_A = \frac{w \cdot L^2}{2L} = \frac{w \cdot L}{2}$$

The bending moment at the section X at a distance x from the free end B is given by,

$$\begin{aligned}
 M_x &= -(\text{Total load for a length } x) \times \text{Distance of the load from } X \\
 &= -(\text{Area of triangle } BCX) \times \text{Distance of C.G. of the triangle from } X \\
 &= -\left(\frac{wx^2}{2L} \right) \times \frac{x}{3} = -\frac{wx^3}{6L} \quad \dots(ii)
 \end{aligned}$$

The equation (ii) shows that the B.M. varies according to the cubic law.

$$\text{At } B, x = 0 \text{ hence } M_B = -\frac{w \times 0}{6L} = 0$$

$$\text{At } A, x = L \text{ hence } M_A = -\frac{w \cdot L^3}{6L} = -\frac{w \cdot L^2}{6}$$

Problem 8.7. A cantilever of length 4 m carries a gradually varying load, zero at the free end to 2 kN/m at the fixed end. Draw the S.F. and B.M. diagrams for the cantilever.

Sol. Given :

Length, $L = 4 \text{ m}$

Load at fixed end, $w = 2 \text{ kN/m}$

Shear Force Diagram

The shear force is zero at B . The shear force at C will be equal to the area of load diagram ABC .

$$\therefore \text{Shear force at } C = \frac{4 \times 2}{2} = 4 \text{ kN}$$

The shear force between A and B varies according to parabolic law.

Bending Moment Diagram

The B.M. at B is zero. The bending moment at A is equal to $-\frac{w \cdot L^2}{6}$.

$$\therefore M_A = -\frac{w \cdot L^2}{6} = -\frac{2 \times 4^2}{6} = -5.33 \text{ kNm.}$$

The B.M. between A and B varies according to cubic law.

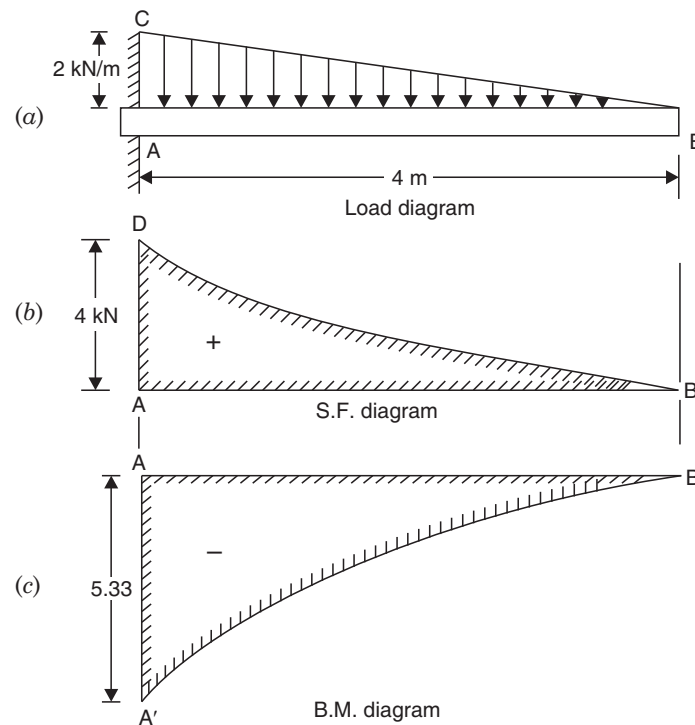


Fig. 8.23

8.7. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A SIMPLY SUPPORTED BEAM

For a simply supported beam, first the reactions at the supports are obtained. Then calculation of shear force and bending moment are done from left end. Also shear force and bending moment diagrams are drawn starting from the left end of the beam.

8.7.1. Shear Force and Bending Moment Diagrams for a Simply Supported Beam with a Point Load at Mid-point. Fig. 8.24 shows a beam AB of length L simply supported at the ends A and B and carrying a point load W at its middle point C .

The reactions at the support will be equal to $\frac{W}{2}$ as the load is acting at the middle point of the beam. Hence $R_A = R_B = \frac{W}{2}$.

Take a section X at a distance x from the end A between A and C .

Let F_x = Shear force at X ,

and M_x = Bending moment at X .

Here we have considered the *left portion* of the section. The shear force at X will be equal to the resultant force acting on the left portion of the section. But the resultant force on the left portion is $\frac{W}{2}$ acting upwards. But according to the sign convention, the resultant force on the *left portion* acting upwards is considered positive. Hence shear force at X is positive and its magnitude is $\frac{W}{2}$.

$$\therefore F_x = +\frac{W}{2}$$

Hence the shear force between A and C is constant and equal to $+\frac{W}{2}$.

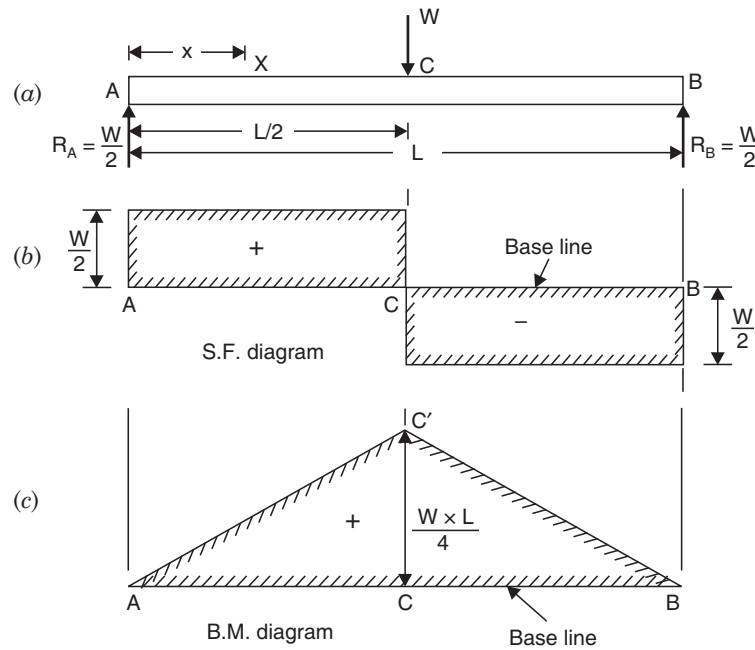


Fig. 8.24

Now consider any section between C and B at distance x from end A. The resultant force on the left portion will be

$$\left(\frac{W}{2} - W\right) = -\frac{W}{2}.$$

This force will also remain constant between C and B. Hence shear force between C and B is equal to $-\frac{W}{2}$.

At the section C the shear force changes from $+\frac{W}{2}$ to $-\frac{W}{2}$.

The shear force diagram is shown in Fig. 8.24 (b).

Bending Moment Diagram

(i) The bending moment at any section between A and C at a distance of x from the end A, is given by

$$M_x = R_A \cdot x \quad \text{or} \quad M_x = +\frac{W}{2} \cdot x \quad \dots(i)$$

(B.M. will be positive as for the *left portion* of the section, the moment of all forces at X is clockwise. Moreover, the bending of beam takes place in such a manner that concavity is at the top of the beam).

At A, $x = 0$ hence $M_A = \frac{W}{2} \times 0 = 0$

At C, $x = \frac{L}{2}$ hence $M_C = \frac{W}{2} \times \frac{L}{2} = \frac{W \times L}{4}$.

From equation (i), it is clear that B.M. varies according to straight line law between A and C. B.M. is zero at A and it increases to $\frac{W \times L}{4}$ at C.

(ii) The bending moment at any section between C and B at a distance x from the end A, is given by

$$M_x = R_A \cdot x - W \times \left(x - \frac{L}{2}\right) = \frac{W}{2} \cdot x - Wx + W \times \frac{L}{2} = \frac{WL}{2} - \frac{2x}{2}$$

At C, $x = \frac{L}{2}$ hence $M_C = \frac{WL}{2} - \frac{W}{2} \times \frac{L}{2} = \frac{W \times L}{4}$

At B, $x = L$ hence $M_B = \frac{WL}{2} - \frac{W}{2} \times L = 0$.

Hence bending moment at C is $\frac{WL}{4}$ and it decreases to zero at B. Now the B.M. diagram can be completed as shown in Fig. 8.24 (c).

Note. The bending moment is maximum at the middle point C, where the shear force changes its sign.

Problem 8.8. A simply supported beam of length 6 m, carries point load of 3 kN and 6 kN at distances of 2 m and 4 m from the left end. Draw the shear force and bending moment diagrams for the beam.

Sol. First calculate the reactions R_A and R_B .

Taking moments of the force about A, we get

$$R_B \times 6 = 3 \times 2 + 6 \times 4 = 30$$

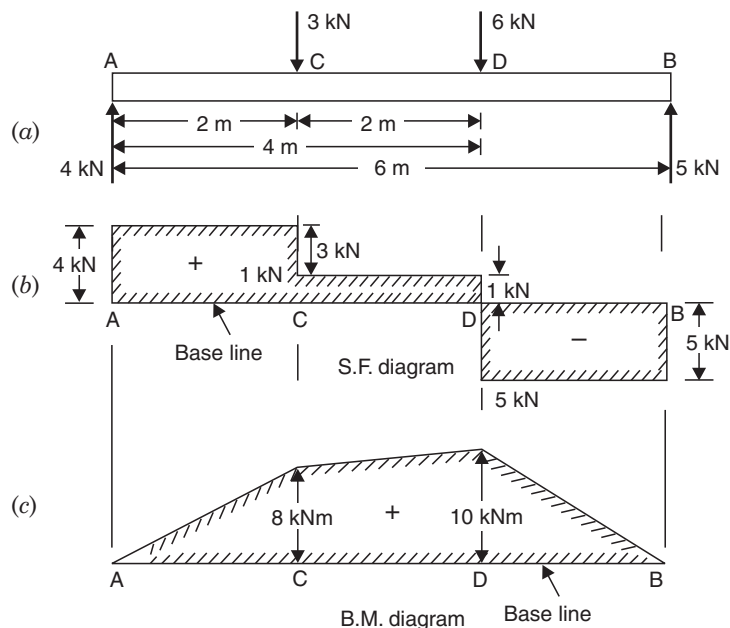


Fig. 8.25

$$\therefore R_B = \frac{30}{6} = 5 \text{ kN}$$

$$\therefore R_A = \text{Total load on beam} - R_B = (3 + 6) - 5 = 4 \text{ kN}$$

Shear Force Diagram

Shear force at A, $F_A = +R_A = +4 \text{ kN}$

Shear force between A and C is constant and equal to +4 kN.

Shear force at C, $F_C = +4 - 3.0 = +1 \text{ kN}$

Shear force between C and D is constant and equal to +1 kN.

Shear force at D, $F_D = +1 - 6 = -5 \text{ kN}$

The shear force between D and B is constant and equal to -5 kN.

Shear force at B, $F_B = -5 \text{ kN}$

The shear force diagram is drawn as shown in Fig. 8.25 (b).

Bending Moment Diagram

B.M. at A, $M_A = 0$

B.M. at C, $M_C = R_A \times 2 = 4 \times 2 = +8 \text{ kNm}$

B.M. at D, $M_D = R_A \times 4 - 3 \times 2 = 4 \times 4 - 3 \times 2 = +10 \text{ kNm}$

B.M. at B, $M_B = 0$

The bending moment diagram is drawn as shown in Fig. 8.25 (c).

8.7.2. Shear Force and Bending Moment Diagrams for a Simply Supported Beam Carrying a Uniformly Distributed Load. Fig. 8.26 shows a beam AB of length L simply supported at the ends A and B and carrying a uniformly distributed load of w per unit length over the entire length. The reactions at the supports will be equal and their magnitude will be half the total load on the entire length.

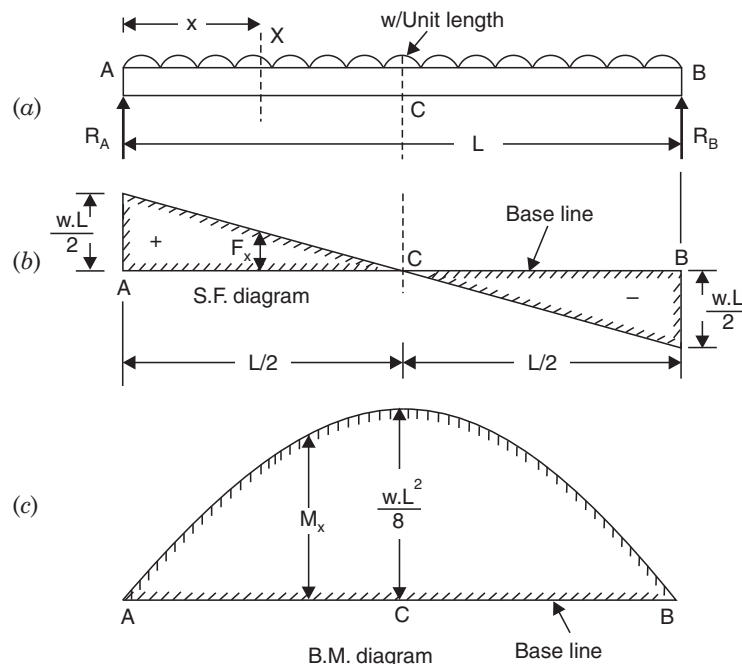


Fig. 8.26

Let R_A = Reaction at A, and R_B = Reaction at B

$$\therefore R_A = R_B = \frac{w.L}{2}$$

Consider any section X at a distance x from the left end A. The shear force at the section (i.e., F_x) is given by,

$$F_x = +R_A - w \cdot x = +\frac{w.L}{2} - w \cdot x \quad \dots(i)$$

From equation (i), it is clear that the shear force varies according to straight line law. The values of shear force at different points are :

$$\text{At A, } x = 0 \text{ hence } F_A = +\frac{w.L}{2} - \frac{w.0}{2} = +\frac{w.L}{2}$$

$$\text{At B, } x = L \text{ hence } F_B = +\frac{w.L}{2} - w.L = -\frac{w.L}{2}$$

$$\text{At C, } x = \frac{L}{2} \text{ hence } F_C = +\frac{w.L}{2} - w \cdot \frac{L}{2} = 0$$

The shear force diagram is drawn as shown in Fig. 8.26 (b).

The bending moment at the section X at a distance x from left end A is given by,

$$\begin{aligned} M_x &= +R_A \cdot x - w \cdot x \cdot \frac{x}{2} \\ &= \frac{w.L}{2} \cdot x - \frac{w.x^2}{2} \quad \left(\because R_A = \frac{w.L}{2} \right) \dots(ii) \end{aligned}$$

From equation (ii), it is clear that B.M. varies according to parabolic law.

The values of B.M. at different points are :

$$\text{At A, } x = 0 \text{ hence } M_A = \frac{w.L}{2} \cdot 0 - \frac{w.0}{2} = 0$$

$$\text{At B, } x = L \text{ hence } M_B = \frac{w.L}{2} \cdot L - \frac{w}{2} \cdot L^2 = 0$$

$$\text{At C, } x = \frac{L}{2} \text{ hence } M_C = \frac{w.L}{2} \cdot \frac{L}{2} - \frac{w}{2} \cdot \left(\frac{L}{2}\right)^2 = \frac{w.L^2}{4} - \frac{w.L^2}{8} = +\frac{w.L^2}{8}$$

Thus the B.M. increases according to parabolic law from zero at A to $+\frac{w.L^2}{8}$ at the middle point of the beam and from this value the B.M. decreases to zero at B according to the parabolic law.

Now the B.M. diagram is drawn as shown in Fig. 8.26 (c).

Problem 8.9. Draw the shear force and bending moment diagram for a simply supported beam of length 9 m and carrying a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Also calculate the maximum B.M. on the section.

Sol. First calculate reactions R_A and R_B .

Taking moments of the forces about A, we get

$$R_B \times 9 = 10 \times 6 \times \frac{6}{2} = 180$$

$$\therefore R_B = \frac{180}{9} = 20 \text{ kN}$$

$$\therefore R_A = \text{Total load on beam} - R_B = 10 \times 6 - 20 = 40 \text{ kN.}$$

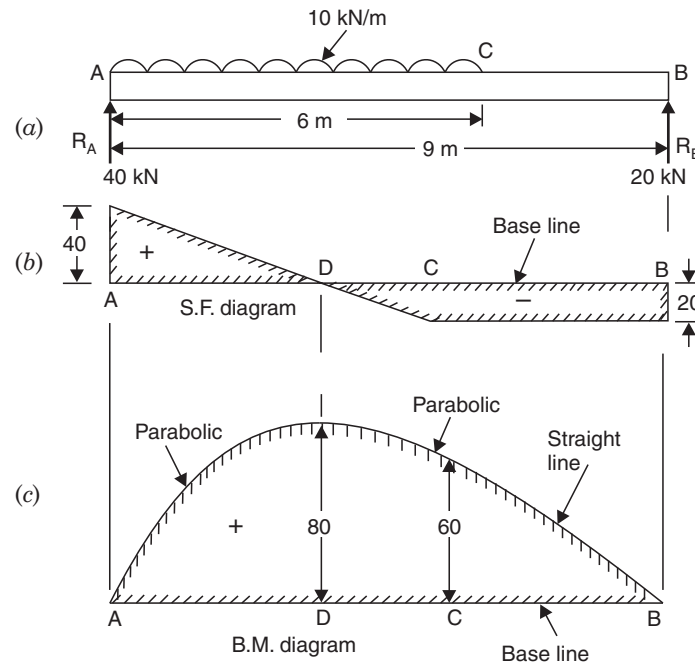


Fig. 8.27

Shear Force Diagram

Consider any section at a distance x from A between A and C . The shear force at the section is given by,

$$F_x = +R_A - 10x = +40 - 10x \quad \dots(i)$$

Equation (i) shows that shear force varies by a straight line law between A and C .

At A , $x = 0$ hence $F_A = +40 - 0 = 40$ kN

At C , $x = 6$ m hence $F_C = +40 - 10 \times 6 = -20$ kN

The shear force at A is $+40$ kN and at C is -20 kN. Also shear force between A and C varies by a straight line. This means that somewhere between A and C , the shear force is zero. Let the S.F. is zero at x metre from A . Then substituting the value of S.F. (*i.e.*, F_x) equal to zero in equation (i), we get

$$0 = 40 - 10x$$

$$\therefore x = \frac{40}{10} = 4 \text{ m}$$

Hence shear force is zero at a distance 4 m from A .

The shear force is constant between C and B . This equal to -20 kN.

Now the shear force diagram is drawn as shown in Fig. 8.27 (b). In the shear force diagram, distance $AD = 4$ m. The point D is at a distance 4 m from A .

B.M. Diagram

The B.M. at any section between A and C at a distance x from A is given by,

$$M_x = R_A \times x - 10 \cdot x \cdot \frac{x}{2} = 40x - 5x^2 \quad \dots(ii)$$

Equation (ii) shows that B.M. varies according to parabolic law between A and C.

At A, $x = 0$ hence $M_A = 40 \times 0 - 5 \times 0 = 0$

At C, $x = 6$ m hence $M_C = 40 \times 6 - 5 \times 6^2 = 240 - 180 = + 60$ kNm

At D, $x = 4$ m hence $M_D = 40 \times 4 - 5 \times 4^2 = 160 - 80 = + 80$ kNm

The bending moment between C and B varies according to linear law.

B.M. at B is zero whereas at C is 60 kNm.

The bending moment diagram is drawn as shown in Fig. 8.27 (c).

Maximum Bending Moment

The B.M. is maximum at a point where shear force changes sign. This means that the point where shear force becomes zero from positive value to the negative or *vice-versa*, the B.M. at that point will be maximum. From the shear force diagram, we know that at point D, the shear force is zero after changing its sign. Hence B.M. is maximum at point D. But the B.M. at D is + 80 kNm.

$\therefore$ Max. B.M. = + 80 kN. Ans.

Problem 8.10. Draw the shear force and B.M. diagrams for a simply supported beam of length 8 m and carrying a uniformly distributed load of 10 kN/m for a distance of 4 m as shown in Fig. 8.28.

Sol. First calculate the reactions R_A and R_B .

Taking moments of the forces about A, we get

$$R_B \times 8 = 10 \times 4 \times \left(1 + \frac{4}{2}\right) = 120$$

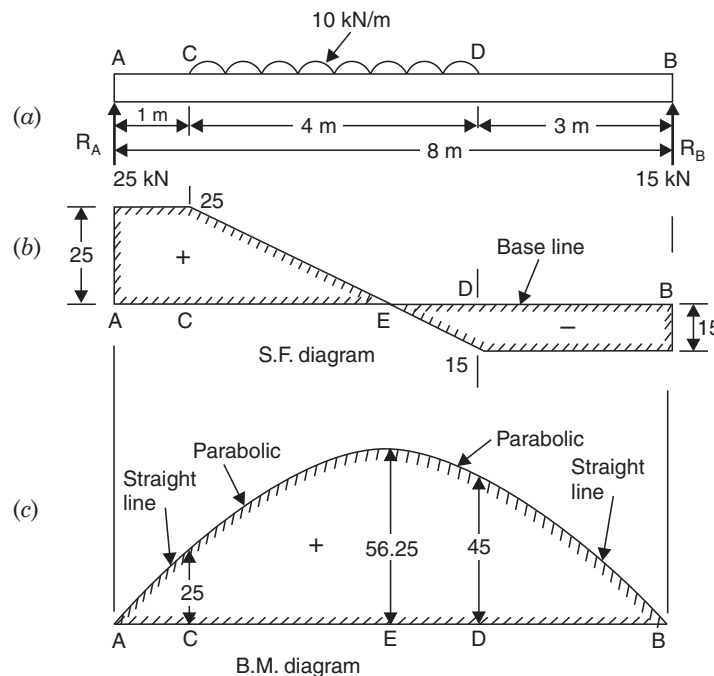


Fig. 8.28

$$\begin{aligned}\therefore R_B &= \frac{120}{8} = 15 \text{ kN} \\ \therefore R_A &= \text{Total load on beam} - R_B \\ &= 10 \times 4 - 15 = 25 \text{ kN}\end{aligned}$$

Shear Force Diagram

The shear force at A is + 25 kN. The shear force remains constant between A and C and equal to + 25 kN. The shear force at B is – 15 kN. The shear force remains constant between B and D and equal to – 15 kN. The shear force at any section between C and D at a distance x from A is given by,

$$\begin{aligned}F_x &= + 25 - 10(x - 1) \quad \dots(i) \\ \text{At } C, x = 1 \text{ hence } F_C &= + 25 - 10(1 - 1) = + 25 \text{ kN} \\ \text{At } D, x = 5 \text{ hence } F_D &= + 25 - 10(5 - 1) = - 15 \text{ kN}\end{aligned}$$

The shear force at C is + 25 kN and at D is – 15 kN. Also shear force between C and D varies by a straight line law. This means that somewhere between C and D, the shear force is zero. Let the S.F. be zero at x metre from A. Then substituting the value of S.F. (i.e., F_x) equal to zero in equation (i), we get

$$\begin{aligned}0 &= 25 - 10(x - 1) \\ \text{or } 0 &= 25 - 10x + 10 \quad \text{or} \quad 10x = 35 \\ \therefore x &= \frac{35}{10} = 3.5 \text{ m}\end{aligned}$$

Hence the shear force is zero at a distance 3.5 m from A.

Hence the distance $AE = 3.5$ m in the shear force diagram shown in Fig. 8.28 (b).

B.M. Diagram

B.M. at A is zero

B.M. at B is also zero

B.M. at C = $R_A \times 1 = 25 \times 1 = 25$ kNm

The B.M. at any section between C and D at a distance x from A is given by,

$$\begin{aligned}M_x &= R_A \cdot x - 10(x - 1) \cdot \frac{(x - 1)}{2} = 25 \times x - 5(x - 1)^2 \quad \dots(ii) \\ \text{At } C, x = 1 \text{ hence } M_C &= 25 \times 1 - 5(1 - 1)^2 = 25 \text{ kNm} \\ \text{At } D, x = 5 \text{ hence } M_D &= 25 \times 5 - 5(5 - 1)^2 = 125 - 80 = 45 \text{ kNm} \\ \text{At } E, x = 3.5 \text{ hence } M_E &= 25 \times 3.5 - 5(3.5 - 1)^2 = 87.5 - 31.25 = 56.25 \text{ kNm}\end{aligned}$$

B.M. will increase from 0 at A to 25 kNm at C by a straight line law. Between C and D the B.M. varies according to parabolic law as is clear from equation (ii). Between C and D, the B.M. will be maximum at E. From D to B the B.M. will decrease from 45 kNm at D to zero at B according to straight line law.

Problem 8.11. A simply supported beam of length 10 m, carries the uniformly distributed load and two point loads as shown in Fig. 8.29. Draw the S.F. and B.M. diagram for the beam. Also calculate the maximum bending moment.

Sol. First calculate the reactions R_A and R_B .

Taking moments of all forces about A, we get

$$\begin{aligned}R_B \times 10 &= 50 \times 2 + 10 \times 4 \times \left(2 + \frac{4}{2}\right) + 40(2 + 4) \\ &= 100 + 160 + 240 = 500\end{aligned}$$

$\therefore R_B = \frac{500}{10} = 50 \text{ kN}$
 and $R_A = \text{Total load on beam} - R_B$
 $= (50 + 10 \times 4 + 40) - 50 = 130 - 50 = 80 \text{ kN}$

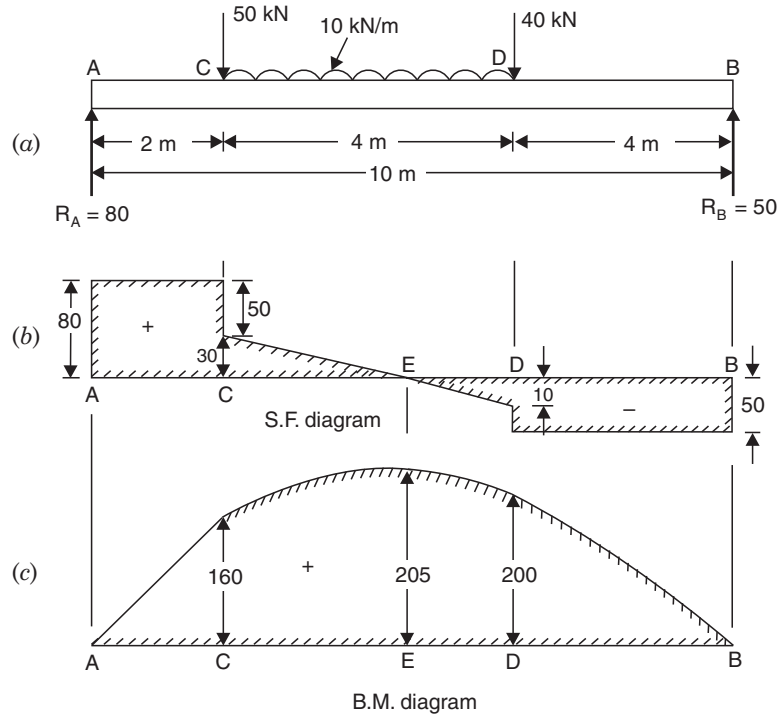


Fig. 8.29

S.F. Diagram

The S.F. at A, $F_A = R_A = +80 \text{ kN}$

The S.F. will remain constant between A and C and equal to $+80 \text{ kN}$

The S.F. just on R.H.S. of C $= R_A - 50 = 80 - 50 = 30 \text{ kN}$

The S.F. just on L.H.S. of D $= R_A - 50 - 10 \times 4 = 80 - 50 - 40 = -10 \text{ kN}$

The S.F. between C and D varies according to straight line law.

The S.F. just on R.H.S. of D $= R_A - 50 - 10 \times 4 - 40 = 80 - 50 - 40 - 40 = -50 \text{ kN}$

The S.F. at B $= -50 \text{ kN}$

The S.F. remains constant between D and B and equal to -50 kN

The shear force diagram is drawn as shown in Fig. 8.29 (b).

The shear force is zero at point E between C and D.

Let the distance of E from point A is x .

$$\begin{aligned}
 \text{Now shear force at E} &= R_A - 50 - 10 \times (x - 2) \\
 &= 80 - 50 - 10x + 20 = 50 - 10x
 \end{aligned}$$

$$\text{But shear force at E} = 0$$

$$\therefore 50 - 10x = 0 \quad \text{or} \quad x = \frac{50}{10} = 5 \text{ m}$$

B.M. Diagram

B.M. at A is zero

B.M. at B is zero

B.M. at C, $M_C = R_A \times 2 = 80 \times 2 = 160 \text{ kNm}$

B.M. at D, $M_D = R_A \times 6 - 50 \times 4 - 10 \times 4 \times \frac{4}{2}$
 $= 80 \times 6 - 200 - 80 = 480 - 200 - 80 = 200 \text{ kNm}$

At E, $x = 5 \text{ m}$ and hence B.M. at E,

$$M_E = F_A \times 5 - 50(5 - 2) - 10 \times (5 - 2) \times \left(\frac{5 - 2}{2}\right)$$

$$= 80 \times 5 - 50 \times 3 - 10 \times 3 \times \frac{3}{2} = 400 - 150 - 45 = 205 \text{ kNm}$$

The B.M. between C and D varies according to parabolic law reaching a maximum value at E. The B.M. between A and C and also between B and D varies according to linear law. The B.M. diagram is shown in Fig. 8.29 (c).

Maximum B.M.

The maximum B.M. is at E, where S.F. becomes zero after changing its sign.

 $\therefore$ Max. B.M. = $M_E = 205 \text{ kNm}$. **Ans.**

8.7.3. Shear Force and B.M. Diagrams for a Simply Supported Beam Carrying a Uniformly Varying Load from Zero at One End to w per Unit Length at the other End. Fig. 8.30 shows a beam AB of length L simply supported at the ends A and B and carrying a uniformly varying load from zero at end A to w per unit length at B. First calculate the reactions R_A and R_B .

Taking moments about A, we get

$$R_B \times L = \left(\frac{w \cdot L}{2}\right) \cdot \frac{2}{3} L \quad \left[\text{Total load } \left(= \frac{w \cdot L}{2}\right) \text{ is acting } \frac{2}{3} L \text{ from A} \right]$$

$$\therefore R_B = \frac{w \cdot L}{3}$$

and

$$R_A = \text{Total load on beam} - R_B = \frac{w \cdot L}{2} - \frac{w \cdot L}{3} = \frac{w \cdot L}{6}$$

Consider any section X at a distance x from end A. The shear force at X is given by,

$$F_x = R_A - \text{load on length AX} = \frac{w \cdot L}{6} - \frac{w \cdot x}{L} \cdot \frac{x}{2}$$

$$\left(\text{Load on AX} = \frac{AX \cdot CX}{2} = \frac{x \cdot w \cdot x}{2 \cdot L} \right)$$

$$= \frac{wL}{6} - \frac{wx^2}{2L} \quad \dots(i)$$

Equation (i) shows that S.F. varies according to parabolic law.

At A, $x = 0$ hence $F_A = \frac{w \cdot L}{6} - \frac{w}{2L} \times 0 = \frac{w \cdot L}{6}$

At B, $x = L$, hence $F_B = \frac{w \cdot L}{6} - \frac{w \cdot L^2}{2L} = \frac{w \cdot L}{6} - \frac{w \cdot L}{2} = \frac{w \cdot L - 3w \cdot L}{6} = -\frac{2w \cdot L}{6} = -\frac{w \cdot L}{3}$

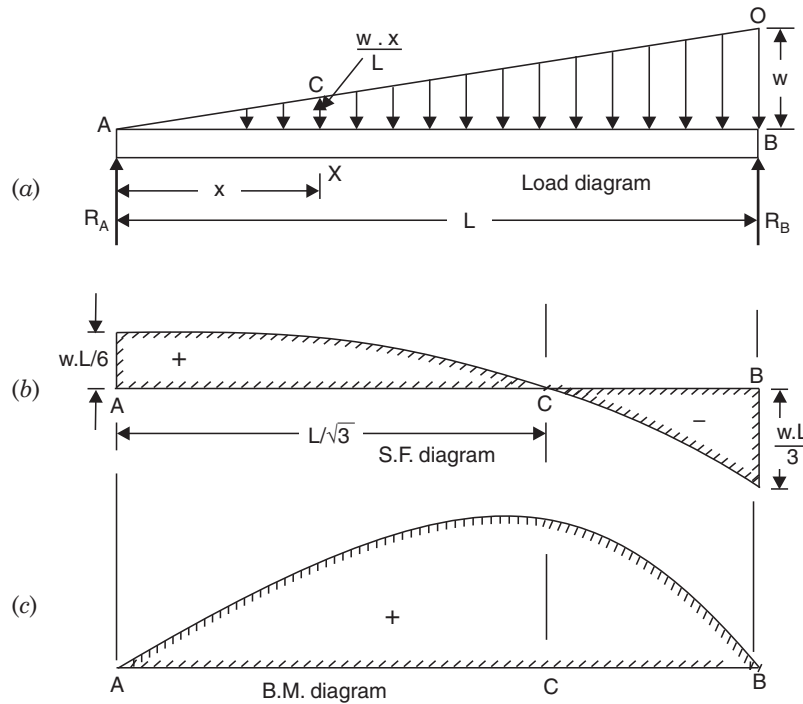


Fig. 8.30

The shear force is $+\frac{w.L}{6}$ at A and it decreases to $-\frac{w.L}{3}$ at B according to parabolic law. Somewhere between A and B, the S.F. must be zero. Let the S.F. be zero to a distance x from A. Equating the S.F. to zero in equation (i), we get

$$0 = \frac{wL}{6} - \frac{wx^2}{2L} \quad \text{or} \quad \frac{wx^2}{2L} = \frac{w.L}{6}$$

or

$$x^2 = \frac{w.L}{6} \times \frac{2L}{w} = \frac{L^2}{3}$$

$\therefore$

$$x = \frac{L}{\sqrt{3}} = 0.577 L$$

B.M. Diagram

The B.M. is zero at A and B.

The B.M. at the section X at a distance x from the end A is given by,

$$M_x = R_A \cdot x - \text{Load on length AX} \cdot \frac{x}{3} \quad \left(\because \text{Load on AX is acting at } \frac{x}{3} \text{ from X} \right)$$

$$= \frac{w.L}{6} \cdot x - \frac{wx^2}{2L} \cdot \frac{x}{3} = \frac{wL}{6} \cdot x - \frac{wx^3}{6L} \quad \dots(ii)$$

Equation (ii) shows the B.M. varies between A and B according to cubic law.

Max. B.M. occurs at a point where S.F. becomes zero after changing its sign.

That point is at a distance of $\frac{L}{\sqrt{3}}$ from A. Hence substituting $x = \frac{L}{\sqrt{3}}$ in equation (ii), we get maximum B.M.

$$\begin{aligned}
 \therefore \text{Max. B.M.} &= \frac{w \cdot L}{6} \cdot \frac{L}{\sqrt{3}} - \frac{w}{6L} \cdot \left(\frac{L}{\sqrt{3}} \right)^3 \\
 &= \frac{wL^2}{6\sqrt{3}} - \frac{wL^2}{18\sqrt{3}} = \frac{3w \cdot L^2 - wL^2}{18\sqrt{3}} = \frac{wL^2}{9\sqrt{3}}.
 \end{aligned}$$

Problem 8.12. Fig. 8.31 shows a beam pivoted at A and simply supported at B and carrying a load varying from 0 at A to 12 kN/m at B. Determine the reactions at A and B, and draw the bending moment (B.M.) diagram. (U.P. Tech. University, I Sem. 2002–2003)

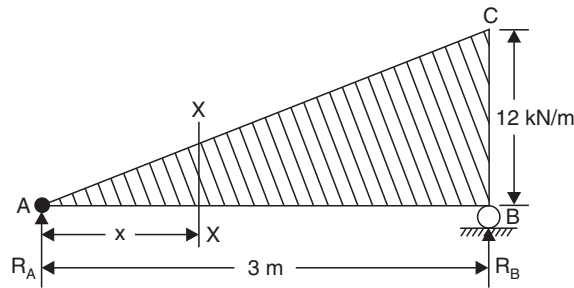


Fig. 8.31

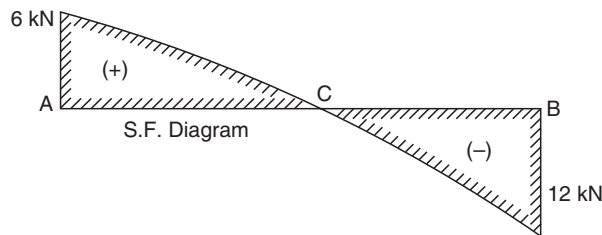


Fig. 8.32

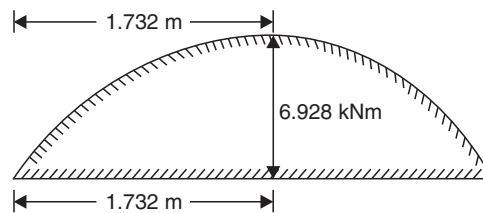


Fig. 8.33

Sol. Let R_A = Reaction at A,
 R_B = Reaction at B.

Total load on the beam

= Area of ABC in Fig. 8.33

$$= \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 3 \times 12 = 18 \text{ kN}$$

For equilibrium of beam, $\Sigma F_y = 0$

$$\therefore R_A + R_B = 18 \text{ kN} \quad \dots(i)$$

Also for equilibrium, $\Sigma M_A = 0$

$$\therefore R_B \times 3 - 18 \times \left(\frac{2}{3} \times 3\right) = 0 \quad (\because \text{Total load will be acting at the C.G. of the triangle } ABC, \text{ which is } 2/3 \text{ of } 3 \text{ from point } A)$$

or $3R_B - 36 = 0$

$$\therefore R_B = \frac{36}{3} = 12 \text{ kN. Ans.}$$

From equation (i), we have

$$R_A = 18 - R_B = 18 - 12 = 6 \text{ kN. Ans.}$$

To draw *BMD*, we have to take the help of *SFD*.

Shear Force Diagram

$$\text{S.F. at } A = R_A = +6 \text{ kN}$$

$$\text{S.F. at } B \text{ due to U.D.L.} = 6 - \frac{1}{2} \times 3 \times 12 = -12 \text{ kN}$$

$$\text{S.F. at } B \text{ due to reaction} = -12 + 12 = 0$$

S.F. at a section *X-X* at a distance *x* from *A* is given by

$$F_x = R_A - \text{Area of triangle } A-X-X.$$

$$\left[\because \text{base} = AX = x; \text{height } X-X \text{ which is given by, } \frac{X-X}{12} = \frac{x}{3} \therefore XX = 4x \right]$$

$$= 6 - \frac{1}{2} \times x \times 4x = 6 - 2x^2 \quad \dots(i)$$

$\therefore$ S.F. diagram will be parabolic as shown in Fig. 8.32.

The S.F. at *A* is +ve whereas S.F. is -ve at *B*. Let us find the point at which the S.F. is zero. The distance of that point at which S.F. is zero, is obtained by putting $F_x = 0$ in equation (i).

$$\therefore 0 = 6 - 2x^2$$

or $x = \sqrt{\frac{6}{2}} = \sqrt{3} = 1.732 \text{ m.}$

B.M. Diagram

$$\text{B.M. at } A = 0$$

$$\begin{aligned} \text{B.M. at section } X-X &= R_A \times x - (\text{Load on } AXX) \times \frac{x}{3} \\ &= 6x - \left(\frac{1}{2} \times x \times 4x\right) \times \frac{x}{3} \\ &= 6x - \frac{2x^3}{3} \quad \dots(ii) \end{aligned}$$

$\therefore$ Variation of B.M. from *A* to *B* follows cubic law.

Let us find the point at which B.M. is maximum.

B.M. will be maximum where S.F. is zero after changing its sign.

S.F. is given by equation (i) as $F_x = 6 - 2x^2$

For max. B.M., $F_x = 0 \therefore 0 = 6 - 2x^2$

or $x = \sqrt{\frac{6}{2}} = \sqrt{3} = 1.732 \text{ m}$

Substituting $x = 1.732$ in equation (ii), we get max. B.M. as

$$(\text{B.M.})_{\max} = 6 \times 1.732 - \frac{2}{3} \times 1.732 = 6.928 \text{ kNm}$$

Now B.M. diagram can be drawn as shown in Fig. 8.33.

8.8. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR OVERHANGING BEAMS

If the end portion of a beam is extended beyond the support, such beam is known as overhanging beam. In case of overhanging beams, the B.M. is positive between the two supports, whereas the B.M. is negative for the overhanging portion. Hence at some point, the B.M. is zero after changing its sign from positive to negative or *vice-versa*. That point is known as the *point of contraflexure* or *point of inflexion*.

8.8.1. Point of Contraflexure. It is the point where the B.M. is zero after changing its sign from positive to negative or *vice-versa*.

Problem 8.13. Draw the shear force and bending moment diagrams for the overhanging beam carrying uniformly distributed load of 2 kN/m over the entire length as shown in Fig. 8.34. Also locate the point of contraflexure.

Sol. First calculate the reactions R_A and R_B

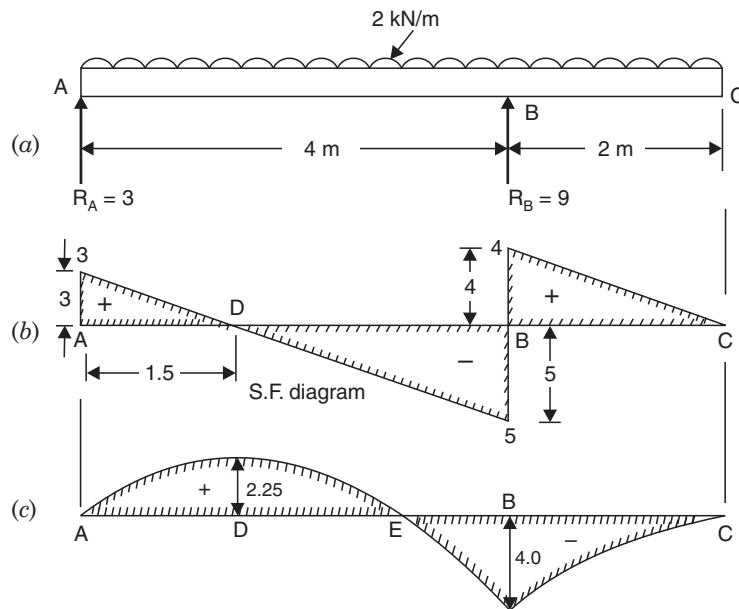
Taking moments of all forces about A, we get

$$R_B \times 4 = 2 \times 6 \times \frac{6}{2} = 36 \quad (\because \text{Total load on beam} = 2 \times 6 = 12 \text{ kN. This load is acting at a distance 3 m from A})$$

$$\therefore R_B = \frac{36}{4} = 9 \text{ kN}$$

and

$$R_A = \text{Total load} - R_B = 2 \times 6 - 9 = 3 \text{ kN}$$



B.M. diagram

Fig. 8.34

Shear Force Diagram

Shear force at A $= + R_A = + 3 \text{ kN}$

(i) The shear force at any section between A and B at a distance x from A is given by,

$$\begin{aligned} F_A &= R_A - 2x & (\because R_A = 3) \\ &= 3 - 2x & \dots(i) \end{aligned}$$

At A, $x = 0$ hence $F_A = 3 \text{ kN}$

At B, $x = 4$ hence $F_B = 3 - 2 \times 4 = - 5 \text{ kN}$

The shear force varies according to straight line law between A and B. At A, the shear force is positive whereas at B, the shear force is negative. Between A and B somewhere S.F. is zero. The point, where S.F. is zero, is obtained by substituting $F_x = 0$ in equation (i).

$$\therefore 0 = 3 - 2x \quad \text{or} \quad x = \frac{3}{2} = 1.5 \text{ m}$$

Hence S.F. is zero at a distance of 1.5 m from A (or S.F. is zero at point D).

(ii) The S.F. at any section between B and C at a distance x from A is given by,

$$\begin{aligned} F_x &= + R_A - 4 \times 2 + R_B - (x - 4) \times 2 = 3 - 8 + 9 - 2(x - 4) \\ &= 4 - 2(x - 4) & \dots(ii) \end{aligned}$$

At B, $x = 4 \text{ m}$ hence $F_B = 4 - 2(4 - 4) = + 4 \text{ kN}$

At C, $x = 6 \text{ m}$ hence $F_C = 4 - 2(6 - 4) = 0$

Between B and C also S.F. varies by a straight line law. At B, S.F. is $+ 4 \text{ kN}$ and at C, S.F. is zero.

The S.F. diagram is shown in Fig. 8.34 (b).

B.M. Diagram

The B.M. at A is zero.

(i) The B.M. at any section between A and B at a distance x is given by,

$$\begin{aligned} M_x &= R_A \times x - 2 \times x \times \frac{x}{2} \\ &= 3x - x^2 & \dots(iii) \end{aligned}$$

At A, $x = 0$ hence $M_A = 0$

At B, $x = 4$ hence $M_B = 3 \times 4 - 4^2 = - 4 \text{ kNm}$

Max. B.M. occurs at D, where S.F. is zero after changing its sign.

At D, $x = 1.5$ hence $M_D = 3 \times 1.5 - 1.5^2 = 4.5 - 2.25 = 2.25 \text{ kNm}$

The B.M. between A and B varies according to parabolic law.

(ii) The B.M. at any section between B and C at a distance x is given by,

$$\begin{aligned} M_x &= R_A \times x - 2 \times x \times \frac{x}{2} + R_B \times (x - 4) \\ &= 3x - x^2 + 9(x - 4) & \dots(iv) \end{aligned}$$

At B, $x = 4$ hence $M_B = 3 \times 4 - 4^2 + 9(4 - 4) = 4 \text{ kNm}$

At C, $x = 6$ hence $M_C = 3 \times 6 - 6^2 + 9(6 - 4) = 18 - 36 + 18 = 0$

The B.M. diagram is shown in Fig. 8.34 (c).

Point of Contraflexure

This point will be between A and B where B.M. is zero after changing its sign. But B.M. at any section at a distance x from A between A and B is given by equation (iii) as

$$M_x = 3x - x^2$$

Equation M_x to zero for point of contraflexure, we get

$$0 = 3x - x^2 = x(3 - x)$$

or

$$3 - x = 0$$

($\because$ x cannot be zero as B.M. is not changing sign at this point)

$$\therefore x = 3$$

Hence point of contraflexure will be at a distance of 3 m from A.

Problem 8.14. Draw the shear force diagram of the beam shown in Fig. 8.35.

(U.P. Tech. University, 2000–2001)

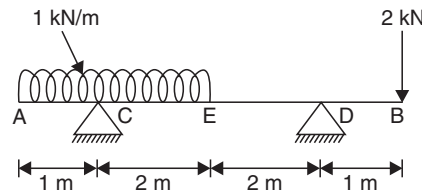


Fig. 8.35

Sol. First calculate the reactions R_C and R_D .

Total load on the portion $AE = 1 \times (1 + 2) = 3$ kN. This load will be acting at the middle point of AE i.e. at a distance of 1.5 m from point A as shown in Fig. 8.35 (a).

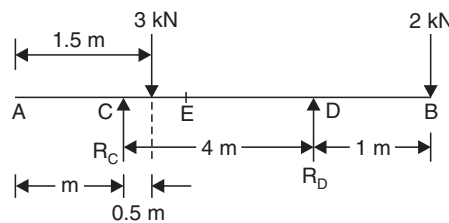


Fig. 8.35 (a)

Taking moments about point C and equating the resultant moment to zero for equilibrium, we get,

$$3 \times 0.5 + 2 \times (4 + 1) - R_D \times 4 = 0$$

or

$$1.5 + 10 - 4R_D = 0$$

or

$$4R_D = 11.5 \quad \therefore R_D = \frac{11.5}{4} = 2.875 \text{ kN}$$

$$\text{Also } \Sigma F_y = 0 \quad \text{or} \quad R_D + R_C = 3 + 2 = 5 \text{ kN}$$

$$\therefore R_C = 5 - R_D = 5 - 2.875 = 2.125 \text{ kN}$$

Now the shear force diagram can be drawn.

SFD of the given beam [Refer to Fig. 8.35 (b)]

(i) S.F. at A = 0.

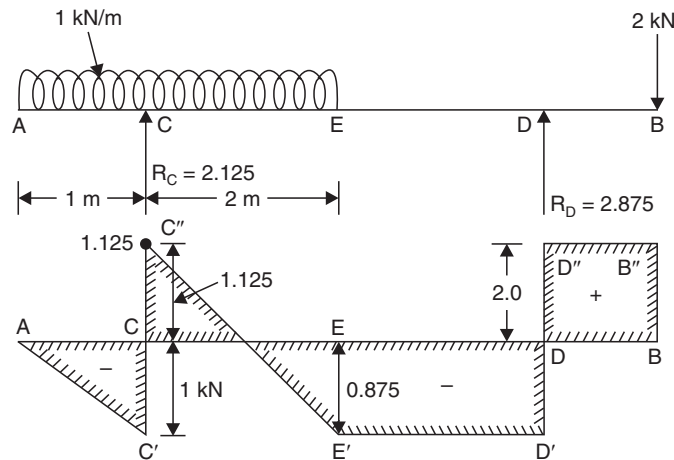


Fig. 8.35 (b)

(ii) S.F. from A to C goes on decreasing at the rate of 1 kN/m. Hence shear force at C due to U.D.L. is $1 \times 1 \text{ kN/m} = 1 \text{ kN}$ downward. This is shown by point C' in which $CC' = 1 \text{ kN}$ downward. Join A to C'.

(iii) At point C, there is reaction R_C acting upward. Hence shear force at C due to reaction R_C will be $R_C = 2.125 \text{ kN}$ upward. Hence take $C'C'' = 2.125 \text{ kN}$. Then CC'' will be $(2.125 - 1) = 1.125 \text{ kN}$ upward.

(iv) S.F. from C to E goes on decreasing at the rate of 1 kN/m. Hence shear force at E will be $1 \text{ (kN/m)} \times 2 \text{ m} = 2 \text{ kN}$ less than that C''. Hence EE' will be $1.125 - 2 = -0.875 \text{ kN}$. Negative sign means, the point E' will be below point E.

(v) No load is acting between ED on the beam. Hence shear force will be constant. The shear force between ED is represented by horizontal line $E'D'$ in SFD.

(vi) At point D, the reaction $R_D = 2.875 \text{ kN}$ is acting upwards. Hence $D'D''$ will be equal to 2.875 kN . But $DD' = 0.875$

$$\therefore DD'' = 2.875 - 0.875 = 2 \text{ kN}.$$

(vii) The S.F. between DB will be constant and equal to 2 kN. It is represented by $D''B'$.

(viii) At point B, the load 2 kN is acting downward. Hence $B'B$ will be equal to 2 kN.

$$\therefore A-C'-C''-E'-D'-D''-B'-B \text{ is a SFD.}$$

Problem 8.15. Draw the S.F. and B.M. diagrams for the overhanging beam carrying uniformly distributed load of 2 kN/m over the entire length and a point load of 2 kN as shown in Fig. 8.36. Locate the point of contraflexure.

Sol. First calculate the reactions R_A and R_B .

Taking moments of all forces about A, we get

$$R_B \times 4 = 2 \times 6 \times 3 + 2 \times 6 = 36 + 12 = 48$$

$$\therefore R_B = \frac{48}{4} = 12 \text{ kN}$$

and

$$R_A = \text{Total load} - R_B = (2 \times 6 + 2) - 12 = 2 \text{ kN}$$

S.F. Diagram

$$\text{S.F. at A} = + R_A = + 2 \text{ kN}$$

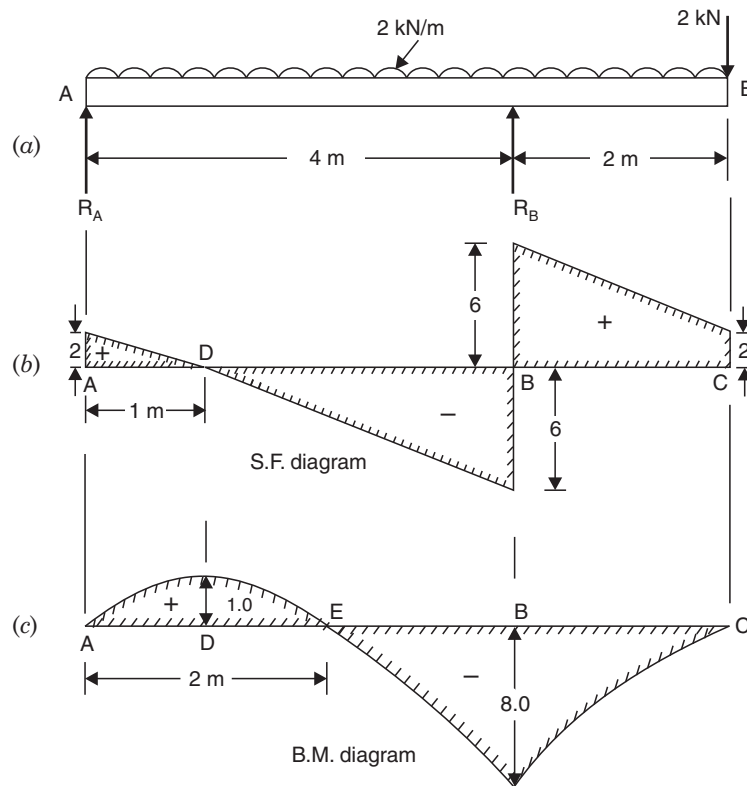


Fig. 8.36

(i) The S.F. at any section between A and B at a distance x from A is given by,

$$\begin{aligned} F_x &= +R_A - 2 \times x \\ &= 2 - 2x \end{aligned} \quad \dots(i)$$

At A , $x = 0$ hence $F_A = 2 - 2 \times 0 = 2 \text{ kN}$

At B , $x = 4$ hence $F_B = 2 - 2 \times 4 = -6 \text{ kN}$

The S.F. between A and B varies according to straight line law. At A , S.F. is positive and at B , S.F. is negative. Hence between A and B , S.F. is zero. The point of zero S.F. is obtained by substituting $F_x = 0$ in equation (i).

$$\therefore 0 = 2 - 2x \quad \text{or} \quad x = \frac{2}{2} = 1 \text{ m}$$

The S.F. is zero at point D . Hence distance of D from A is 1 m.

(ii) The S.F. at any section between B and C at a distance x from A is given by,

$$\begin{aligned} F_x &= +R_A - 2 \times 4 + R_B - 2(x - 4) \\ &= 2 - 8 + 12 - 2(x - 4) = 6 - 2(x - 4) \end{aligned} \quad \dots(ii)$$

At B , $x = 4$ hence $F_B = 6 - 2(4 - 4) = +6 \text{ kN}$

At C , $x = 6$ hence $F_C = 6 - 2(6 - 4) = 6 - 4 = 2 \text{ kN}$

The S.F. diagram is drawn as shown in Fig. 8.36 (b).

B.M. Diagram

B.M. at A is zero

(i) B.M. at any section between A and B at a distance x from A is given by,

$$M_x = R_A \times x - 2 \times x \times \frac{x}{2} = 2x - x^2 \quad \dots(iii)$$

The above equation shows that the B.M. between A and B varies according to parabolic law.

At A, $x = 0$ hence $M_A = 0$

At B, $x = 4$ hence $M_B = 2 \times 4 - 4^2 = -8 \text{ kNm}$

Max. B.M. is at D where S.F. is zero after changing sign

At D, $x = 1$ hence $M_D = 2 \times 1 - 1^2 = 1 \text{ kNm}$

The B.M. at C is zero. The B.M. also varies between B and C according to parabolic law. Now the B.M. diagram is drawn as shown in Fig. 8.36 (c).

Point of Contraflexure

This point is at E between A and B, where B.M. is zero after changing its sign. The distance of E from A is obtained by putting $M_x = 0$ in equation (iii).

$$\begin{aligned} \therefore 0 &= 2x - x^2 = x(2 - x) \\ 2 - x &= 0 \end{aligned}$$

and

$$x = 2 \text{ m. Ans.}$$

Problem 8.16. A beam of length 12 m is simply supported at two supports which are 8 m apart, with an overhang of 2 m on each side as shown in Fig. 8.37. The beam carries a concentrated load of 1000 N at each end. Draw S.F. and B.M. diagrams.

Sol. As the loading on the beam is symmetrical. Hence reactions R_A and R_B will be equal and their magnitude will be half of the total load.

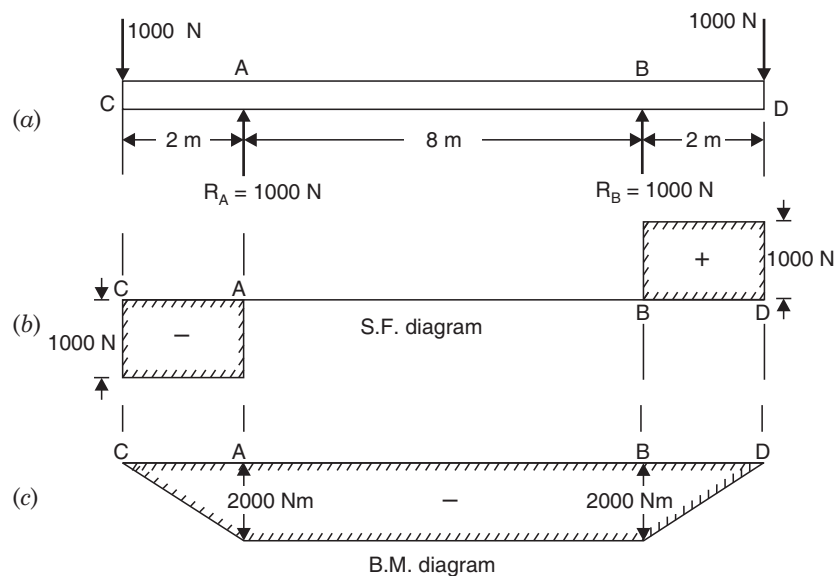


Fig. 8.37

$$\therefore R_A = R_B = \frac{(1000 + 1000)}{2} = 1000 \text{ N}$$

S.F. at C = - 1000 N
 S.F. remains constant (i.e., = - 1000 N) between C and A
 S.F. at A = - 1000 + R_A = - 1000 + 1000 = 0
 S.F. remains constant (i.e., = 0) between A and B
 S.F. at B = 0 + 1000 = + 1000 N
 S.F. remains constant (i.e., = 1000 N) between B and D
 S.F. diagram is drawn as shown in Fig. 8.37 (b).

B.M. Diagram

B.M. at C = 0

B.M. at A = - 1000 × 2 = - 2000 Nm (-ve sign is due to hogging moment)

B.M. between C and A varies according to straight line law.

The B.M. at any section in AB at a distance of x from C is given by,

$$\begin{aligned} M_x &= - 1000 \times x + R_A(x - 2) \\ &= - 1000 \times x + 1000(x - 2) = - 2000 \text{ Nm} \end{aligned}$$

Hence B.M. between A and B is constant and equal to - 2000 Nm.

B.M. at D = 0.

$\therefore$ B.M. diagram is shown in Fig. 8.37 (c).

Note. In this particular case, the S.F. is zero between AB and B.M. is constant. Hence length AB is subjected to only constant B.M. The length between A and B is absolutely free from shear force.

Problem 8.17. Draw the S.F. and B.M. diagrams for the beam which is loaded as shown in Fig. 8.38. Determine the points of contraflexure within the span AB.

Sol. First calculate the reactions R_A and R_B .

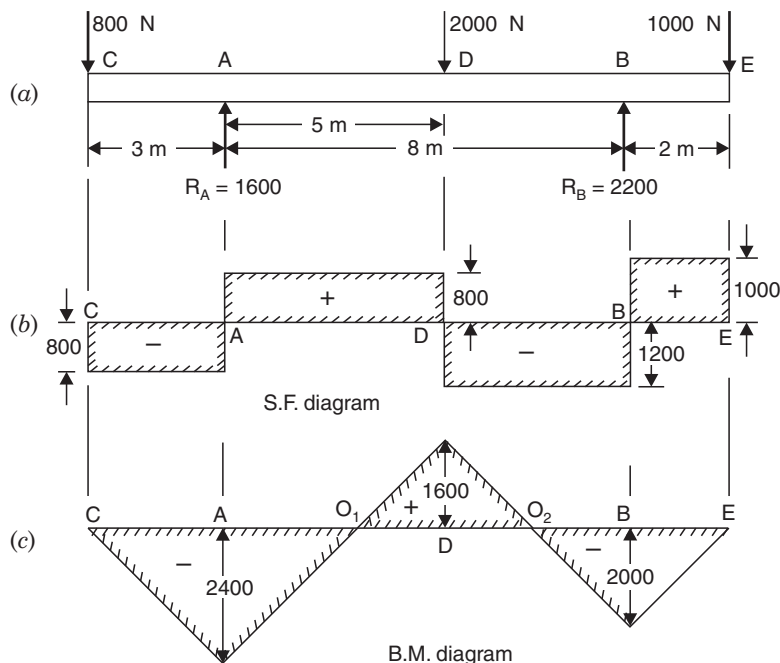


Fig. 8.38

Taking moments about A, we have

$$R_B \times 8 + 800 \times 3 = 2000 \times 5 + 1000(8 + 2)$$

or

$$8R_B + 2400 = 10000 + 10000$$

$$\therefore R_B = \frac{20000 - 2400}{8} = \frac{17600}{8} = 2200 \text{ N}$$

and

$$R_A = \text{Total load} - R_B = 3800 - 2200 = 1600$$

S.F. Diagram

$$\text{S.F. at } C = -800 \text{ N}$$

$$\text{S.F. between } C \text{ and } A \text{ remains } -800 \text{ N}$$

$$\text{S.F. at } A = -800 + R_A = -800 + 1600 = +800 \text{ N}$$

$$\text{S.F. between } A \text{ and } D \text{ remains } +800 \text{ N}$$

$$\text{S.F. at } D = +800 - 2000 = -1200 \text{ N}$$

$$\text{S.F. between } D \text{ and } B \text{ remains } -1200 \text{ N}$$

$$\text{S.F. at } B = -1200 + R_B = -1200 + 2200 = +1000 \text{ N}$$

$$\text{S.F. between } B \text{ and } E \text{ remains } +1000 \text{ N}$$

S.F. diagram is shown in Fig. 8.38 (b).

B.M. Diagram

$$\text{B.M. at } C = 0$$

$$\text{B.M. at } A = -800 \times 3 = -2400 \text{ Nm}$$

$$\begin{aligned} \text{B.M. at } D &= -800 \times (3 + 5) + R_A \times 5 \\ &= -800 \times 8 + 1600 \times 5 \\ &= -6400 + 8000 = +1600 \text{ Nm} \end{aligned}$$

$$\text{B.M. at } B = -1000 \times 2 = -2000 \text{ Nm}$$

$$\text{B.M. at } E = 0$$

The B.M. diagram is drawn as shown in Fig. 8.38 (c).

Points of Contraflexure

There will be two points of contraflexure O_1 and O_2 , where B.M. becomes zero after changing its sign. Point O_1 lies between A and D, whereas the point O_2 lies between D and B.

(i) Let the point O_1 is x metre from A.

$$\begin{aligned} \text{Then B.M. at } O_1 &= -800(3 + x) + R_A \times x = -800(3 + x) + 1600x \\ &= -2400 - 800x + 1600x = -2400 + 800x \end{aligned}$$

But B.M. at O_1 is zero

$$\therefore O = -2400 + 800x \quad \text{or} \quad x = \frac{2400}{800} = 3 \text{ m. Ans.}$$

(ii) Let the point O be x metre from B.

$$\text{Then B.M. at } O_2 = 1000(x + 2) - R_B \times x = 1000x + 2000 - 2200 \times x = 2000 - 1200x$$

$$\text{But B.M. at } O_2 = 0$$

$$\therefore O = 2000 - 1200x$$

$$\therefore x = \frac{2000}{1200} = \frac{5}{3} = 1.67 \text{ m from B. Ans.}$$

8.9. SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR BEAMS SUBJECTED TO COUPLES

When a beam is subjected to a couple at a section, only the bending moment at the section of the couple changes suddenly in magnitude equal to that of the couple. But the S.F. does not change at the section of the couple as there is no change in load due to couple at the section. But while calculating the reactions, the magnitude of the couple is taken into account.

The sudden change in B.M. at the section of the couple can also be obtained by calculating B.M. separately with the help of both the reactions.

Problem 8.17(A). A simply supported beam AB of length 6 m is hinged at A and B . It is subjected to a clockwise couple of 24 kNm at a distance of 2 m from the left end A . Draw the S.F. and B.M. diagrams.

Sol. Fig. 8.39 (a) shows the simply supported beam AB , hinged at A and B . The clockwise couple at C will try to lift the beam up at the support A , and to depress the beam down at the support B . Hence the reaction at A will be downwards and at B the reaction will be upwards as shown in Fig. 8.39 (b).

To find reactions of R_A and R_B , take the moments about A .

$$\therefore R_B \times 6 - 24 = 0 \quad (\because \text{Moment due to } R_B \text{ is anti-clockwise and moment at } C \text{ is clockwise})$$

$$\therefore R_B = \frac{24}{6} = 4 \text{ kN } \uparrow$$

Since there is no external vertical load on the beam, therefore the reaction at A will be the same, as that of B , but in opposite direction.

$$\begin{aligned} \therefore R_A &= \text{Load on beam} - R_B & (\because \text{Load on beam} = 0) \\ &= -R_B = -4 \text{ kN}. \end{aligned}$$

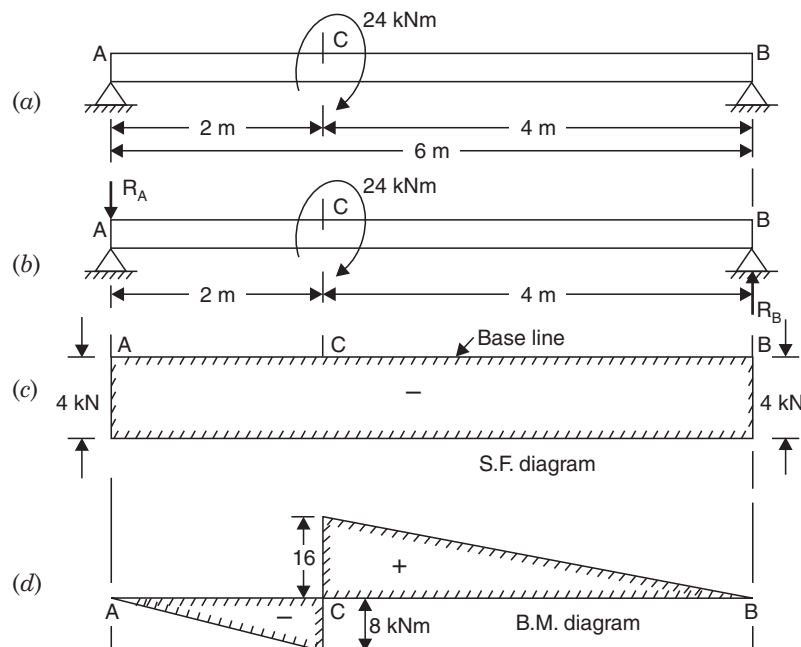


Fig. 8.39

S.F. Diagram

$$\text{S.F. at A} = R_A = -4 \text{ kN}$$

The S.F. remains constant (*i.e.*, equal to -4 kN) between *A* and *B*.

The S.F. diagram is shown in Fig. 8.39 (*c*).

B.M. Diagram

$$\text{B.M. at A} = 0$$

$$\text{B.M. just on the L.H.S. of C} = R_A \times 2 = -4 \times 2 = -8 \text{ kNm}$$

$$\text{B.M. just on the R.H.S. of C} = R_B \times 4 = 4 \times 4 = +16 \text{ kNm}$$

(B.M. just on the R.H.S. of *C* can also be calculated as the sum of moments due to R_A and moment due to couple. But moment due to R_A is anti-clockwise whereas due to couple is clockwise. Hence net B.M. on R.H.S. of *C* $= -8 + 24 = +16 \text{ kNm}$).

There is a sudden change in B.M. at *C* due to couple.

$$\text{B.M. at B} = 0$$

B.M. diagram is shown in Fig. 8.39 (*d*).

Problem 8.18. Draw the shear force diagram for the beam *AB* shown in Fig. 8.40 loaded through the attached strut.
(U.P. Tech. University, 2002–2003)

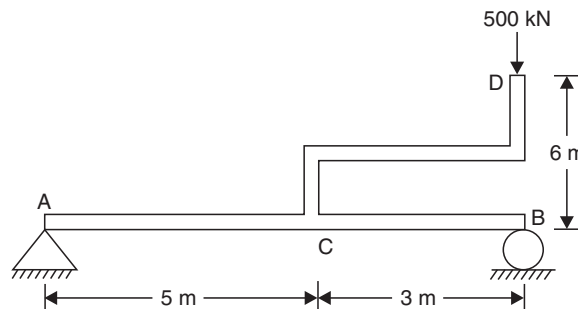


Fig. 8.40

Sol. Due to attached strut, the load of 500 kN will be acting downward at the point *C* and also there will be a clockwise moment of magnitude ($500 \times 3 = 1500 \text{ kNm}$) acting at point *C* as shown in Fig. 8.41 below.

Let R_A = Reaction at *A*

R_B = Reaction at *B*.

Then $R_A + R_B = 500 \text{ kN}$.

The moments about point *A* should be zero for the equilibrium of the beam.

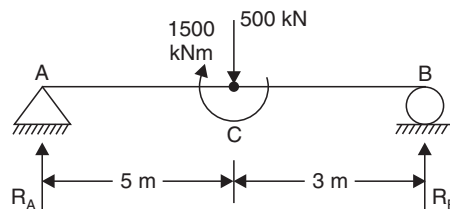


Fig. 8.41

Taking moments about A, we get

$$R_B \times 8 - 500 \times 5 - 1500 = 0$$

$$\therefore 8R_B = 2500 + 1500 = 4000$$

$$\therefore R_B = \frac{4000}{8} = 500 \text{ kN}$$

But $R_A + R_B = 500$

$$\therefore R_A = 500 - R_B = 500 - 500 = 0$$

SFD

(i) Shear force at A = $R_A = 0$

(ii) Shear force will be zero from A to C as there is no load from A to C.

(iii) Shear force at C = 500 kN

(iv) Shear force will be equal to 500 kN from C to B.

(v) Shear force at B = $-500 + 500 = 0$.

Now the shear force diagram can be drawn as shown in Fig. 8.42.

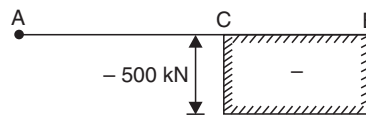


Fig. 8.42

Problem 8.19. A beam 10 m long and simply supported at each end, has a uniformly distributed load of 1000 N/m extending from the left end upto the centre of the beam. There is also an anti-clockwise couple of 15 kNm at a distance of 2.5 m from the right end. Draw the S.F. and B.M. diagrams.

Sol. The reaction at A will be upwards. To find whether the reaction at B is upwards or downwards, take the moments about A.

The following are the moments at A :

(i) Moment due to U.D.L. = $1000 \times 5 \times \frac{5}{2} = 12500 \text{ Nm}$ (clockwise)

(ii) Moment of couple = 15000 Nm (Anti-clockwise)

$$\therefore \text{Net moment} = 15000 - 12500 = 2500 \text{ Nm (Anti-clockwise)}$$

This moment must be balanced by the moments due to reaction at B. Hence the moment about A due to reaction at B should be equal to 2500 Nm (clockwise). This is only possible when R_B is acting downwards. This is shown in Fig. 8.43 (b).

$$\therefore R_B \times 10 = 2500$$

$$\therefore R_B = \frac{2500}{10} = 250 \text{ N}$$

and

$$\begin{aligned} R_A &= \text{Total load on beam} + R_B \\ &\quad \text{(Here } R_B \text{ is +ve as acting downwards)} \\ &= 1000 \times 5 + 250 = 5250 \text{ N.} \end{aligned}$$

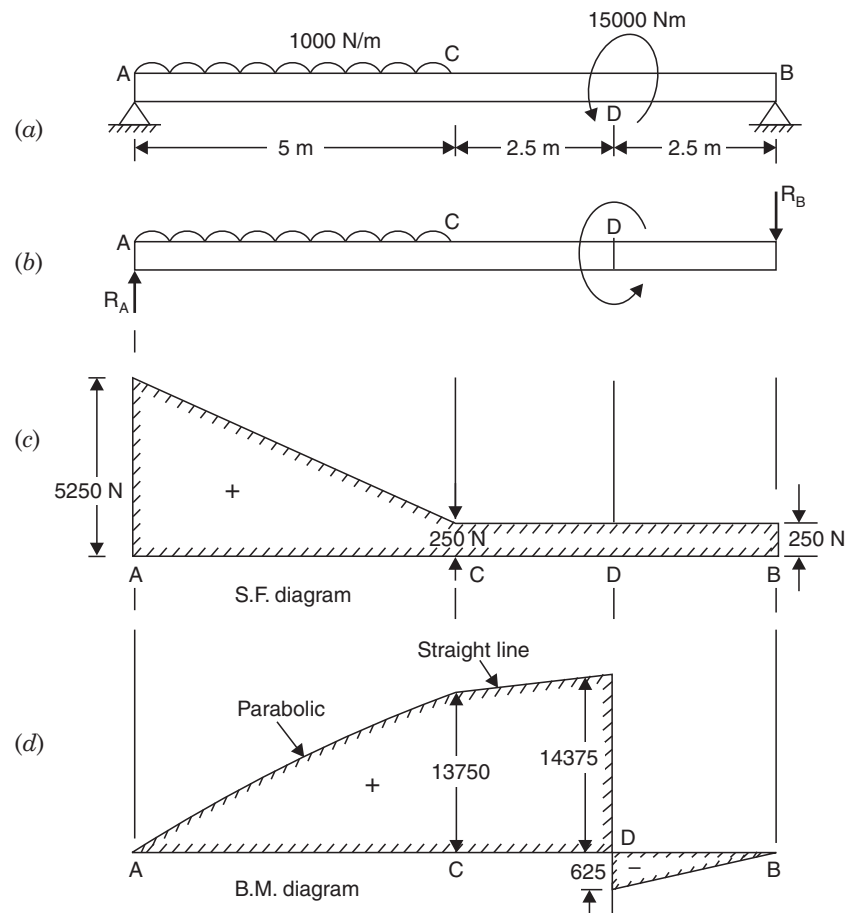


Fig. 8.43

S.F. Diagram

$$\text{S.F. at A} = +R_A = 5250 \text{ N}$$

$$\text{S.F. at C} = 5250 - 5 \times 1000 = +250 \text{ N}$$

S.F. between A and C varies according to straight line law.

S.F. between C and B remains constant and equal to +250 N

S.F. diagram is shown in Fig. 8.43 (c).

B.M. Diagram

$$\text{B.M. at A} = 0$$

$$\begin{aligned} \text{B.M. at C} &= R_A \times 5 - 1000 \times 5 \times \frac{5}{2} \\ &= 5250 \times 5 - 12500 = 13750 \text{ Nm} \end{aligned}$$

B.M. just on the left hand side of D

$$\begin{aligned} &= 5250 \times 7.5 - 1000 \times 5 \times \left(\frac{5}{2} + 2.5 \right) \\ &= 39375 - 25000 = 14375 \text{ Nm} \end{aligned}$$

B.M. just on the right hand side of D

$$= -R_B \times 2.5 = -250 \times 2.5 = -625 \text{ Nm}$$

B.M. at B = 0

The B.M. diagram is shown in Fig. 8.43 (d).

8.10. RELATIONS BETWEEN LOAD, SHEAR FORCE AND BENDING MOMENT

Fig. 8.44 shows a beam carrying a uniformly distributed load of w per unit length. Consider the equilibrium of the portion of the beam between sections 1-1 and 2-2. This portion is at a distance of x from left support and is of length dx .

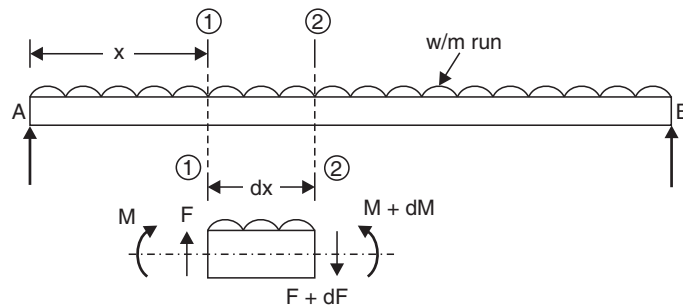


Fig. 8.44

Let
 F = Shear force at the section 1-1,
 $F + dF$ = Shear force at the section 2-2,
 M = Bending moment at the section 1-1,
 $M + dM$ = Bending moment at the section 2-2.

The forces and moments acting on the length ' dx ' of the beam are :

- (i) The force F acting vertically up at the section 1-1.
- (ii) The force $F + dF$ acting vertically downwards at the section 2-2.
- (iii) The load $w \times dx$ acting downwards.
- (iv) The moments M and $(M + dM)$ acting at section 1-1 and section 2-2 respectively.

The portion of the beam of length dx is in equilibrium. Hence resolving the forces acting on this part vertically, we get

$$F - w \cdot dx - (F + dF) = 0$$

$$\text{or} \quad -dF = w \cdot dx \quad \text{or} \quad \frac{dF}{dx} = -w.$$

The above equation shows that the rate of change of shear force is equal to the rate of loading.

Taking the moments of the forces and couples about the section 2-2, we get

$$M - w \cdot dx \cdot \frac{dx}{2} + F \cdot dx = M + dM$$

$$\text{or} \quad -\frac{w(dx)^2}{2} + F \cdot dx = dM$$

Neglecting the higher powers of small quantities, we get

$$F \cdot dx = dM$$

or
$$F = \frac{dM}{dx} \quad \text{or} \quad \frac{dM}{dx} = F.$$

The above equation shows that the rate of change of bending moment is equal to the shear force at the section.

8.11. SUPPORT REACTIONS FROM GIVEN SHEAR FORCE DIAGRAM OR GIVEN BENDING MOMENT DIAGRAM

Sometimes the shear force diagram (SFD) or B.M. Diagram is given and it is required to find the support reactions. The procedure is described by taking the following solved problems.

Problem 8.20. The shear force diagram (SFD) of a simply supported beam is given in Fig. 8.45. Calculate the support reactions of the beam and also draw the bending moment diagram of the beam. (U.P. Tech. University, 2001)

Sol.

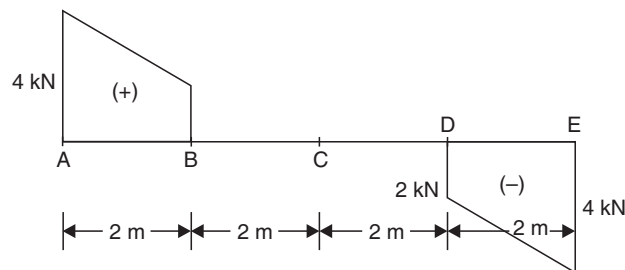


Fig. 8.45

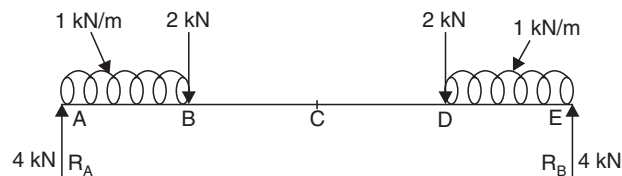


Fig. 8.46

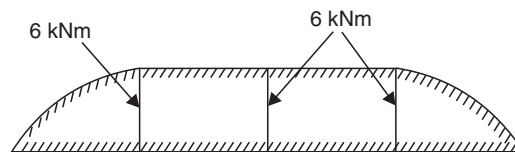


Fig. 8.46 (a)

(i) Support Reactions

The shear force diagram is given in Fig. 8.45. As the shear force at A is 4 kN (+ve). Hence reaction at A will be equal to 4 kN. Hence $R_A = 4$ kN acting vertically upward. The shear force at E is 4 kN (-ve). Hence reaction at E will be equal to 4 kN and acting vertically upward.

Hence $R_A = R_E = 4$ kN. **Ans.**

(ii) B.M. Diagram (Bending Moment Diagram)

First calculate the load on the simply supported beam from shear force diagram. The reactions R_A and R_E have been obtained already. From Fig. 8.46, it is clear that shear force decreases linearly from A to B. This is only possible, if U.D.L. is acting between length AB. The decrease in shear force is $4 - 2 = 2$ kN in a length of 2 m. Hence rate of loading (or U.D.L.)

between AB is $\frac{2 \text{ kN}}{\text{m}} = 1 \text{ kN/m}$. The net S.F. at B zero. This is possible if a point load of magnitude

2 kN is acting downward at point B. The S.F. for portion BD is zero. Hence there is no load between BD. At point D, there is net shear force = -2 kN. Hence a vertical load of 2 kN is acting downward at D. Again the shear force changes linearly between DE. This is only possible if U.D.L. is acting between DE. The rate of loading (or U.D.L.) = Change of S.F. between DE

divided by length DE = $(4 \text{ kN} - 2 \text{ kN}) \div 2 = \frac{2 \text{ kN}}{2 \text{ m}} = 1 \text{ kN/m}$.

Hence the load acting on the simply support beam is shown in Fig. 8.46.

B.M. Diagram

B.M. at A = 0

B.M. at B = $4 \times 2 - 2 \times 1 \times \frac{2}{2} = 6 \text{ kNm}$

B.M. at C (Middle point) = $4 \times 4 - 2 \times 1 \times (4 - 1) - 2 \times 2 = 16 - 6 - 4 = 6 \text{ kNm}$

B.M. at D = $4 \times 6 - 2 \times 1 \times (6 - 1) - 2 \times 4 = 24 - 10 - 8 = 6 \text{ kN}$.

Now the B.M. Diagram can be drawn as shown in Fig. 8.46 (a).

Problem 8.21. The B.M.D. of a simply supported beam is given below. Calculate the support reactions of the beam. (U.P. Tech. University, 2000–2001)

Sol. Given :

B.M. at A and B = 0,

B.M. at C = 7 kNm and at D = 5 kNm

As the B.M.D. is positive and B.M. at A and B are zero, hence the beam is simply supported at A and B. (For a simply supported beam, B.M. is always zero at supports)

Let R_A = Reaction at A and
 R_B = Reaction at B.

Now as the B.M.D. curve is shown by straight lines, hence only point loads are acting on the beam. From the given B.M.D. curve, it is clear that there are two point loads at point C and D.

Let W_C = Point load at C and
 W_D = Point load at D

The reactions and point loads are shown in Fig. 8.47 (a).

The B.M. at C = $R_A \times 1$

But B.M. at C is given as = 7 kN m

$\therefore R_A \times 1 = 7$ or $R_A = 7 \text{ kN}$

The B.M. at D = $R_A \times 2 - W_C \times 1$
 $= 7 \times 2 - W_C$

($\because R_A = 7 \text{ kN}$)

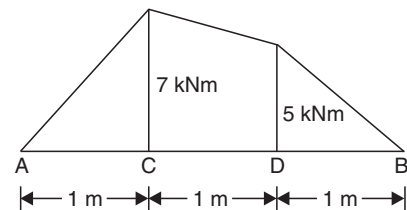


Fig. 8.47

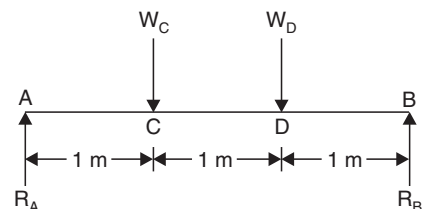


Fig. 8.47 (a)

But B.M. at D is given as = 5 kNm

$$\therefore 5 = 7 \times 2 - W_C$$

or
$$W_C = 14 - 5 = 9 \text{ N}$$

The B.M. at D is also = $R_B \times 1$

But B.M. at D is given as = 5 kNm

$$\therefore R_B \times 1 = 5$$

$$\therefore R_B = 5 \text{ kN.}$$

Hence support reactions are :

$$R_A = 7 \text{ kN. Ans.}$$

$$R_B = 5 \text{ kN. Ans.}$$

HIGHLIGHTS

1. Shear force at a section is the resultant vertical force to the right or left of the section.
2. The diagram which shows the variation of the shear force along the length of a beam, is known as shear force diagram.
3. Bending moment at a section is algebraic sum of the moments of all the forces acting to the left or right of the section.
4. The diagram which shows the variation of the bending moment along the length of a beam, is known as bending moment diagram.
5. A beam which is fixed at one end and free at the other is known as cantilever beam. But a beam supported or resting freely on the supports at its both ends, is known as simply supported beam.
6. If the end portion of a beam is extended beyond the support, such beam is known as overhanging beam.
7. A load acting at a point, is known as concentrated load or a point load.
8. If a left portion of a section is considered, then S.F. will be positive at the section if the resultant of the vertical forces (including reactions) to the left of the section is upwards. But if the resultant is acting downwards then S.F. at the section will be negative.
9. If a right portion of a section is considered, the S.F. will be positive at the section if the resultant of the vertical forces to the right of the section is downwards. But if the resultant is acting upwards then S.F. at the section will be negative.
10. If a left portion of a section is considered, the B.M. will be positive at the section if the moment of all vertical forces and of reaction, at the section is clockwise. But if the resultant moment at the section is anti-clockwise, then B.M. will be negative.
11. If a right portion of a section is considered, the B.M. will be positive at the section if the resultant moment at the section is anti-clockwise. But if the resultant moment at the section is clockwise, then B.M. will be positive.
12. The shear force changes suddenly at a section where there is a vertical point load.
13. The shear force between any two vertical loads remains constant.
14. Shear force for a uniformly distributed load varies according to a straight line law whereas B.M. varies according to a parabolic curve.
15. B.M. is maximum at a section where S.F. is zero after changing its sign.
16. The point where B.M. is zero after changing its sign, is known as point of contraflexure or point of inflexion.
17. When an inclined load is acting on a beam, then inclined load is resolved into two components. Vertical component will cause S.F. and B.M. whereas horizontal component will cause axial thrust in the beam.

18. When a beam is subjected to a couple at a section, then B.M. changes suddenly at the section but S.F. remains unaltered at the section.

EXERCISE 8

(A) Theoretical Questions

- Define and explain the following terms :
Shear force, bending moment, shear force diagram and bending moment diagram.
- What are the different types of beams ? Differentiate between a cantilever and a simply supported beam.
- What are the different types of loads acting on a beam ? Differentiate between a point load and a uniformly distributed load.
- What are the sign conventions for shear force and bending moment in general ?
- Draw the S.F. and B.M. diagrams for a cantilever of length L carrying a point load W at the free end.
- Draw the S.F. and B.M. diagrams for a cantilever of length L carrying a uniformly distributed load of w per m length over its entire length.
- Draw the S.F. and B.M. diagrams for a cantilever of length L carrying a gradually varying load from zero at the free end to w per unit length at the fixed end.
- Draw the S.F. and B.M. diagrams for a simply supported beam of length L carrying a point load W at its middle point.
- Draw the S.F. and B.M. diagrams for a simply supported beam carrying a uniformly distributed load of w per unit length over the entire span. Also calculate the maximum B.M.
- Column II gives maximum bending moments for the following loads. Match the following columns :

<i>Column I</i>	<i>Column II</i>
(i) Cantilever with point load at free end	(P) $W \times L/4$
(ii) Cantilever with uniformly distributed load on the entire cantilever	(Q) $W \times L^2/8$
(iii) Simple supported beam with point load at mid-span	(R) $W \times L$
(iv) Simple supported beam with uniformly distributed load on the entire span	(S) $W \times L^2/8$

[Ans. (i) and (R) ; (ii) and (S) ; (iii) and (P) ; (iv) and (Q)]

(U.P. Tech. University, May 2009)

- What do you mean by point of contraflexure ? Is the point of contraflexure and point of inflexion different ?
- How many points of contraflexure you will have for simply supported beam overhanging at one end only ?

(B) Numerical Problems

- A cantilever beam of length 2 m carries a point load of 1 kN at its free end, and another load of 2 kN at a distance of 1 m from the free end. Draw the S.F. and B.M. diagrams for the cantilever.
[Ans. $F_{max} = + 3 \text{ kN}$; $M_{max} = - 4 \text{ kNm}$]
- A cantilever beam of length 4 m carries point loads of 1 kN, 2 kN and 3 kN at 1, 2 and 4 m from the fixed end. Draw the shear force and B.M. diagrams for the cantilever.
[Ans. $F_{max} = + 6 \text{ kN}$; $M_{max} = - 17 \text{ kNm}$]

3. A cantilever of length 2 m carries a uniformly distributed load of 3 kN/m run over a length of 1 m from the fixed end. Draw the S.F. and B.M. diagrams. [Ans. $F_{max} = + 3 \text{ kN}$; $M_{max} = - 1.5 \text{ kNm}$]
4. A cantilever of length 5 m carries a uniformly distributed load of 2 kN/m length over the whole length and a point load of 4 kN at the free end. Draw the S.F. and B.M. diagrams for the cantilever.
[Ans. $F_{max} = + 14 \text{ kN}$; $M_{max} = - 45 \text{ kNm}$]
5. A cantilever of length 4 m carries a uniformly distributed load of 1 kN/m run over the whole length and a point load of 2 kN at a distance of 1 m from the free end. Draw the S.F. and B.M. diagrams for the cantilever.
[Ans. $F_{max} = + 14 \text{ kN}$; $M_{max} = - 14 \text{ kNm}$]
6. A cantilever 2 m long is loaded with a uniformly distributed load of 2 kN/m run over a length of 1 m from the free end. It also carries a point load of 4 kN at a distance of 0.5 m from the free end. Draw the shear force and B.M. diagrams.
[Ans. $F_{max} = + 6 \text{ kN}$; $M_{max} = - 9 \text{ kNm}$]
7. A cantilever of length 6 m carries two point loads of 2 kN and 3 kN at a distance of 1 m and 6 m from the fixed end respectively. In addition to this the beam also carries a uniformly distributed load of 1 kN/m over a length of 2 m at a distance of 3 m from the fixed end. Draw the S.F. and B.M. diagrams.
[Ans. $F_{max} = + 7 \text{ kN}$; $M_{max} = - 28 \text{ kNm}$]
8. A cantilever of length 6 m carries a gradually varying load, zero at the free end to 2 kN/m at the fixed end. Draw the S.F. and B.M. diagrams for the cantilever.
[Ans. $F_{max} = + 6 \text{ kN}$; $M_{max} = - 12 \text{ kNm}$]
9. A simply supported beam of length 8 m carries point loads of 4 kN and 6 kN at a distance of 2 m and 4 m from the left end. Draw the S.F. and B.M. diagrams for the beam.
[Ans. $M_{max} = + 20 \text{ kNm}$]
10. A simply supported beam of length 10 m carries point loads of 30 kN and 50 kN at a distance of 3 m and 7 m from the left end. Draw the S.F. and B.M. diagrams for the beam.
[Ans. $M_{max} = + 132 \text{ kNm}$]
11. A simply supported beam of length 8 m carries point loads of 4 kN, 10 kN and 7 kN at a distance of 1.5 m, 2.5 m and 2 m respectively from left end A. Draw the S.F. and B.M. diagrams for the simply supported beam.
[Ans. $M_{max} = + 90 \text{ kNm}$]
12. A simply supported beam is carrying a uniformly distributed load of 2 kN/m over a length of 3 m from the right end. The length of the beam is 6 m. Draw the S.F. and B.M. diagrams for the beam and also calculate the maximum B.M. on the section.
[Ans. $M_{max} = + 5.06 \text{ kNm}$]
13. A beam of length 6 m is simply supported at the ends and carries a uniformly distributed load of 1.5 kN/m run and three concentrated loads of 1 kN, 2 kN and 3 kN acting at a distance of 1.5 m, 3 m and 4.5 m respectively from left end. Draw the S.F. and B.M. diagrams and determine the maximum bending moment.
[Ans. 12.75 kNm]
14. A beam of length 10 m is simply supported and carries point loads of 5 kN each at a distance of 3 m and 7 m from left support and also a uniformly distributed load of 1 kN/m between the point loads. Draw S.F. and B.M. diagrams for the beam.
[Ans. $M_{max} = + 23 \text{ kNm}$]
15. A beam of length 6 m is simply supported at its ends. It is loaded with a gradually varying load of 750 N/m from left hand support to 1500 N/m to the right hand support. Construct the S.F. and B.M. diagrams and find the amount and position of the maximum B.M. over the beam.
[Ans. $M_{max} = 5077.5 \text{ Nm}$ at 3.165 m from left hand support]
16. A simply supported beam of length 8 m rests on supports 6 m apart, the right hand end is overhanging by 2 m. The beam carries a uniformly distributed load of 1500 N/m over the entire length. Draw S.F. and B.M. diagrams and find the point of contraflexure, if any.
[Ans. $M_{max} = 5.33 \text{ kNm}$, 5.33 from left hand support]
17. A simply supported beam of length 8 m rests on supports 5 m apart, the right hand end is overhanging by 2 m and the left hand end is overhanging by 1 m. The beam carries a uniformly distributed load of 5 kN/m over the entire length. It also carries two point loads of 4 kN and 6 kN

at each end of the beam. The load of 4 kN is at the extreme left of the beams, whereas the load of 6 kN is at the extreme right of the beam. Draw S.F. and B.M. diagrams for the beam and find the points of contraflexure. [Ans. 1.405 m and 4.955 from the extreme left of the beam]

18. A beam is loaded as shown in Fig. 8.48. Draw the S.F. and B.M. diagrams and find :

(i) maximum S.F.

(ii) maximum B.M.

(iii) point of inflexion.

[Ans. (i) 50 kN ; (ii) 100 kN ; (iii) none]

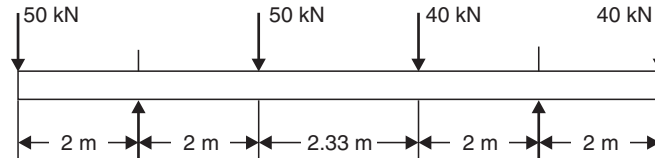


Fig. 8.48

19. A uniformly loaded beam with equal overhang on both sides of the support is shown in Fig. 8.49.

Draw the bending moment diagram when $a = \frac{L}{4}$. (U.P. Tech. University, 2002–2003)

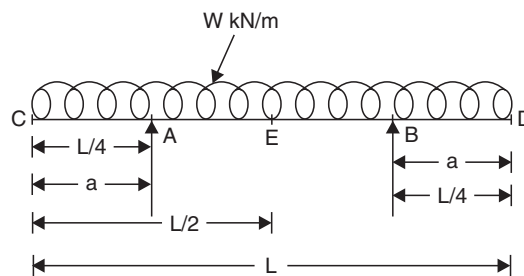


Fig. 8.49

[Hint. Due to symmetry, $R_A = R_B = \frac{W \times L}{2}$]

B.M. calculation

$$M_C = 0, M_D = 0$$

$$M_A = \left(W \times \frac{L}{4} \right) \times \frac{L}{8} = -\frac{WL^2}{32}, M_B = -\frac{WL^2}{32}$$

$$M_E = -\left(W \times \frac{L}{2} \right) \times \frac{L}{4} + R_A \times \frac{L}{4} \\ = -\frac{WL^2}{8} + \frac{WL}{2} \times \frac{L}{4} = -\frac{WL^2}{8} + \frac{WL^2}{8} = 0$$

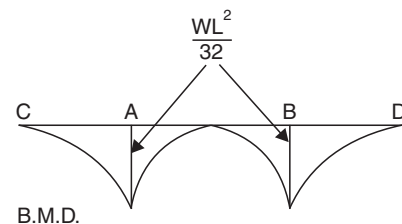


Fig. 8.50

UNIT III

CENTROID AND MOMENT OF INERTIA

Chapter

9. Centroid and Moment of Inertia

Centroid and Moment of Inertia

9.1. CENTRE OF GRAVITY

Centre of gravity of a body is the point through which the whole weight of the body acts. A body is having only one centre of gravity for all positions of the body. It is represented by C.G. or simply G.

9.1.1. Centroid. The point at which the total area of a plane figure (like rectangle, square, triangle, quadrilateral, circle etc.) is assumed to be concentrated, is known as the centroid of that area. The centroid is also represented by C.G. or simply G. The centroid and centre of gravity are at the same point.

9.2. CENTRE OF MASS

The point at which the total mass of a body is assumed to be concentrated. A body is having only one centre of mass for all positions of the body.

9.3. CENTROID OR CENTRE OF GRAVITY OF SIMPLE PLANE FIGURES

- (i) The centre of gravity (C.G.) of a uniform rod lies at its middle point.
- (ii) The centre of gravity of a triangle lies at the point where the three medians* of the triangle meet.
- (iii) The centre of gravity of a rectangle or of a parallelogram is at the point, where its diagonal meet each other. It is also the point of intersection of the lines joining the middle points of the opposite sides.
- (iv) The centre of gravity of a circle is at its centre.

9.4. CENTROID (OR CENTRE OF GRAVITY) OF AREAS OF PLANE FIGURES BY THE METHOD OF MOMENTS

Fig. 9.1 shows a plane figure of total area A whose centre of gravity is to be determined. Let this area A is composed of a number of small areas $a_1, a_2, a_3, a_4, \dots$ etc.

$$\therefore A = a_1 + a_2 + a_3 + a_4 + \dots$$

Let x_1 = The distance of the C.G. of the area a_1 from axis OY

x_2 = The distance of the C.G. of the area a_2 from axis OY

x_3 = The distance of the C.G. of the area a_3 from axis OY

x_4 = The distance of the C.G. of the area a_4 from axis OY

and so on.

The moments of all small areas about the axis OY

$$= a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \dots \quad \dots(i)$$

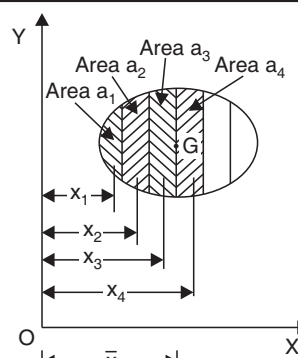


Fig. 9.1

*The line connecting the vertex and the middle point of the opposite side of a triangle is known as median of the triangle.

Let G is the centre of gravity of the total area A whose distance from the axis OY is $\bar{x}$.

Then moment of total area about $OY = A\bar{x}$... (ii)

The moments of all small areas about the axis OY must be equal to the moment of total area about the same axis. Hence equating equations (i) and (ii), we get

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \dots = A\bar{x}$$

$$\text{or } \bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \dots}{A} \quad \dots(9.1)$$

where $A = a_1 + a_2 + a_3 + a_4 \dots$

If we take the moments of the small areas about the axis OX and also the moment of total area about the axis OX , we will get

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + \dots}{A} \quad \dots(9.2)$$

where $\bar{y}$ = The distance of G from axis OX

y_1 = The distance of C.G. of the area a_1 from axis OX

y_2, y_3, y_4 = The distance of C.G. of area a_2, a_3, a_4 from axis OX respectively.

9.4.1. Centre of Gravity of Areas of Plane Figures by Integration Method. The equations (9.1) and (9.2) can be written as

$$\bar{x} = \frac{\sum a_i x_i}{\sum a_i} \quad \text{and} \quad \bar{y} = \frac{\sum a_i y_i}{\sum a_i}$$

where $i = 1, 2, 3, 4, \dots$

x_i = Distance of C.G. of area a_i from axis OY and

y_i = Distance of C.G. of area a_i from axis OX .

The value of i depends upon the number of small areas. If the small areas are large in number (mathematically speaking infinite in number), then the summations in the above equations can be replaced by integration. Let the small areas are represented by dA instead of ' a ', then the above equations are written as :

$$\bar{x} = \frac{\int x^* dA}{\int dA} \quad \dots(9.2 A)$$

$$\text{and } \bar{y} = \frac{\int y^* dA}{\int dA} \quad \dots(9.2 B)$$

where $\int x^* dA = \sum x_i a_i$

$\int dA = \sum a_i$

$\int y^* dA = \sum y_i a_i$

Also x^* = Distance of C.G. of area dA from axis OY

y^* = Distance of C.G. of area dA from axis OX .

9.4.2. Centroid (or Centre of Gravity) of a Line. The centre of gravity of a line which may be straight or curve, is obtained by dividing the given line, into a large number of small lengths as shown in Fig. 9.1 (a).

The centre of gravity is obtained by replacing dA by dL in equations (9.2 A) and (9.2 B).

$$\text{Then these equations become } \bar{x} = \frac{\int x^* dL}{\int dL} \quad \dots(9.2 C)$$

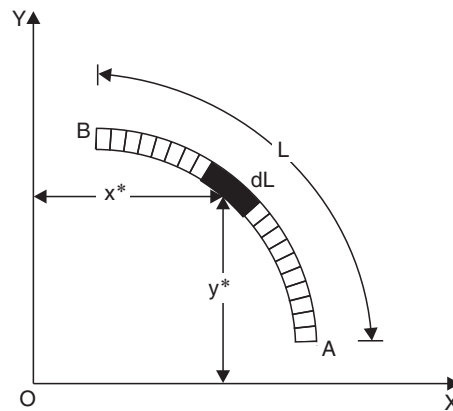


Fig. 9.1 (a)

and

$$\bar{y} = \frac{\int y^* dL}{\int dL} \quad \dots(9.2 D)$$

where x^* = Distance of C.G. of length dL from y -axis, and
 y^* = Distance of C.G. of length dL from x -axis.

If the lines are straight, then the above equations are written as :

$$\bar{x} = \frac{L_1 x_1 + L_2 x_2 + L_3 x_3 + \dots}{L_1 + L_2 + L_3 + \dots} \quad \dots(9.2 E)$$

and

$$\bar{y} = \frac{L_1 y_1 + L_2 y_2 + L_3 y_3 + \dots}{L_1 + L_2 + L_3 + \dots} \quad \dots(9.2 F)$$

9.4.3. Important Points

(i) The axis, about which moments of areas are taken, is known as axis of reference. In the above article, axis OX and OY are called axis of reference.

(ii) The axis of reference, of plane figures, is generally taken as the lowest line of the figure for determining $\bar{y}$, and left line of the figure for calculating $\bar{x}$.

(iii) If the given section is symmetrical about $X-X$ axis or $Y-Y$ axis, then the C.G. of the section will lie on the axis is symmetry.

9.5. CENTROID OF COMPOSITE BODIES

The centre of gravity of composite bodies or sections like T -section, I -section, L -sections etc. are obtained by splitting them into rectangular components. Then equations (9.1) and (9.2) are used.

Problem 9.1. Find the centre of gravity of the T -section shown in Fig. 9.2 (a).

Sol. The given T -section is split up into two rectangles $ABCD$ and $EFGH$ as shown in Fig. 9.2 (b). The given T -section is symmetrical about $Y-Y$ axis. Hence the C.G. of the section will lie on this axis. The lowest line of the figure is line GF . Hence the moments of the areas are taken about this line GF , which is the axis of reference in this case.

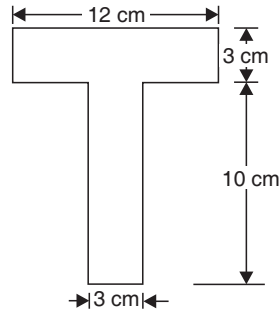
Let $\bar{y}$ = The distance of the C.G. of the T -section from the bottom line GF
 (which is axis of reference)

$$a_1 = \text{Area of rectangle } ABCD = 12 \times 3 = 36 \text{ cm}^2$$

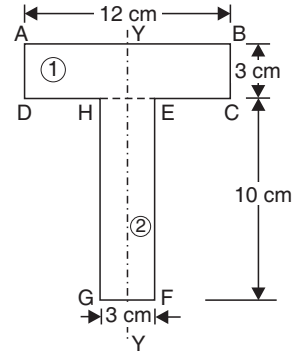
$$y_1 = \text{Distance of C.G. of area } a_1 \text{ from bottom line } GF = 10 + \frac{3}{2} = 11.5 \text{ cm}$$

$$a_2 = \text{Area of rectangle } EFGH = 10 \times 3 = 30 \text{ cm}^2$$

$$y_2 = \text{Distance of C.G. of area } a_2 \text{ from bottom line } GF = \frac{10}{2} = 5 \text{ cm.}$$



(a)



(b)

Fig. 9.2

Using equation (9.2), we have

$$\begin{aligned} \bar{y} &= \frac{a_1 y_1 + a_2 y_2}{A} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} \quad (\because A = a_1 + a_2) \\ &= \frac{36 \times 11.5 + 30 \times 5}{36 + 30} = \frac{414 + 150}{66} = 8.545 \text{ cm. Ans.} \end{aligned}$$

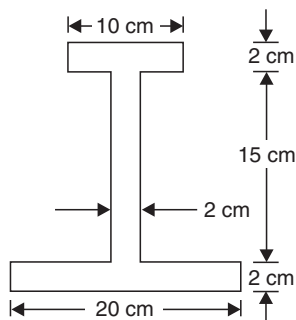
Problem 9.2. Find the centre of gravity of the I-section shown in Fig. 9.3 (a).

Sol. The I-section is split up into three rectangles $ABCD$, $EFGH$ and $JKLM$ as shown in Fig. 9.3 (b). The given I-section is symmetrical about $Y-Y$ axis. Hence the C.G. of the section will lie on this axis. The lowest line of the figure line is ML . Hence the moment of areas are taken about this line, which is the axis of reference.

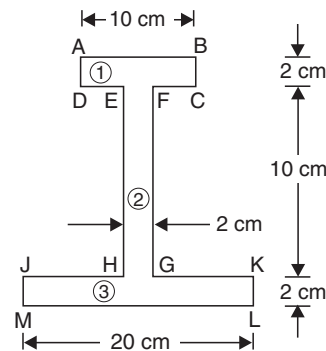
Let $\bar{y}$ = Distance of the C.G. of the I-section from the bottom line ML

$$a_1 = \text{Area of rectangle } ABCD = 10 \times 2 = 20 \text{ cm}^2$$

$$y_1 = \text{Distance of C.G. of rectangle } ABCD \text{ from bottom line } ML = 2 + 15 + \frac{2}{2} = 18 \text{ cm}$$



(a)



(b)

Fig. 9.3

$$a_2 = \text{Area of rectangle } EFGH = 15 \times 2 = 30 \text{ cm}^2$$

$$y_2 = \text{Distance of C.G. of rectangle } EFGH \text{ from bottom line } ML = 2 + \frac{15}{2} = 2 + 7.5 = 9.5 \text{ cm}$$

$$a_3 = \text{Area of rectangle } JKLM = 20 \times 2 = 40 \text{ cm}^2$$

$$y_3 = \text{Distance of C.G. of rectangle } JKLM \text{ from bottom line } ML = \frac{2}{2} = 1.0 \text{ cm}$$

Now using equation (9.2), we have

$$\begin{aligned} \bar{y} &= \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{A} \\ &= \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} \quad (\because A = a_1 + a_2 + a_3) \\ &= \frac{20 \times 18 + 30 \times 9.5 + 40 \times 1}{20 + 30 + 40} \\ &= \frac{360 + 285 + 40}{90} = \frac{685}{90} \\ &= 7.611 \text{ cm. Ans.} \end{aligned}$$

Problem 9.3. Find the centre of gravity of the L-section shown in Fig. 9.4.

Sol. The given L-section is not symmetrical about any section. Hence in this case, there will be two axis of references. The lowest line of the figure (i.e., line GF) will be taken as axis of reference for calculating $\bar{y}$. And the left line of the L-section (i.e., line AG) will be taken as axis of reference for calculating $\bar{x}$.

The given L-section is split up into two rectangles $ABCD$ and $DEFG$, as shown in Fig. 9.4.

To Find $\bar{y}$

Let $\bar{y}$ = Distance of the C.G. of the L-section from bottom line GF

$$a_1 = \text{Area of rectangle } ABCD = 10 \times 2 = 20 \text{ cm}^2$$

$$y_1 = \text{Distance of C.G. of rectangle } ABCD \text{ from bottom line } GF$$

$$= 2 + \frac{10}{2} = 2 + 5 = 7 \text{ cm}$$

$$a_2 = \text{Area of rectangle } DEFG = 8 \times 2 = 16 \text{ cm}^2$$

$$y_2 = \text{Distance of C.G. of rectangle } DEFG \text{ from bottom line } GF$$

$$= \frac{2}{2} = 1.0 \text{ cm.}$$

Using equation (9.2), we have

$$\begin{aligned} \bar{y} &= \frac{a_1 y_1 + a_2 y_2}{A}, \quad \text{where } A = a_1 + a_2 \\ &= \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{20 \times 7 + 16 \times 1}{20 + 16} = \frac{140 + 16}{36} \\ &= \frac{156}{36} = \frac{13}{3} = 4.33 \text{ cm.} \end{aligned}$$

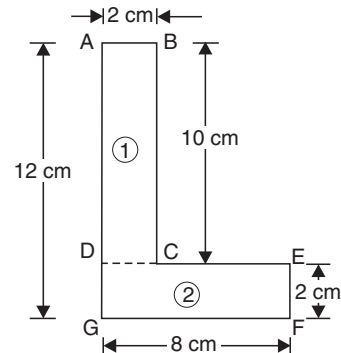


Fig. 9.4

To Find $\bar{x}$

Let $\bar{x}$ = Distance of the C.G. of the L -section from left line AG

x_1 = Distance of the rectangle $ABCD$ from left line AG

$$= \frac{2}{2} = 1.0 \text{ cm}$$

x_2 = Distance of the rectangle $DEFG$ from left line AG

$$= \frac{8}{2} = 4.0 \text{ cm.}$$

Using equation (9.1), we get

$$\begin{aligned}\bar{x} &= \frac{a_1 x_1 + a_2 x_2}{A}, \quad \text{where } A = a_1 + a_2 \\ &= \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{20 \times 1 + 16 \times 4}{20 + 16} \quad (\because a_1 = 20 \text{ and } a_2 = 16) \\ &= \frac{20 + 64}{36} = \frac{84}{36} = \frac{7}{3} = 2.33 \text{ cm.}\end{aligned}$$

Hence the C.G. of the L -section is at a distance of 4.33 cm from the bottom line GF and 2.33 cm from the left line AG . **Ans.**

Problem 9.4. Using the analytical method, determine the centre of gravity of the plane uniform lamina shown in Fig. 9.5. (U.P. Tech. University, 2001–2002, May 2009)

Sol. Let $\bar{y}$ be the distance between c.g. of the lamina and the bottom line AB .

Area 1

$$a_1 = 10 \times 5 = 50 \text{ cm}^2$$

$$y_1 = \frac{5}{2} = 2.5 \text{ cm}$$

Area 2

$$a_2 = \frac{\pi}{2} \times r^2 = \frac{\pi}{2} \times 2.5^2 = 9.82 \text{ cm}^2$$

$$y_2 = \frac{5}{2} = 2.5 \text{ cm}$$

Area 3

$$a_3 = \frac{5 \times 5}{2} = 12.5 \text{ cm}^2$$

$$y_3 = 5 + \frac{5}{3} = 6.67 \text{ cm.}$$

Using the relation,

$$\begin{aligned}\bar{y} &= \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} \\ &= \frac{50 \times 2.5 + 9.82 \times 2.5 + 12.5 \times 6.67}{50 + 9.82 + 12.5} \text{ cm} = \frac{232.9}{72.32} = 3.22 \text{ cm.}\end{aligned}$$

Similarly, let $\bar{x}$ be the distance between c.g. of the lamina and the left line CD .

Area 1

$$a_1 = 50 \text{ cm}^2$$

$$x_1 = 2.5 + \frac{10}{2} = 7.5 \text{ cm}$$

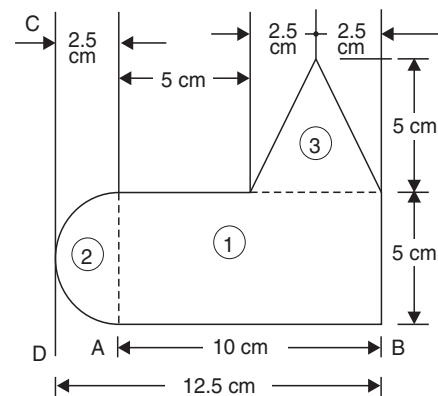


Fig. 9.5

Area 2

$$a_2 = 9.82 \text{ cm}^2$$

$$x_2 = 2.5 - \frac{4r}{3\pi} = 2.5 - \frac{4.25}{3\pi} \text{ cm} = 1.44 \text{ cm}$$

Area 3

$$a_3 = 12.5 \text{ cm}^2$$

$$x_3 = 2.5 + 5 + 2.5 = 10 \text{ cm.}$$

Now using the relation,

$$\begin{aligned} \bar{x} &= \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = \frac{50 \times 7.5 + 9.82 \times 1.44 + 12.5 \times 10}{50 + 9.82 + 12.5} \text{ cm} \\ &= \frac{514.14}{72.32} = 7.11 \text{ cm.} \end{aligned}$$

Hence the C.G. of the uniform lamina is at a distance of 3.22 cm from the bottom line AB and 7.11 cm from the left line CD . **Ans.**

Problem 9.5. From a rectangular lamina $ABCD$ $10 \text{ cm} \times 12 \text{ cm}$ a rectangular hole of $3 \text{ cm} \times 4 \text{ cm}$ is cut as shown in Fig. 9.6.

Find the c.g. of the remainder lamina.

Sol. The section shown in Fig. 9.6, is having a cut hole. The centre of gravity of a section with a cut hole is determined by considering the main section first as a complete one, and then subtracting the area of the cut-out hole, i.e., by taking the area of the cut-out hole as negative.

Let $\bar{y}$ is the distance between the C.G. of the section with a cut hole from the bottom line DC .

$$a_1 = \text{Area of rectangle } ABCD = 10 \times 12 = 120 \text{ cm}^2$$

y_1 = Distance of C.G. of the rectangle $ABCD$ from bottom line DC

$$= \frac{12}{2} = 6 \text{ cm}$$

$$\begin{aligned} a_2 &= \text{Area of cut-out hole, i.e., rectangle } EFGH, \\ &= 4 \times 3 = 12 \text{ cm}^2 \end{aligned}$$

y_2 = Distance of C.G. of cut-out hole from bottom line DC

$$= 2 + \frac{4}{2} = 2 + 2 = 4 \text{ cm.}$$

Now using equation (9.2) and taking the area (a_2) of the cut-out hole as negative, we get

$$\bar{y} = \left(\frac{a_1 y_1 - a_2 y_2}{A} \right)^* \quad \text{where } A = a_1 - a_2$$

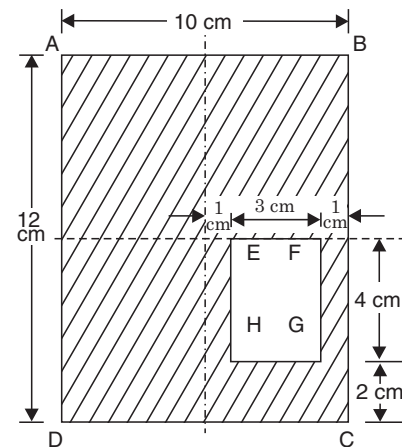


Fig. 9.6

$$*\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} \text{ but for cut-hole area } a_2 \text{ is taken -ve. Hence}$$

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2}.$$

$$\begin{aligned}
 &= \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} \quad (\text{-ve sign is taken due to cut-out hole}) \\
 &= \frac{120 \times 6 - 12 \times 4}{120 - 12} = \frac{720 - 48}{108} = 6.22 \text{ cm.}
 \end{aligned}$$

To Find $\bar{x}$

Let $\bar{x}$ = Distance between the C.G. of the section with a cut hole from the left line AD

x_1 = Distance of the C.G. of the rectangle $ABCD$ from the left line AD

$$= \frac{10}{2} = 5 \text{ cm}$$

x_2 = Distance of the C.G. of the cut-out hole from the left line AD

$$= 5 + 1 + \frac{3}{2} = 7.5 \text{ cm.}$$

Using equation (9.1) and taking area (a_2) of the cut hole as negative, we get

$$\begin{aligned}
 \bar{x} &= \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2} \quad (\because A = a_1 - a_2) \\
 &= \frac{120 \times 5 - 12 \times 7.5}{120 - 12} = \frac{600 - 90}{108} = \frac{510}{108} = 4.72 \text{ cm.}
 \end{aligned}$$

Hence the C.G. of the section with a cut hole will be at a distance of 6.22 cm from bottom line DC and 4.72 cm from the line AD . **Ans.**

Problem 9.5 (A). Determine the co-ordinates X_C and Y_C of the centre of a 100 mm diameter circular hole cut in a thin plate so that this point will be the centroid of the remaining shaded area shown in Fig. 9.6 (a).

(U.P. Tech. University, 2001–2002)

Sol. The given shaded area is equal to area of a thin rectangular plate of size 200 mm $\times$ 150 mm minus the area of a triangle of length 100 mm and height 75 mm minus the area of circular hole of dia. 100 mm as shown in Fig. 9.6 (b).

Let A_1 = Area of rectangular plate
 $= 200 \times 150 = 30000 \text{ mm}^2$

A_2 = Area of triangle
 $= \frac{100 \times 75}{2} = 3750 \text{ mm}^2$

A_3 = Area of hole $= \frac{\pi}{4} (100^2) = 2500\pi \text{ mm}^2$

The centre of hole is the centroid of the shaded area. Hence X_C and Y_C is the co-ordinates of the centre of the hole and also the co-ordinates of the centroid of the shaded area.

For area A_1 ,

$$x_1 = \frac{200}{2} = 100 \text{ mm, } y_1 = \frac{150}{2} = 75 \text{ mm}$$

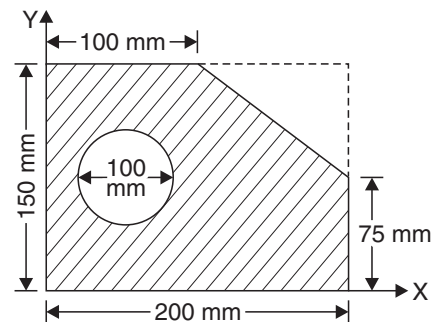


Fig. 9.6 (a)

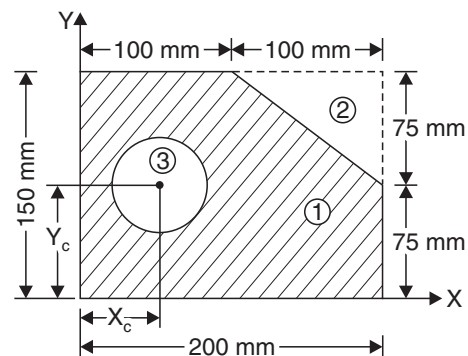


Fig. 9.6 (b)

For area A_2 ,

$$x_2 = 100 + \frac{2}{3} \times 100 = 166.67,$$

$$y_2 = 75 + \frac{2}{3} \times 75 = 125 \text{ mm}$$

For area A_3 , $x_3 = X_C$ and $y_3 = Y_C$

Now using equation (9.1) and taking areas A_2 and A_3 as negative, we get

$$\bar{x} - X_C = \frac{A_1 x_1 - A_2 x_2 - A_3 x_3}{A_1 - A_2 - A_3} = \frac{30000 \times 100 - 3750 \times 166.67 - 2500\pi \times X_C}{(30000 - 3750 - 2500\pi)}$$

$$\text{or} \quad X_C (30000 - 3750 - 2500\pi) = 30000 \times 100 - 3750 \times 166.67 - 2500\pi \times X_C$$

$$X_C (30000 - 3750) - 2500\pi \times X_C = 30000 \times 100 - 3750 \times 166.67 - 2500\pi \times X_C$$

$$\text{or} \quad X_C (30000 - 3750) = 30000 \times 100 - 3750 \times 166.67$$

(Cancelling $2500 \times \pi \times X_C$ on both sides)

$$\therefore 26250 X_C = 3000000 - 625012.5 = 2374987.5$$

$$\therefore X_C = \frac{2374987.5}{26250} = 90.47 \text{ mm.} \quad \text{Ans.}$$

$$\text{Similarly} \quad \bar{y} = Y_C = \frac{A_1 y_1 - A_2 y_2 - A_3 y_3}{A_1 - A_2 - A_3}$$

$$= \frac{30000 \times 75 - 3750 \times 125 - 2500\pi \times Y_C}{(30000 - 3750 - 2500\pi)}$$

$$\text{or} \quad Y_C (30000 - 3750 - 2500\pi) = 30000 \times 75 - 3750 \times 125 - 2500\pi \times Y_C$$

$$\text{or} \quad Y_C (30000 - 3750) = 30000 \times 75 - 3750 \times 125$$

(Cancelling $2500\pi \times Y_C$ on both sides)

$$\text{or} \quad 26250 Y_C = 30000 \times 75 - 3750 \times 125$$

$$= 225000 - 468750 = 1781250$$

$$\therefore Y_C = \frac{1781250}{26250} = 67.85 \text{ mm.} \quad \text{Ans.}$$

Problem 9.5 (B). A semi-circular area is removed from the trapezoid as shown in Fig. 9.6 (c). Determine the centroid of the remaining area.

(U.P. Tech. University, 2000–2001)

Sol. The given shaded area is equal to the area of a thin rectangular plate of size $100 \text{ mm} \times (150 + 100) \text{ mm}$ plus the area of the triangle of length 250 mm and of height $(150 - 100) = 50 \text{ mm}$ minus the area of semi-circular area of diameter 100 mm as shown in Fig. 9.6 (c).

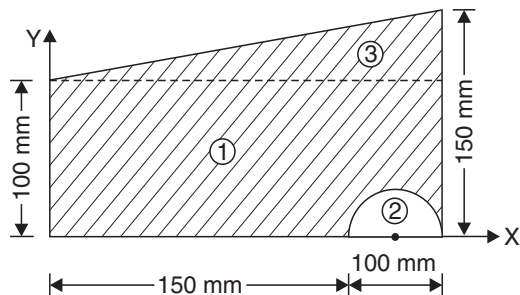


Fig. 9.6 (c)

Let A_1 = Area of rectangular plate $= 100 \times 250 = 25000 \text{ mm}^2$

$$A_2 = \text{Area of semi-circle} = \frac{\pi r^2}{2} = \frac{\pi \times 50^2}{2} = 1250\pi \text{ mm}^2$$

$$A_3 = \text{Area of the triangle} = \frac{250 \times 50}{2} = 6250 \text{ mm}^2$$

$$x_1 = \text{Distance of C.G. of area } A_1 \text{ from } y\text{-axis} = \frac{250}{2} = 125 \text{ mm}$$

$$y_1 = \text{Distance of C.G. of area } A_1 \text{ from } x\text{-axis} = \frac{100}{2} = 50 \text{ mm}$$

$$x_2 = \text{Distance of C.G. of area } A_2 \text{ from } y\text{-axis} = 150 + \frac{100}{2} = 200 \text{ mm}$$

$$y_2 = \text{Distance of C.G. of area } A_2 \text{ from } x\text{-axis} = \frac{4r}{3\pi} = \frac{4 \times 50}{3\pi} = \frac{200}{3\pi}$$

$$x_3 = \text{Distance of C.G. of area } A_3 \text{ from } y\text{-axis} = 250 \times \frac{2}{3} = \frac{500}{3}$$

$$y_3 = \text{Distance of C.G. of area } A_3 \text{ from } x\text{-axis} = 100 + \frac{50}{3} = \frac{350}{3} \text{ mm}$$

$$\bar{x}, \bar{y} = \text{Distance of C.G. of the shaded area from } y \text{ and } x\text{-axis.}$$

Now using equation (9.1) and taking area A_2 as negative, we get

$$\begin{aligned} \bar{x} &= \frac{A_1 x_1 - A_2 x_2 + A_3 x_3}{A_1 - A_2 + A_3} \\ &= \frac{25000 \times 125 - 1250\pi \times 200 + 6250 \times \frac{500}{3}}{25000 - 1250\pi + 6250} \\ &= \frac{3125000 - 785398 + 1041666}{27323} = 123.75 \text{ mm. Ans.} \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{y} &= \frac{A_1 y_1 - A_2 y_2 + A_3 y_3}{A_1 - A_2 + A_3} = \frac{25000 \times 50 - 1250\pi \times \frac{200}{3\pi} + 6250 \times \frac{350}{3}}{27323} \\ &= \frac{1250000 - 83333 + 729166}{27323} = 69.38 \text{ mm. Ans.} \end{aligned}$$

$\therefore$ Centroid of the given section = $(\bar{x}, \bar{y}) = (123.75 \text{ mm}, 69.38 \text{ mm})$.

9.6. DETERMINATION OF CENTROID BY INTEGRATION

The centroid of any area by integration is obtained by using equations (9.2A) and (9.2B) as

$$\bar{x} = \frac{\int x^* dA}{\int dA} \quad \text{and} \quad \bar{y} = \frac{\int y^* dA}{\int dA}$$

where $\bar{x}$ = Distance of C.G. of whole area from axis-OY

$\bar{y}$ = Distance of C.G. of whole area from axis-OX

x^* = Distance of C.G. of area dA from axis-OY and

y^* = Distance of C.G. of area dA from axis-OX.

Problem 9.6. Determine the co-ordinates of the C.G. of the area OAB shown in Fig. 9.7, if the curve OB represents the equation of a parabola, given by

$$y = kx^2$$

in which

$$OA = 6 \text{ units}$$

and

$$AB = 4 \text{ units.}$$

Sol. The equation of parabola is $y = kx^2$... (i)

First determine the value of constant k . The point B is lying on the curve and having co-ordinates

$$x = 6 \text{ and } y = 4$$

Substituting these values of equation (i), we get

$$4 = k \times 6^2 = 36k$$

$$\therefore k = \frac{4}{36} = \frac{1}{9}$$

Substituting the value of k in equation (i), we get

$$y = \frac{1}{9} x^2 \quad \dots (ii)$$

or

$$x^2 = 9y$$

or

$$x = 3\sqrt{y} \quad \dots (iii)$$

Consider a strip of height y and width dx as shown in Fig. 9.7. The area dA of the strip is given by

$$dA = y \times dx$$

The co-ordinates of the C.G. of this area dA are x and $\frac{y}{2}$

$\therefore$ Distance of C.G. of area dA from y -axis = x

and distance of C.G. of area dA from x -axis = $\frac{y}{2}$

$$\therefore x^* = x \quad \text{and} \quad y^* = \frac{y}{2}$$

Let $\bar{x}$ = Distance of C.G. of total area OAB from axis OY

$\bar{y}$ = Distance of C.G. of total area OAB from axis OX .

Using equation (9.2 A), we get

$$\bar{x} = \frac{\int x^* dA}{\int dA} = \frac{\int_0^6 x \times y dx}{\int_0^6 y dx} \quad (\because dA = y dx, x^* = x)$$

But $y = \frac{x^2}{9}$ from equation (ii)

$$\begin{aligned} \therefore \bar{x} &= \frac{\int_0^6 x \times \frac{x^2}{9} \times dx}{\int_0^6 \frac{x^2}{9} dx} = \frac{\frac{1}{9} \int_0^6 x^3 dx}{\frac{1}{9} \int_0^6 x^2 dx} \\ &= \frac{\int_0^6 x^3 dx}{\int_0^6 x^2 dx} = \frac{\left[\frac{x^4}{4} \right]_0^6}{\left[\frac{x^3}{3} \right]_0^6} = \frac{\frac{1}{4} \times 6^4}{\frac{1}{3} \times 6^3} \\ &= \frac{1}{4} \times \frac{3}{1} \times 6 = 4.5. \quad \text{Ans.} \end{aligned}$$

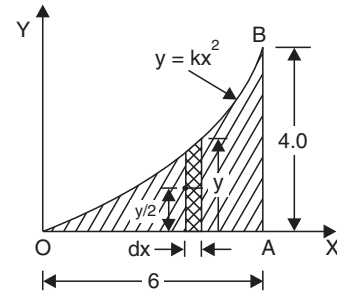


Fig. 9.7

Using equation (9.2 B), we get

$$\bar{y} = \frac{\int y^* dA}{\int dA}$$

where y^* = Distance of C.G. of area dA from x -axis

$$= \frac{y}{2} \text{ (here)}$$

$$dA = ydx$$

$$\begin{aligned} \therefore \int y^* dA &= \int \frac{y}{2} \times dA = \int_0^6 \frac{y}{2} \times ydx = \int_0^6 \frac{y^2}{2} dx \\ &= \frac{1}{2} \int_0^6 y^2 dx = \frac{1}{2} \int_0^6 \left(\frac{x^2}{9} \right)^2 dx \quad \left(\because y = \frac{x^2}{9} \right) \\ &= \frac{1}{2} \int_0^6 \frac{x^4}{81} dx = \frac{1}{2} \times \frac{1}{81} \int_0^6 x^4 dx = \frac{1}{2} \times \frac{1}{81} \left[\frac{x^5}{5} \right]_0^6 \\ &= \frac{1}{2} \times \frac{1}{81} \times \frac{6^5}{5} = \frac{6^5}{810} \end{aligned}$$

$$\text{Also } \int dA = \int_0^6 ydx = \int_0^6 \frac{x^2}{9} dx = \frac{1}{9} \left[\frac{x^3}{3} \right]_0^6 = \frac{1}{9} \times \frac{6^3}{3} = \frac{6^3}{27}$$

$$\begin{aligned} \therefore \bar{y} &= \frac{\int y^* dA}{\int dA} = \frac{\frac{6^5}{810}}{\frac{6^3}{27}} = \frac{27}{810} \times \frac{6^5}{6^3} \\ &= \frac{1}{30} \times 6^2 = \frac{36}{30} = \frac{6}{5} . \quad \text{Ans.} \end{aligned}$$

Problem 9.7. Determine the co-ordinates of the C.G. of the shaded area between the parabola $y = \frac{x^2}{4}$ and the straight line $y = x$ as shown in Fig. 9.8.

Sol. The equations of parabola and straight line are

$$y = \frac{x^2}{4} \quad \dots(i)$$

$$y = x \quad \dots(ii)$$

The point A is lying on the straight line as well as on the given parabola. Hence both the above equations holds good for point A. Let the co-ordinates of point A are x, y .

Substituting the value of y from equation (ii) in equation (i), we get

$$x = \frac{x^2}{4} \quad \text{or} \quad 4 = \frac{x^2}{x} = x$$

Substituting the value of $x = 4$, in equation (ii),

$$y = 4$$

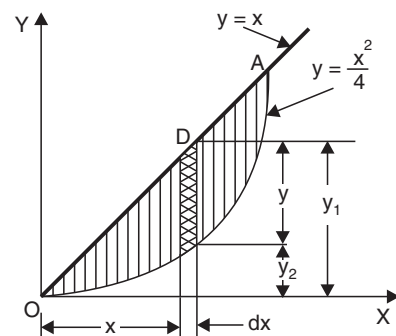


Fig. 9.8

Hence the co-ordinates of point A are 4, 4.

Now divide the shaded area into large small areas each of height y and width dx as shown in Fig. 9.8. Then area dA of the strip is given by

$$dA = ydx = (y_1 - y_2) dx \quad \dots(iii)$$

where y_1 = Co-ordinate of point D which lies on the straight line OA

y_2 = Co-ordinate of the point E which lies on the parabola OA .

The horizontal co-ordinates of the points D and E are same.

The values of y_1 and y_2 can be obtained in terms of x from equations (ii) and (i),

$$y_1 = x \quad \text{and} \quad y_2 = \frac{x^2}{4}$$

Substituting these values in equation (iii),

$$dA = \left(x - \frac{x^2}{4} \right) dx \quad \dots(iv)$$

The distance of the C.G. for the area dA from y -axis is given by,

$$x^* = x$$

And the distance of the C.G. of the area dA from x -axis is given by,

$$\begin{aligned} y^* &= y_2 + \frac{y}{2} = y_2 + \frac{y_1 - y_2}{2} & (\because y = y_1 - y_2) \\ &= \frac{2y_2 + y_1 - y_2}{2} = \frac{y_1 + y_2}{2} \\ &= \frac{x + \frac{x^2}{4}}{2} & \left(\because y_1 = x \text{ and } y_2 = \frac{x^2}{4} \right) \\ &= \frac{1}{2} \left(x + \frac{x^2}{4} \right) & \dots(v) \end{aligned}$$

Now, let $\bar{x}$ = Distance of C.G. of shaded area of Fig. 9.8 from y -axis

$\bar{y}$ = Distance of C.G. of shaded area of Fig. 9.8 from x -axis.

Now using equation (9.2 A),

$$\bar{x} = \frac{\int x^* dA}{\int dA}, \quad \text{where } x^* = x$$

$$dA = \left(x - \frac{x^2}{4} \right) dx \quad [\text{see equation (iv)}]$$

$$\therefore \int x^* dA = \int_0^4 x \left(x - \frac{x^2}{4} \right) dx \quad (\because x \text{ varies from 0 to 4})$$

$$\begin{aligned} &= \int_0^4 \left(x^2 - \frac{x^3}{4} \right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4 \times 4} \right]_0^4 \\ &= \frac{4^3}{3} - \frac{4^4}{4 \times 4} = \frac{64}{3} - 16 = \frac{64 - 48}{3} = \frac{16}{3} \end{aligned}$$

and

$$\int dA = \int_0^4 \left(x - \frac{x^2}{4} \right) dx$$

$$\begin{aligned}
 &= \left[\frac{x^2}{2} - \frac{x^3}{3 \times 4} \right]_0^4 = \frac{4^2}{2} - \frac{4^3}{3 \times 4} \\
 &= \frac{16}{2} - \frac{16}{3} = \frac{48 - 32}{6} = \frac{16}{6} \quad \dots(vi)
 \end{aligned}$$

$$\therefore \bar{x} = \frac{\int x^* dA}{\int dA} = \frac{\frac{16}{3}}{\frac{16}{6}} = \frac{16}{3} \times \frac{6}{16} = 2. \quad \text{Ans.}$$

Now using equation (9.2 B), $\bar{y} = \frac{\int y^* dA}{\int dA}$

where $y^* = \frac{1}{2} \left[x + \frac{x^2}{4} \right]$ [From equation (v)]

$$dA = \left(x - \frac{x^2}{4} \right) dx$$

$$\begin{aligned}
 \therefore \int y^* dA &= \int_0^4 \frac{1}{2} \left(x + \frac{x^2}{4} \right) \left(x - \frac{x^2}{4} \right) dx \\
 &= \frac{1}{2} \int_0^4 \left(x^2 - \frac{x^4}{16} \right) dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5 \times 16} \right]_0^4 \\
 &= \frac{1}{2} \left[\frac{4^3}{3} - \frac{4^5}{5 \times 16} \right] = \frac{1}{2} \left[\frac{64}{3} - \frac{64}{5} \right] \\
 &= \frac{64}{2} \left[\frac{1}{3} - \frac{1}{5} \right] = 32 \left(\frac{5-3}{15} \right) = 32 \times \frac{2}{15} = \frac{64}{15}
 \end{aligned}$$

and $\int dA = \frac{16}{6}$ [From equation (vi)]

$$\therefore \bar{y} = \frac{\int y^* dA}{\int dA} = \frac{\frac{64}{15}}{\frac{16}{6}} = \frac{64}{15} \times \frac{6}{16} = \frac{8}{5}. \quad \text{Ans.}$$

9.6.1. Problems of Finding Centroid or Centre of Gravity of Line-Segment by Integration Method

Problem 9.8. Determine the centre of gravity of a quadrant AB of the arc of a circle of radius R as shown in Fig. 9.9 (a).

Sol. 1st Method. The centre of gravity of the line AB, which is an arc of a circle radius R, is obtained by dividing the curved line AB into a large number of elements of length dL as shown in Fig. 9.9 (a).

The equation of curve AB is the equation of circle of radius R.

$\therefore$ The equation of curve AB is given by

$$x^2 + y^2 = R^2$$

Differentiating the above equation,

$$2x dx + 2y dy = 0 \quad [\because R \text{ is constant}]$$

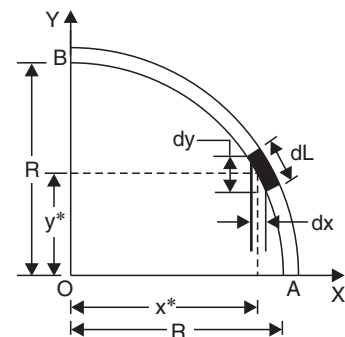


Fig. 9.9 (a)

or

$$2y \, dy = -2x \, dx$$

or

$$dy = \frac{-2x \, dx}{2y} = \frac{-x \, dx}{y} \quad \dots(i)$$

Consider an element of length dL as shown in Fig. 9.9 (a). The C.G. of the length dL is at a distance x^* from y -axis and y^* from x -axis.

Now using equation (9.2 D) for $\bar{y}$, we get

$$\bar{y} = \frac{\int y^* \, dL}{\int dL} \quad \dots(ii)$$

Let us express dL in terms of dx and dy .

But

$$\begin{aligned} dL &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{dx^2 + \left(\frac{-x \, dx}{y}\right)^2} \quad \left(\because \text{From (i), } dy = \frac{-x \, dx}{y}\right) \\ &= \sqrt{dx^2 + \frac{x^2}{y^2} dx^2} \\ &= dx \sqrt{1 + \frac{x^2}{y^2}} = dx \sqrt{\frac{y^2 + x^2}{y^2}} \\ &= dx \sqrt{\frac{R^2}{y^2}} \quad (\because x^2 + y^2 = R^2) \\ &= \frac{R}{y} \cdot dx. \end{aligned}$$

Substituting the value of dL in equation (ii),

$$\begin{aligned} \bar{y} &= \frac{\int y^* \times \frac{R}{y} \cdot dx}{\int dL} = \frac{\int y \times \frac{R}{y} \, dx}{\int dL} \quad (\because y^* = y) \\ &= \frac{\int R \, dx}{\int dL} = \frac{R \int_0^R dx}{\int dL} = \frac{R \left[x \right]_0^R}{\left(\frac{2\pi R}{4} \right)} \\ &\quad (\because \int dL \text{ is total length of arc of one quadrant of a circle}) \\ &= \frac{R \times R}{\frac{2\pi R}{4}} = \frac{2R}{\pi}. \quad \text{Ans.} \end{aligned}$$

Similarly, the value of $\bar{x}$ can be calculated. Due to symmetry this value will also be equal to $\frac{2R}{\pi}$.

$$\therefore \bar{x} = \bar{y} = \frac{2R}{\pi}. \quad \text{Ans.}$$

2nd Method

Here

$$dL = R d\theta$$

$$y^* = R \sin \theta$$

$$x^* = R \cos \theta$$

Now

$$\begin{aligned} \bar{y} &= \frac{\int y^* dL}{\int dL} = \frac{\int_0^{\pi/2} (R \sin \theta) \times (R d\theta)}{\int_0^{\pi/2} R d\theta} \\ &= \frac{\int_0^{\pi/2} R^2 \sin \theta d\theta}{\int_0^{\pi/2} R d\theta} = \frac{R^2 \int_0^{\pi/2} \sin \theta d\theta}{R \int_0^{\pi/2} d\theta} \\ &= \frac{R^2 \left[-\cos \theta \right]_0^{\pi/2}}{R \left[\theta \right]_0^{\pi/2}} = \frac{-R \left[\cos \left(\frac{\pi}{2} \right) - \cos 0 \right]}{\left[\frac{\pi}{2} - 0 \right]} \\ &= \frac{-R[0 - 1]}{\frac{\pi}{2}} = \frac{2R}{\pi}. \quad \text{Ans.} \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{x} &= \frac{\int x^* dL}{\int dL} = \frac{\int_0^{\pi/2} (R \cos \theta) \times (R d\theta)}{\int_0^{\pi/2} R d\theta} = \frac{R^2 \int_0^{\pi/2} \cos \theta d\theta}{R \int_0^{\pi/2} d\theta} \\ &= \frac{R \left[\sin \theta \right]_0^{\pi/2}}{R \left[\theta \right]_0^{\pi/2}} = \frac{R[\sin 90^\circ - \sin 0^\circ]}{\left(\frac{\pi}{2} - 0 \right)} = \frac{R}{\left(\frac{\pi}{2} \right)} = \frac{2R}{\pi}. \quad \text{Ans.} \end{aligned}$$

Problem 9.9. Determine the centre of gravity of the area of the circular sector OAB of radius R and central angle α as shown in Fig. 9.10.

Sol. The given area is symmetrical about x -axis. Hence the C.G. of the area will lie on x -axis. This means $\bar{y} = 0$. To find $\bar{x}$, the moment of small areas are to be taken along y -axis. Divide the area OAB into a large number of triangular elements each of altitude R and base $Rd\theta$ as shown in Fig. 9.10. Such triangular element is shown by OCD in which altitude $OC = R$ and base $CD = Rd\theta$. The area dA of this triangular element is given by,

$$\begin{aligned} dA &= \frac{OC \times CD}{2} = \frac{R \times Rd\theta}{2} \\ &= \frac{R^2 d\theta}{2} \end{aligned}$$

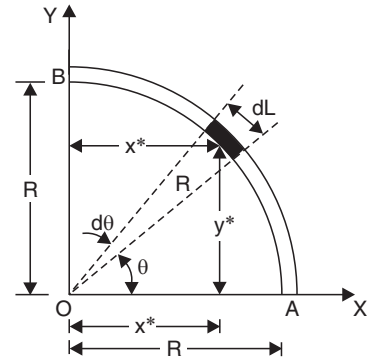


Fig. 9.9 (b)

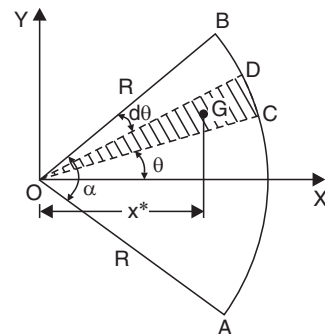


Fig. 9.10

The C.G. of this triangular element is at G

where $OG = \frac{2}{3} \times OC = \frac{2}{3} \times R$

The distance of C.G. of area dA from y -axis is given by,

$$x^* = OG \times \cos \theta = \frac{2}{3} R \times \cos \theta$$

Now using equation (9.2 A),

$$\begin{aligned} \bar{x} &= \frac{\int x^* dA}{\int dA} = \frac{2 \int_0^{\alpha/2} \left(\frac{2}{3} R \cos \theta \right) \left(\frac{R^2 d\theta}{2} \right)}{2 \int_0^{\alpha/2} \frac{R^2}{2} d\theta} \\ &= \frac{\frac{R^3}{3} \int_0^{\alpha/2} \cos \theta d\theta}{\frac{R^2}{2} \int_0^{\alpha/2} d\theta} = \frac{2R \left[\sin \theta \right]_0^{\alpha/2}}{3 \left[\theta \right]_0^{\alpha/2}} \\ &= \frac{2R \sin \left(\frac{\alpha}{2} \right)}{3 \left(\frac{\alpha}{2} \right)} = \frac{4R}{3\alpha} \sin \left(\frac{\alpha}{2} \right) . \quad \text{Ans.} \end{aligned}$$

The area OAB is symmetrical about the x -axis, hence

$$\bar{y} = 0. \quad \text{Ans.}$$

For a semi-circle, $\alpha = \pi = 180^\circ$, hence

$$\begin{aligned} \bar{x} &= \frac{4R}{3\alpha} \sin \left(\frac{\pi}{2} \right) \\ &= \frac{4R}{3 \times \pi} \sin \left(\frac{180}{2} \right) = \frac{4R}{3\pi} . \quad \text{Ans.} \end{aligned}$$

Problem 9.10. Determine the centre of gravity of a semi-circle of radius R as shown in Fig. 9.10 (a).

Sol. This problem can also be solved by the method given in problem 9.9. The following other methods can also be used. Due to symmetry, $\bar{x} = 0$. The area AOB is symmetrical about the Y -axis, hence $\bar{x} = 0$. The value of $\bar{y}$ is obtained by taking the moments of small areas and total area about x -axis.

1. Considering the strip parallel to Y -axis

Area of strip, $dA = y \cdot dx$

The distance of the C.G. of the area dA from

x -axis is equal to $\frac{y}{2}$

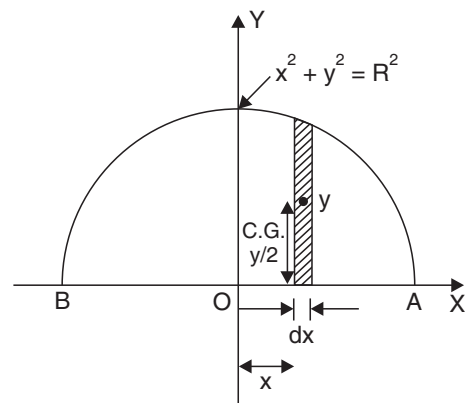


Fig. 9.10 (a)

Moment of area dA about x -axis

$$\begin{aligned}
 &= dA \cdot \frac{y}{2} \\
 &= \frac{y}{2} \cdot dA \\
 &= \frac{y}{2} \cdot y dx \quad (\because dA = y \cdot dx) \\
 &= \frac{y^2}{2} \cdot dx
 \end{aligned}$$

Moment of total area A about x -axis is obtained by integrating the above equation.

$\therefore$ Moment of total area A about x -axis

$$\begin{aligned}
 &= \int \frac{y^2}{2} \cdot dx \\
 &= \int_{-R}^R \frac{y^2}{2} dx \quad (\because x \text{ varies from } -R \text{ to } R)
 \end{aligned}$$

But equation of semi-circle is

$$x^2 + y^2 = R^2 \quad \text{or} \quad y^2 = R^2 - x^2$$

Substituting this value of y^2 in the above equation, we get

Moment of total area A about x -axis

$$\begin{aligned}
 &= \int_{-R}^R \frac{(R^2 - x^2)}{2} dx \\
 &= \frac{1}{2} \left[R^2 \cdot x - \frac{x^3}{3} \right]_{-R}^R \\
 &= \frac{1}{2} \left[\left(R^2 \cdot R - \frac{R^3}{3} \right) - \left\{ R^2 (-R) - \frac{(-R)^3}{3} \right\} \right] \\
 &= \frac{1}{2} \left[\left(R^3 - \frac{R^3}{3} \right) - \left\{ -R^3 - \frac{(-R^3)}{3} \right\} \right] \\
 &= \frac{1}{2} \left[\frac{2R^3}{3} - \left(-R^3 + \frac{R^3}{3} \right) \right] = \frac{1}{2} \left[\frac{2R^3}{3} - \left(-\frac{2R^3}{3} \right) \right] \\
 &= \frac{1}{2} \left[\frac{2R^3}{3} + \frac{2R^3}{3} \right] = \frac{1}{2} \times \frac{4R^3}{3} = \frac{2R^3}{3} \quad \dots(i)
 \end{aligned}$$

Let $\bar{y}$ = Distance of C.G. of the total area of semi-circle from x -axis.

The total area of semi-circle is also equal to $\frac{\pi R^2}{2}$

$\therefore$ Moment of this total area about x -axis

$$= \bar{y} \times \frac{\pi R^2}{2} \quad \dots(ii)$$

Equating the two values given by equations (i) and (ii), we get

$$\bar{y} \times \frac{\pi R^2}{3} = \frac{2R^3}{3}$$

$$\therefore \bar{y} = \frac{2R^3}{3} \times \frac{2}{\pi R^2} = \frac{4R}{3\pi} \quad \text{Ans.}$$

Hence the location of C.G. of semi-circle is $\left(0, \frac{4R}{3\pi}\right)$. **Ans.**

2. Considering the strip parallel to x-axis

Area of strip, $dA = 2x \cdot dy$

The distance of the C.G. of this area from x-axis is y

$\therefore$ Moment of this area about x-axis

$$= y \cdot dA$$

$$= y \cdot 2x dy$$

$$= 2xy dy$$

But, we know $x^2 + y^2 = R^2$

$$\therefore x^2 = R^2 - y^2$$

or

$$x = \sqrt{R^2 - y^2}$$

Substituting the above value of x in equation (i), we get

Moment of area dA about x-axis,

$$= 2\sqrt{R^2 - y^2} \cdot y \cdot dy$$

Moment of total area A about x-axis will be obtained by integrating the above equation from O to R .

$\therefore$ Moment of area A about x-axis

$$= \int_0^R 2\sqrt{R^2 - y^2} \cdot y \cdot dy \quad (\because y \text{ varies from } O \text{ to } R)$$

$$= - \int_0^R \sqrt{R^2 - y^2} \cdot (-2y) dy = - \left[\frac{(R^2 - y^2)^{3/2}}{3/2} \right]_0^R$$

$$= - \frac{2}{3} [0 - R^3] = \frac{2R^3}{3} \quad \dots(i)$$

Also the moment of total area A about x-axis $= A \times \bar{y}$

where $A = \text{Total area of semi-circle} = \frac{\pi R^2}{2}$

$\bar{y} = \text{Distance of C.G. of area } A \text{ from } x\text{-axis}$

$$\therefore \text{Moment of total area } A \text{ about } x\text{-axis} = \frac{\pi R^2}{2} \times \bar{y} \quad \dots(ii)$$

Equating the two values given by equations (i) and (ii),

$$\frac{\pi R^2}{2} \times \bar{y} = \frac{2R^3}{3}$$

or

$$\bar{y} = \frac{2R^3}{3} \times \frac{2}{\pi R^2} = \frac{4R}{3\pi} \quad \text{Ans.}$$

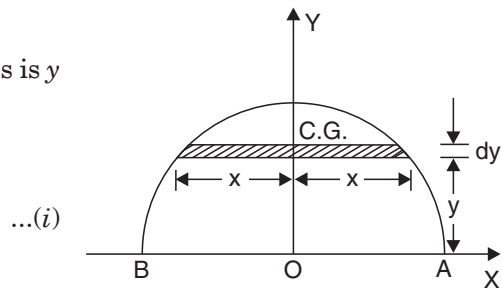


Fig. 9.10 (b)

Problem 9.11. To determine the centre of gravity of the area shown in Fig. 9.10 (c)

given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Consider a small strip of thickness dx parallel to y -axis at a distance x from the y -axis.

Area of the strip, $dA = y \cdot dx$

The C.G. of area dA is at a distance $\frac{y}{2}$ from x -axis.

Moment of the area dA about x -axis

$$\begin{aligned} &= \frac{y}{2} \cdot dA \\ &= \frac{y}{2} \cdot y dx \quad (\because dA = y \cdot dx) \\ &= \frac{y^2}{2} \cdot dx \end{aligned}$$

$\therefore$ Moment of the total area about x -axis

$$= \int_0^a \frac{y^2}{2} \cdot dx$$

($\because x$ varies from O to a) ... (i)

Let us substitute the value of y^2 in terms of x .

The given equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

or
$$y^2 = \frac{b^2}{a^2} (a^2 - x^2) \quad \dots(ii)$$

Substituting the value of y^2 in equation (i), we get

Moment of total area about x -axis

$$\begin{aligned} &= \frac{1}{2} \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx = \frac{b^2}{2a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a \\ &= \frac{b^2}{2a^2} \left[a^3 - \frac{a^3}{3} \right] = \frac{b^2}{2a^2} \times \frac{2a^3}{3} = \frac{ab^2}{3} \quad \dots(iii) \end{aligned}$$

The total area A of the given figure is given by

$$A = \int dA = \int y \cdot dx \quad \dots(iv)$$

From equation (ii), $y = \left[\frac{b^2}{a^2} (a^2 - x^2) \right]^{1/2} = \frac{b}{a} (a^2 - x^2)^{1/2} \quad \dots(v)$

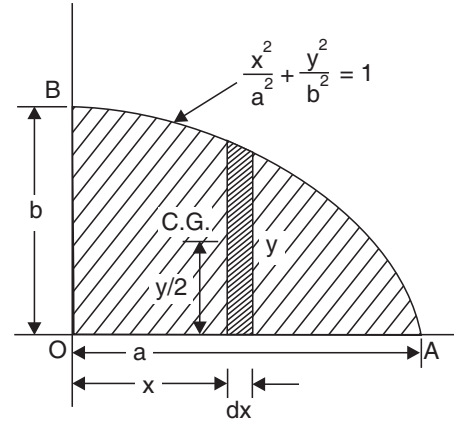


Fig. 9.10 (c)

Now equation (iv) is, $A = \int_0^a \frac{b}{a} (a^2 - x^2)^{1/2} \cdot dx$

$$= \frac{b}{a} \left[\int_0^a (a^2 - x^2)^{1/2} \cdot dx \right] = \frac{b}{a} \left[\frac{\pi a^2}{4} \right]^*$$

$$= \frac{\pi \cdot ab}{4} \quad \left(\because \int_0^a \sqrt{a^2 - x^2} \, dx = \frac{\pi a^2}{4} \right) \dots(vi)$$

Let $\bar{y}$ = the distance of C.G. of the total area A from x -axis.

Then moment of total area A about x -axis

$$= A \times \bar{y}$$

$$= \frac{\pi ab}{4} \cdot \bar{y} \dots(vii)$$

The equations (iii) and (vii) give the moment of total area about x -axis. Hence equating these equations, we get

$$\frac{\pi ab}{4} \cdot \bar{y} = \frac{ab^2}{3}$$

$$\therefore \bar{y} = \frac{ab^2}{3} \cdot \frac{4}{\pi ab} = \frac{4b}{3\pi} \text{ . Ans.}$$

To find $\bar{x}$, take the moment of small area dA about y -axis.

The C.G. of area dA is at a distance of x from y -axis.

$\therefore$ Moment of area dA about y -axis = $x \cdot dA$

$$= x \cdot y \cdot dx \quad (\because dA = y \cdot dx)$$

Moment of total area A about y -axis is obtained by integration

Now moment of total area A about y -axis

$$= \int_0^a x \cdot y \cdot dx \quad (\because x \text{ varies from } O \text{ to } a)$$

$$= \int_0^a x \cdot \frac{b}{a} (a^2 - x^2)^{1/2} \cdot dx \quad \left[\because y = \frac{b}{a} (a^2 - x^2)^{1/2} \text{ from equation (v)} \right]$$

$$= \frac{b}{a} \int_0^a x \cdot (a^2 - x^2)^{1/2} \, dx = \frac{b}{a} \int_0^a \frac{(-2)}{(-2)} \cdot x (a^2 - x^2)^{1/2} \cdot dx$$

$$= \frac{b}{-2a} \left[\frac{(a^2 - x^2)^{3/2}}{3/2} \right]_0^a = \frac{-b}{3a} [0 - a^3] = \frac{ba^2}{3} \dots(viii)$$

Also the moment of total area A about y -axis

$$= A \times \bar{x} \dots(ix)$$

where $\bar{x}$ = Distance of C.G. of total area A from y -axis.

*Please refer some standard Textbook of Mathematics.

$$\int_0^a \sqrt{a^2 - x^2} \, dx = \left[\frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} \right]_0^a = \left[0 + \frac{1}{2} a^2 \sin^{-1} (1) \right]$$

$$= \frac{1}{2} a^2 \left(\frac{\pi}{2} \right) = \frac{\pi a^2}{4} \text{ .}$$

Equating the two values given by equations (viii) and (ix),

$$\begin{aligned}
 A \times \bar{x} &= \frac{ba^2}{3} \\
 \therefore \bar{x} &= \frac{ba^2}{3A} = \frac{ba^2}{3 \times \frac{\pi ab}{4}} \quad \left[\because A = \frac{\pi ab}{4} \text{ see equation (vi)} \right] \\
 &= \frac{4a}{3\pi} \quad \text{Ans.}
 \end{aligned}$$

The co-ordinates of the C.G. of given area are

$$\bar{x} = \frac{4a}{3\pi} \quad \text{and} \quad \bar{y} = \frac{4b}{3\pi}.$$

9.7. CENTROID OF VOLUME

Centroid of volume is the point at which the total volume of a body is assumed to be concentrated. The volume is having three dimensions *i.e.*, length, width and thickness. Hence volume is measured in $[\text{length}]^3$. The centroid [*i.e.*, or centre of gravity] of a volume is obtained by dividing the given volume into a large number of small volumes as shown in Fig. 9.10 (d). Similar method was used for finding the centroid of an area in which case the given area was divided into large number of small areas. The centroid of the volume is hence obtained by replacing dA by dv in equations (9.2A) and (9.2B).

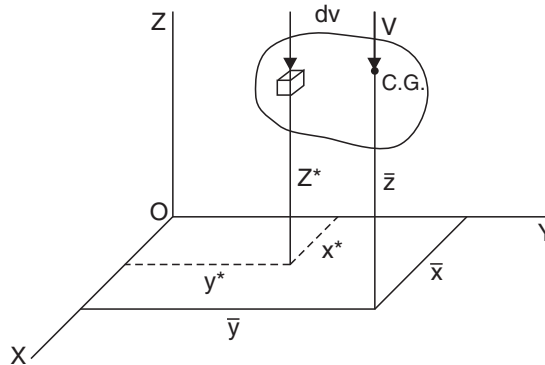


Fig. 9.10 (d)

Then these equations becomes as

$$\bar{x} = \frac{\int x^* dv}{\int dv} \quad \dots(9.3A)$$

and

$$\bar{y} = \frac{\int y^* dv}{\int dv} \quad \dots(9.3B)$$

As volume is having three dimensions, hence third equation is written as

$$\bar{z} = \frac{\int z^* dv}{\int dv} \quad \dots(9.3C)$$

where x^* = Distance of C.G. of small volume dv from y - z plane (*i.e.*, from axis OY)

y^* = Distance of C.G. of small volume dv from x - z plane (i.e., from axis OX)

z^* = Distance of C.G. of small volume dv from x - y plane

and $\bar{x}$, $\bar{y}$, $\bar{z}$ = Location of centroid of total volume.

Note. If a body has a plane of symmetry, the centre of gravity lies in that plane. If it has two planes of symmetry, the line of intersection of the two planes gives the position of centre of gravity. If it has three planes of symmetry, the point of intersection of the three planes gives the position of centre of gravity.

Problem 9.12. A right circular cone of radius R at the base and of height h is placed as shown in Fig. 9.10 (e). Find the location of the centroid of the volume of the cone.

Sol. Given :

Radius of cone = R

Height of cone = h

In the Fig. 9.10 (e), the axis of the cone is along x -axis. The centroid will be at the x -axis. Hence, $\bar{y} = 0$ and $\bar{z} = 0$.

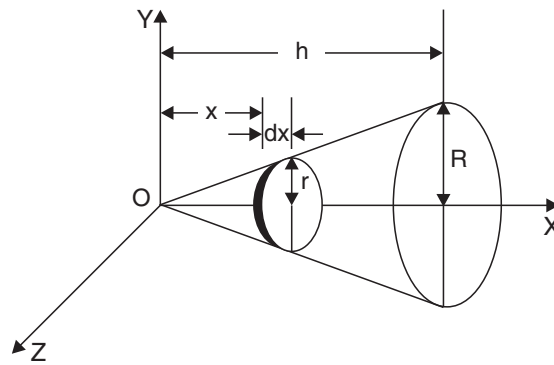


Fig. 9.10 (e)

To find $\bar{x}$, consider a small volume dv . For this, take a thin circular plate at a distance x from O . Let the thickness of the plate is dx as shown in figure and radius of the plate is r . The centroid of the plate is at a distance ' x ' from O . Hence $x^* = x$.

Now volume of the thin plate,

$$dv = \pi r^2 \times dx \quad \dots(i)$$

Let us find the value of r in terms of x .

From similar triangles, we get

$$\frac{R}{r} = \frac{h}{x}$$

or

$$r = \frac{R \times x}{h}$$

Substituting the value of r in equation (i), we get

$$dv = \pi \left(\frac{R \times x}{h} \right)^2 dx \quad \dots(ii)$$

Now $\bar{x}$ is given by equation (9.3A) as

$$\begin{aligned}
 \bar{x} &= \frac{\int x^* dv}{\int dv} = \frac{\int x dv}{\int dv} & [\because \text{Here } x^* = x] \\
 &= \frac{\int x \cdot \pi \left(\frac{R \times x}{h} \right)^2 dx}{\int \pi \left(\frac{R \times x}{h} \right)^2 dx} & \left[\because dv = \pi \left(\frac{R \times x}{h} \right)^2 dx \text{ from equation (ii)} \right] \\
 &= \frac{\frac{\pi \times R^2}{h^2} \int_0^h x^3 dx}{\pi \times \frac{R^2}{h^2} \int_0^h x^2 dx} \\
 & \quad [\because \text{Limits of integration are w.r.t. } x. \text{ And } x \text{ varies from } 0 \text{ to } h] \\
 &= \left[\frac{\left(\frac{x^4}{4} \right)}{\frac{x^3}{3}} \right]_0^h = \frac{3h}{4} \quad \text{Ans.}
 \end{aligned}$$

Problem 9.13. A hemisphere of radius R is placed as shown in Fig. 9.10 (f). The axis of symmetry is along z -axis. Find the centroid of the hemisphere.

Sol. The hemisphere is placed as shown in Fig. 9.10 (f). The axis of symmetry is taken as Z -axis. The centroid will be at the Z -axis. Hence $\bar{x} = 0$ and $\bar{y} = 0$.

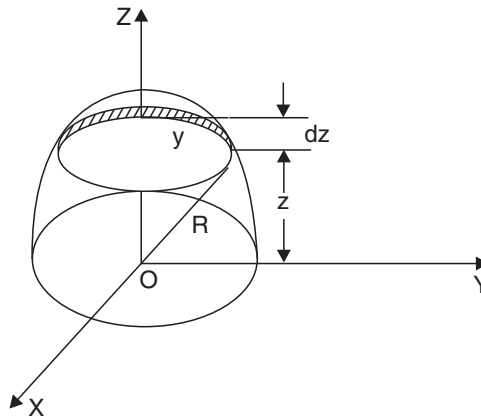


Fig. 9.10 (f)

Radius of hemisphere = R .

To find $\bar{z}$, consider a small volume dv of the hemisphere. For this, take a thin circular plate at a height z and thickness dz . Let 'y' is the radius of this plate.

$$\begin{aligned}
 \text{Then} \quad dv &= \text{Area of section} \times \text{thickness} \\
 &= \pi y^2 \times dz & \dots(i) \\
 & \quad (\because \text{Area of any section for sphere or hemisphere} = \pi r^2 \\
 & \quad \quad \quad \text{Here } r = y)
 \end{aligned}$$

The centre of gravity of the small volume is at a distance z from O .

Let us now, find the value of y in terms of z .

From Fig. 9.10 (f), we have

$$R^2 = z^2 + y^2$$

or

$$y^2 = R^2 - z^2$$

Substituting the value of y^2 in equation (i), we get

$$dv = \pi[R^2 - z^2] \times dz \quad \dots(ii)$$

As in this case, the axis of symmetry is Z -axis. Hence $\bar{x}$ and $\bar{y}$ are zero. The distance of the centroid from x - y plane is given by equation (9.3C) as

$$\bar{z} = \frac{\int z^* dv}{\int dv}$$

where $z^* =$ Distance of centroid of the small volume dv from x - y plane.

$= z$ [In the present case]

$$\begin{aligned} \therefore \bar{z} &= \frac{\int z dv}{\int dv} \\ &= \frac{\int z \times \pi(R^2 - z^2) dz}{\int \pi(R^2 - z^2) dz} \quad [\because \text{From equation (ii), } dv = \pi(R^2 - z^2) \times dz] \\ &= \frac{\int_0^R \pi(R^2 z - z^3) dz}{\int_0^R \pi(R^2 - z^2) dz} \end{aligned}$$

[The limits of integration are according to dz . Here z varies from O to R]

$$= \frac{\pi \left[\frac{R^2 z^2}{2} - \frac{z^4}{4} \right]_0^R}{\pi \left[R^2 z - \frac{z^3}{3} \right]_0^R} = \frac{\left[\frac{R^2 \times R^2}{2} - \frac{R^4}{4} \right]}{\left[R^2 \times R - \frac{R^3}{3} \right]} = \frac{\left(\frac{R^4}{4} \right)}{\frac{2}{3} R^3} = \frac{3}{8} R. \quad \text{Ans.}$$

9.8. DEFINITION OF MOMENT OF INERTIA OF AREA

Consider a thin lamina of area A as shown in Fig. 9.11.

Let x = Distance of the C.G. of area A from the axis OY .

y = Distance of the C.G. of area A from the axis OX .

Then moment of area about the axis OY

$=$ Area $\times$ perpendicular distance of C.G. of area from axis OY

$= Ax \quad \dots(9.3D)$

Equation (9.3D) is known as *first moment of area* about the axis OY . This first moment of area is used to determine the centre of gravity of the area.

If the moment of area given by equation (9.3D) is again multiplied by the perpendicular distance between the C.G. of the area and axis OY (i.e., distance x), then the quantity $(Ax) \cdot x = Ax^2$ is known as moment of the moment of area or *second moment of area* or area moment of inertia about the axis OY . This second moment of area is used in the study of mechanics of fluids and mechanics of solids.

Similarly, the moment of area (or first moment of area) about the axis $OX = Ay$.

And second moment of area (or area moment of inertia) about the axis $OX = (Ay) \cdot y = Ay^2$.

If, instead of area, the mass (m) of the body is taken into consideration then the second moment is known as second moment of mass. This second moment of mass is also known as mass moment of inertia.

Hence moment of inertia when mass is taken into consideration about the axis $OY = mx^2$ and about the axis $OX = my^2$.

Hence the product of the area (or mass) and the square of the distance of the centre of gravity of the area (or mass) from an axis is known as moment of inertia of the area (or mass) about that axis. Moment of inertia is represented by I . Hence moment of inertia about the axis OX is represented by I_{xx} whereas about the axis OY by I_{yy} .

The product of the area (or mass) and the square of the distance of the centre of gravity of the area (or mass) from an axis perpendicular to the plane of the area is known as polar moment of inertia and is represented by J .

Consider a plane area which is split up into small areas $a_1, a_2, a_3, \dots$ etc. Let the C.G. of the small areas from a given axis be at a distance of $r_1, r_2, r_3, \dots$ etc. as shown in Fig. 9.12.

Then the moment of inertia of the plane area about the given axis is given by

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + \dots \quad \dots(9.4)$$

$$\text{or} \quad I = \Sigma ar^2. \quad \dots(9.5)$$

9.8.1. Radius of Gyration. Radius of gyration of a body (or a given lamina) about an axis is a distance such that its square multiplied by the area gives moment of inertia of the area about the given axis.

For the Fig. 9.12, the moment of inertia about the given axis is given by equation (9.4) as

$$I = a_1 r_1^2 + a_2 r_2^2 + a_3 r_3^2 + \dots \quad \dots(i)$$

Let the whole mass (or area) of the body is concentrated at a distance k from the axis of reference, then the moment of inertia of the whole area about the given axis will be equal to Ak^2 .

If $Ak^2 = I$, then k is known as radius of gyration about the given axis.

$$\therefore k = \sqrt{\frac{I}{A}}. \quad \dots(9.6)$$

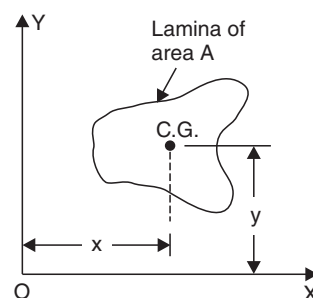


Fig. 9.11

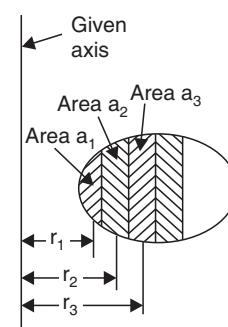


Fig. 9.12

9.9. PERPENDICULAR AXIS THEOREM AND POLAR MOMENT OF INERTIA

Theorem of the perpendicular axis states that if I_{XX} and I_{YY} be the moment of inertia of a plane section about two mutually perpendicular axis $X-X$ and $Y-Y$ in the plane of the section, then the moment of inertia of the section I_{ZZ} about the axis $Z-Z$, perpendicular to the plane and passing through the intersection of $X-X$ and $Y-Y$ is given by

$$I_{ZZ} = I_{XX} + I_{YY}.$$

The moment of inertia I_{ZZ} is also known as **polar moment of inertia**.

Proof. A plane section of area A and lying in plane $x-y$ is shown in Fig. 9.13. Let OX and OY be the two mutually perpendicular axes, and OZ be the perpendicular axis. Consider a small area dA .

Let x = Distance of dA from the axis OY

y = Distance of dA from axis OX

r = Distance of dA from axis OZ

Then $r^2 = x^2 + y^2$.

Now moment of inertia of dA about x -axis

$$= dA \times (\text{Distance of } dA \text{ from } x\text{-axis})^2$$

$$= dA \times y^2.$$

$\therefore$ Moment of inertia of total area A about x -axis,

$$I_{XX} = \Sigma dA y^2.$$

Similarly, moment of inertia of total area A about y -axis, $I_{YY} = \Sigma dA x^2$

and moment of inertia of total area A about z -axis,

$$I_{ZZ} = \Sigma dA r^2$$

$$= \Sigma dA [x^2 + y^2]$$

$$(\because r^2 = x^2 + y^2)$$

$$= \Sigma dA x^2 + \Sigma dA y^2$$

$$= I_{YY} + I_{XX}$$

or

$$I_{ZZ} = I_{XX} + I_{YY}. \quad \dots(9.7)$$

The above equation shows that the moment of inertia of an area about an axis at origin normal to x, y plane is the sum of moments of inertia about the corresponding x and y -axis.

In equation (9.7), I_{ZZ} is known as **Polar Moment of Inertia**.

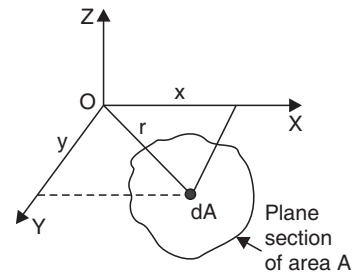


Fig. 9.13

9.10. PARALLEL AXIS THEOREM

It states that if the moment of inertia of a plane area about an axis in the plane of area through the C.G. of the plane area be represented by I_G , then the moment of the inertia of the given plane area about a parallel axis AB in the plane of area at a distance h from the C.G. of the area is given by

$$I_{AB} = I_G + Ah^2.$$

where I_{AB} = Moment of inertia of the given area about AB

I_G = Moment of inertia of the given area about C.G.

A = Area of the section

h = Distance between the C.G. of the section and the axis AB .

Proof. A lamina of plane area A is shown in Fig. 9.14.

Let $X-X$ = The axis in the plane of area A and passing through the C.G. of the area.

AB = The axis in the plane of area A and parallel to axis $X-X$.

h = Distance between AB and $X-X$.

Consider a strip parallel to $X-X$ axis at a distance y from the $X-X$ axis.

Let the area of the strip = dA

Moment of inertia of area dA about $X-X$ axis = dAy^2 .

∴ Moment of inertia of the total area about $X-X$ axis,

$$I_{XX} \text{ or } I_G = \Sigma dAy^2 \quad \dots(i)$$

Moment of inertia of the area dA about AB

$$= dA(h + y)^2$$

$$= dA[h^2 + y^2 + 2hy].$$

∴ Moment of inertia of the total area A about AB ,

$$\begin{aligned} I_{AB} &= \Sigma dA[h^2 + y^2 + 2hy] \\ &= \Sigma dAh^2 + \Sigma dAy^2 + \Sigma dA 2hy. \end{aligned}$$

As h or h^2 is constant and hence they can be taken outside the summation sign. Hence the above equation becomes

$$I_{AB} = h^2 \Sigma dA + \Sigma dAy^2 + 2h \Sigma dAy.$$

But $\Sigma dA = A$. Also from equation (i), $\Sigma dAy^2 = I_G$. Substituting these values in the above equation, we get

$$I_{AB} = h^2 \cdot A + I_G + 2h \Sigma dAy. \quad \dots(ii)$$

But $dA \cdot y$ represents the moment of area of strip about $X-X$ axis. And ΣdAy represents the moments of the total area about $X-X$ axis. But the moments of the total area about $X-X$ axis is equal to the product of total area (A) and the distance of the C.G. of the total area from $X-X$ axis. As the distance of the C.G. of the total area from $X-X$ axis is zero, hence ΣdAy will be equal to zero.

Substituting this value in equation (ii), we get

$$I_{AB} = h^2 \cdot A + I_G + 0$$

$$\text{or} \quad I_{AB} = I_G + Ah^2 \quad \dots(9.8)$$

Thus if the moment of inertia of an area with respect to an axis in the plane of area (and passing through the C.G. of the area) is known, the moment of inertia with respect to any parallel axis in the plane may be determined by using the above equation.

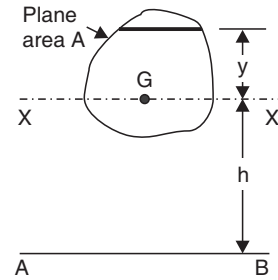


Fig. 9.14

9.11. MOMENT OF INERTIA OF SIMPLE AREAS BY INTEGRATION

The area moment of inertia of the following sections will be determined by the method of integration :

1. Moment of inertia of a rectangular section,
2. Moment of inertia of a circular section,
3. Moment of inertia of a triangular section,
4. Moment of inertia of a uniform thin rod.

9.11.1. Moment of Inertia of a Rectangular Section

1st Case. Moment of inertia of the rectangular section about the $X-X$ axis passing through the C.G. of the section.

Fig. 9.15 shows a rectangular section $ABCD$ having width = b and depth = d . Let $X-X$ is the horizontal axis passing through the C.G. of the rectangular section. We want to determine

the moment of inertia of the rectangular section about X-X axis. The moment of inertia of the given section about X-X axis is represent by I_{XX} .

Consider a rectangular elementary strip of thickness dy at a distance y from the X-X axis as shown in Fig. 9.15.

Area of the strip = $b \cdot dy$.

Moment of inertia of the area of the strip about X-X axis

$$\begin{aligned} &= \text{Area of strip} \times y^2 \\ &= (b \cdot dy) \times y^2 = by^2 dy. \end{aligned}$$

Moment of inertia of the whole section will be obtained by

integrating the above equation between the limits $-\frac{d}{2}$ to $\frac{d}{2}$.

$$\begin{aligned} \therefore I_{XX} &= \int_{-d/2}^{d/2} by^2 dy = b \int_{-d/2}^{d/2} y^2 dy \\ (\because b \text{ is constant and can be taken outside the integral sign}) \\ &= b \left[\frac{y^3}{3} \right]_{-d/2}^{d/2} = \frac{b}{3} \left[\left(\frac{d}{2} \right)^3 - \left(-\frac{d}{2} \right)^3 \right] \\ &= \frac{b}{3} \left[\frac{d^3}{8} - \left(-\frac{d^3}{8} \right) \right] = \frac{b}{3} \left[\frac{d^3}{8} + \frac{d^3}{8} \right] \\ &= \frac{b}{3} \cdot \frac{2d^3}{8} = \frac{bd^3}{12}. \end{aligned} \quad \dots(9.9)$$

Similarly, the moment of inertia of the rectangular section about Y-Y axis passing through the C.G. of the section is given by

$$I_{YY} = \frac{db^3}{12}. \quad \dots(9.10)$$

Refer to Fig. 9.15 (a)

Area of strip, $dA = d \times dx$

M.O.I. of strip above Y-Y axis = $dA \times x^2$

$$\begin{aligned} &= (d \times dx) \times x^2 \quad (\because dA = d \cdot dx) \\ &= d \times x^2 \times dx \end{aligned}$$

$$\begin{aligned} \therefore I_{YY} &= \int_{-b/2}^{b/2} d \times x^2 \times dx = d \left[\frac{x^3}{3} \right]_{-b/2}^{b/2} \\ &= \frac{d}{3} \left[\left(\frac{b}{2} \right)^3 - \left(-\frac{b}{2} \right)^3 \right] \\ &= \frac{d}{3} \left[\frac{b^3}{8} + \frac{b^3}{8} \right] = \frac{d}{3} \cdot \frac{b^3}{4} = \frac{db^3}{12}. \end{aligned}$$

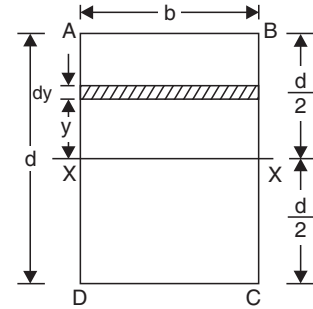


Fig. 9.15

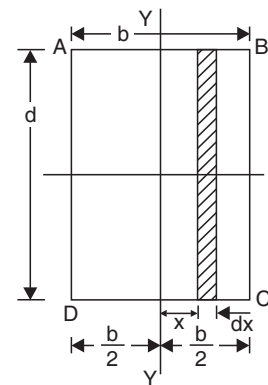


Fig. 9.15 (a)

2nd Case. Moment of inertia of the rectangular section about a line passing through the base.

Fig. 9.16 shows a rectangular section $ABCD$ having width = b and depth = d . We want to find the moment of inertia of the rectangular section about the line CD , which is the base of the rectangular section.

Consider a rectangular elementary strip of thickness dy at a distance y from the line CD as shown in Fig. 9.12.

Area of strip = $b \cdot dy$.

Moment of inertia of the area of strip about the line CD

$$\begin{aligned} &= \text{Area of strip} \cdot y^2 \\ &= b \cdot dy \cdot y^2 = by^2 dy. \end{aligned}$$

Moment of inertia of the whole section about the line CD is obtained by integrating the above equation between the limits 0 to d .

$\therefore$ Moment of inertia of the whole section about the line CD .

$$\begin{aligned} &= \int_0^d by^2 dy = b \int_0^d y^2 dy \\ &= b \left[\frac{y^3}{3} \right]_0^d = \frac{bd^3}{3}. \end{aligned} \quad \dots(9.11)$$

3rd Case. Moment of inertia of a hollow rectangular section.

Fig. 9.17 shows a hollow rectangular section in which $ABCD$ is the main section and $EFGH$ is the cut-out section.

The moment of inertia of the main section $ABCD$ about $X-X$ axis is given by equation (9.7),

$$= \frac{bd^3}{12}$$

where b = Width of main section

d = Depth.

The moment of inertia of the cut-out section $EFGH$ about $X-X$ axis

$$= \frac{b_1 d_1^3}{12}$$

where b_1 = Width of the cut-out section, and

d_1 = Depth of the cut-out section.

Then moment of inertia of hollow rectangular section about $X-X$ axis,

I_{XX} = Moment of inertia of rectangle $ABCD$ about $X-X$ axis—moment of inertia of rectangle $EFGH$ about $X-X$ axis

$$= \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}.$$

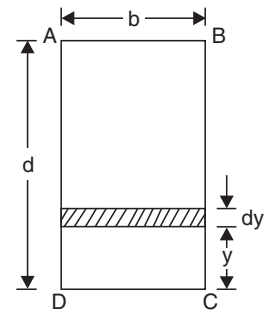


Fig. 9.16

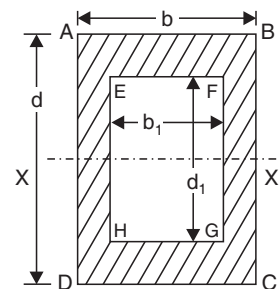


Fig. 9.17

9.11.2. Moment of Inertia of a Circular Section. Fig. 9.18 shows a circular section of radius R with O as centre. Consider an elementary circular ring of radius ' r ' and thickness ' dr '. Area of circular ring

$$= 2\pi r \cdot dr.$$

In this case first find the moment of inertia of the circular section about an axis passing through O and perpendicular to the plane of the paper. This moment of inertia is also known as **polar moment of inertia**. Let this axis be $Z-Z$. (Axis $Z-Z$ is not shown in Fig. 9.18). Then from the theorem of perpendicular axis, the moment of inertia about $X-X$ axis or $Y-Y$ axis is obtained.

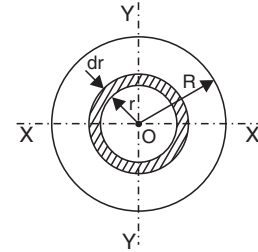


Fig. 9.18

Moment of inertia of the circular ring about an axis passing through O and perpendicular to the plane of the paper

$$\begin{aligned} &= (\text{Area of ring}) \times (\text{radius of ring from } O)^2 \\ &= (2\pi r \cdot dr) \cdot r^2 \\ &= 2\pi r^3 dr \end{aligned} \quad \dots(i)$$

Moment of inertia of the whole circular section is obtained by integrating equation (i) between the limit O to R .

$\therefore$ Moment of inertia of the whole section about an axis passing through O and perpendicular to the plane of the paper is given as

$$\begin{aligned} I_{ZZ} &= \int_0^R 2\pi r^3 dr = 2\pi \int_0^R r^3 dr \\ &= 2\pi \left[\frac{r^4}{4} \right]_0^R = 2\pi \frac{R^4}{4} = \frac{\pi R^4}{2}. \end{aligned}$$

But $R = \frac{D}{2}$

where D = Diameter of the circular section

$$\therefore I_{ZZ} = \frac{\pi}{2} \times \left(\frac{D}{2} \right)^4 = \frac{\pi D^4}{32} \quad \dots(9.12)$$

or Polar moment of inertia = $\frac{\pi D^4}{32}$

But from the theorem of perpendicular axis given by equation (9.7), we have

$$I_{ZZ} = I_{XX} + I_{YY}.$$

But due to symmetry, $I_{XX} = I_{YY}$

$$\begin{aligned} \therefore I_{XX} &= I_{YY} = \frac{I_{ZZ}}{2} \\ &= \frac{\pi D^4}{32} \times \frac{1}{2} = \frac{\pi D^4}{64} \end{aligned} \quad \dots(9.13)$$

Moment of inertia of a hollow circular section

Fig. 9.19 shows a hollow circular section.

Let D = Diameter of outer circle, and

d = Diameter of cut-out circle.

Then from equation (9.13), the moment of inertia of the outer

circle about $X-X$ axis = $\frac{\pi}{64} D^4$.

And moment of inertia of the cut-out circle about $X-X$ axis

$$= \frac{\pi}{64} d^4.$$

$\therefore$ Moment of inertia of the hollow circular section, about $X-X$ axis,

I_{XX} = Moment of inertia of outer circle—moment of inertia of cut-out circle

$$= \frac{\pi}{64} D^4 - \frac{\pi}{64} d^4 = \frac{\pi}{64} [D^4 - d^4]$$

Similarly, $I_{YY} = \frac{\pi}{64} [D^4 - d^4]$.

9.11.3. Moment of Inertia of a Triangular Section

1st Case. Moment of inertia of a triangular section about its base.

Fig. 9.20 shows a triangular section ABC of base width = b and height = h . Consider a small strip of thickness dy at a distance y from the vertex A .

Area of the strip = $DE \cdot dy$... (i)

The distance DE in terms of y , b and h is obtained from two similar triangles ADE and ABC as

$$\frac{DE}{BC} = \frac{y}{h}$$

$$\therefore DE = BC \cdot \frac{y}{h}$$

$$= \frac{b \cdot y}{h}$$

$$(\because BC = b)$$

Substituting this value of DE in equation (i), we get

$$\text{Area of strip} = \frac{by}{h} \cdot dy.$$

Distance of the strip from the base = $(h - y)$

$\therefore$ Moment of inertia of the strip about the base

$$= \text{Area of strip} \times (\text{Distance of strip from base})^2$$

$$= \frac{by}{h} \cdot dy \cdot (h - y)^2 = \frac{by}{h} (h - y)^2 \cdot dy.$$

The moment of inertia of the whole triangular section about the base (I_{BC}) is obtained by integrating the above equation between the limits 0 to h .

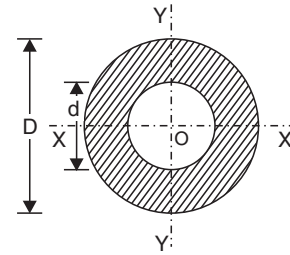


Fig. 9.19

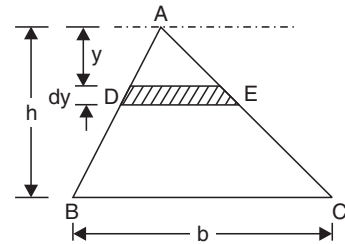


Fig. 9.20

$$\begin{aligned}
 \therefore I_{BC} &= \int_0^h \frac{by}{h} [h-y]^2 dy \\
 &= \frac{b}{h} \int_0^h y(h-y)^2 dy \\
 &\quad (\because b \text{ and } h \text{ are constants and can be taken outside the integral sign}) \\
 &= \frac{b}{h} \int_0^h y(h^2 + y^2 - 2hy) dy = \frac{b}{h} \int_0^h (yh^2 + y^3 - 2hy^2) dy \\
 &= \frac{b}{h} \left[\frac{y^2 h^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3} \right]_0^h \\
 &= \frac{b}{h} \left[\frac{h^2 \cdot h^2}{2} + \frac{h^4}{4} - \frac{2h \cdot h^3}{3} \right] = \frac{b}{h} \left[\frac{h^4}{2} + \frac{h^4}{4} - \frac{2h^4}{3} \right] \\
 &= \frac{b}{h} \cdot h^4 \left[\frac{6+3-8}{12} \right] = bh^3 \cdot \frac{1}{12} \\
 &= \frac{bh^3}{12} \quad \dots(9.14)
 \end{aligned}$$

2nd Case. Moment of inertia of the triangular section about an axis passing through the C.G. and parallel to the base.

Consider a triangular section of base = b and height = h as shown in Fig. 9.21. Let $X-X$ is the axis passing through the C.G. of the triangular section and parallel to the base.

The distance between the C.G. of the triangular section and base $AB = \frac{h}{3}$.

Now from the theorem of parallel axis, given by equation (9.8), we have

Moment of inertia about

$$BC = \text{Moment of inertia about C.G.} + \text{Area} \times (\text{Distance between } X-X \text{ and } BC)^2$$

or

$$I_{BC} = I_G + A \times \left(\frac{h}{3} \right)^2$$

$\therefore$

$$\begin{aligned}
 I_G &= I_{BC} - A \times \left(\frac{h}{3} \right)^2 \\
 &= \frac{bh^3}{12} - \left(\frac{b \times h}{2} \right) \cdot \left(\frac{h}{3} \right)^2 \quad \left(\because I_{BC} = \frac{bh^3}{12} \text{ and Area} = \frac{b \times h}{2} \right) \\
 &= \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3(3-2)}{36} \\
 &= \frac{bh^3}{36} \quad \dots(9.15)
 \end{aligned}$$

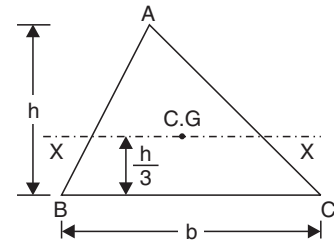


Fig. 9.21

Problem 9.13 (A). Determine the moment of inertia of the section about an axis passing through the base BC of a triangular section shown in Fig. 9.21 (a).

(U.P. Tech. University, 2002–2003)

Sol. Given :

Base, $b = 100$ mm ; height, $h = 90$ mm.

Moment of inertia of a triangular section about an axis passing through the base is given by equation (9.14) as

$$I_{BC} = \frac{bh^3}{12}$$

$$= \frac{100 \times 90^3}{12} = 6.075 \times 10^6 \text{ mm}^4. \quad \text{Ans.}$$

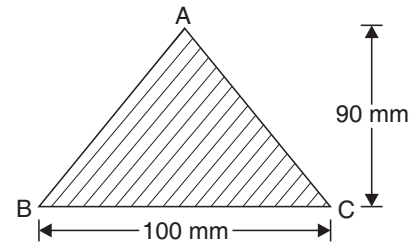


Fig. 9.21(a)

9.11.4. Moment of Inertia of a Uniform Thin Rod.

Consider a uniform thin rod AB of length L as shown in Fig. 9.22.

Let m = Mass per unit length of rod, and

M = Total mass of the rod

$$= m \times L \quad \dots(i)$$

Suppose it is required to find the moment of inertia of the rod about the axis $Y-Y$. Consider a strip of length dx at a distance x from the axis $Y-Y$.

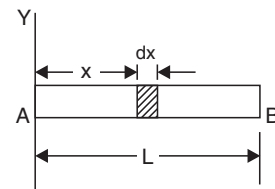


Fig. 9.22

$$\begin{aligned} \text{Mass of the strip} &= \text{Length of strip} \times \text{Mass per unit length} \\ &= dx \cdot m \quad \text{or} \quad m \cdot dx. \end{aligned}$$

$$\begin{aligned} \text{Moment of inertia of the strip about } Y-Y \text{ axis} &= \text{Mass of strip} \times x^2 \\ &= (m \cdot dx) \cdot x^2 \\ &= mx^2 dx. \end{aligned}$$

Moment of inertia of the whole rod (I_{YY}) will be obtained by integrating the above equation between the limits 0 to L .

$$\therefore I_{YY} = \int_0^L mx^2 dx = m \int_0^L x^2 dx \quad (\because m \text{ is constant})$$

$$= m \left[\frac{x^3}{3} \right]_0^L = \frac{mL^3}{3}$$

$$= \frac{mL \cdot L^2}{3} = \frac{ML^2}{3} \quad [\because m \cdot L = M \text{ from equation (i)}]$$

9.11.5. Moment of Inertia of Area Under a Curve of given Equation. Fig. 9.22 (a) shows an area under a curve whose equation is parabolic and is given by

$$x = ky^2$$

in which

$$y = b \quad \text{when} \quad x = a$$

Suppose it is required to find the moment of inertia of this area about y -axis. Consider a strip of thickness dx at a distance x from y -axis.

The area of strip, $dA = y \, dx$... (i)

Let us substitute the value of y in terms of x in the above equation. The equation of curve is

$$x = ky^2 \quad \dots (ii)$$

First find the value of k .

When $y = b$, $x = a$. Hence above equation becomes

$$a = kb^2$$

or

$$k = \frac{a}{b^2}$$

Substituting the value of ' k ' in equation (ii), we get

$$x = \frac{a}{b^2} \cdot y^2 \quad \text{or} \quad y^2 = \frac{b^2 x}{a}$$

or

$$y = \left(\frac{b^2 x}{a} \right)^{1/2} = \frac{b}{\sqrt{a}} \sqrt{x} \quad \dots (iii)$$

Substituting this value of y in equation (i), we get

$$dA = \frac{b}{\sqrt{a}} \cdot \sqrt{x} \cdot dx$$

The moment of inertia of elemental area (dA) about y -axis

$$= x^2 \cdot dA = x^2 \cdot \frac{b}{\sqrt{a}} \cdot \sqrt{x} \, dx$$

$\therefore$ Moment of inertia of the total area about y -axis is obtained by integrating the above equation between the limits O to a . ($\because x$ varies from O to a)

$$\begin{aligned} \therefore I_{yy} &= \int_0^a x^2 \cdot \frac{b}{\sqrt{a}} \cdot \sqrt{x} \cdot dx = \frac{b}{\sqrt{a}} \cdot \int_0^a x^{5/2} \cdot dx \\ &= \frac{b}{\sqrt{a}} \left[\frac{x^{7/2}}{7/2} \right]_0^a = \frac{2}{7} \cdot \frac{b}{\sqrt{a}} \cdot a^{7/2} = \frac{2}{7} ba^2. \quad \text{Ans.} \end{aligned}$$

To find the moment of inertia of the given area about x -axis, the element shown in Fig. 9.22 (a) can be considered to be a rectangle of thickness dx . The moment of inertia of this element about x -axis is equal to the moment of inertia of the rectangle about its base.

$\therefore$ Moment of inertia of the element about x -axis

$$= \frac{dx \cdot y^3}{3} \quad \left[\because \text{it is } \frac{bd^3}{3}, \text{ where } b = dx \text{ and } d = y \right]$$

The moment of inertia of the given area about x -axis is obtained by integrating the above equation between the limits O to a .

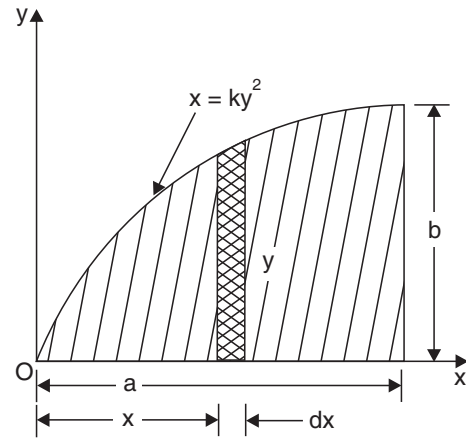


Fig. 9.22 (a)

$$\begin{aligned}
 \therefore I_{XX} &= \int_0^a \frac{dx \cdot y^3}{3} = \int_0^a \frac{y^3}{3} \cdot dx \\
 &= \int_0^a \frac{\left[\frac{b}{\sqrt{a}} \cdot \sqrt{x} \right]^3}{3} dx \quad \left[\because y = \frac{b}{\sqrt{a}} \sqrt{x} \text{ from equation (iii)} \right] \\
 &= \frac{b^3}{3a^{3/2}} \int_0^a x^{3/2} dx = \frac{b^3}{3a^{3/2}} \left[\frac{x^{5/2}}{5/2} \right]_0^a \\
 &= \frac{b^3}{3 \cdot a^{3/2}} \cdot \frac{2}{5} \cdot a^{5/2} = \frac{2}{15} b^3 \cdot a = \frac{2}{15} ab^3. \quad \text{Ans.}
 \end{aligned}$$

9.12. MOMENT OF INERTIA OF COMPOSITE AREAS

Problem 9.14. Fig. 9.23 shows a T-section of dimensions $10 \times 10 \times 2$ cm. Determine the moment of inertia of the section about the horizontal and vertical axes, passing through the centre of gravity of the section. Also find the polar moment of inertia of the given T-section.

Sol. First of all, find the location of centre of gravity of the given T-section. The given section is symmetrical about the axis Y-Y and hence the C.G. of the section will lie on Y-Y axis. The given section is split up into two rectangles ABCD and EFGH for calculating the C.G. of the section.

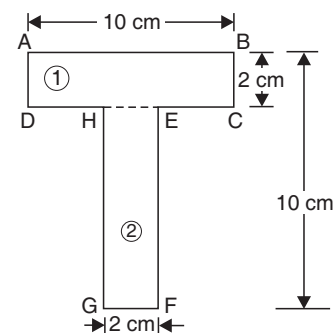


Fig. 9.23

Let $\bar{y}$ = Distance of the C.G. of the section from the bottom line GF

$$a_1 = \text{Area of rectangle } ABCD = 10 \times 2 = 20 \text{ cm}^2$$

$$y_2 = \text{Distance of C.G. of the area } a_1 \text{ from the bottom line } GF = 8 + 1 = 9 \text{ cm}$$

$$a_2 = \text{Area of rectangle } EFGH = 8 \times 2 = 16 \text{ cm}^2$$

$$y_2 = \text{Distance of C.G. of rectangle } EFGH \text{ from the bottom line } GF = \frac{8}{2} = 4 \text{ cm}$$

$$\text{Using the relation, } \bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{20 \times 9 + 16 \times 4}{20 + 16} = \frac{180 + 64}{36} = \frac{244}{36} = 6.777 \text{ cm.}$$

Hence the C.G. of the given section lies at a distance of 6.777 cm from GF. Now find the moment of inertia of the T-section.

Now, Let I_{G_1} = Moment of inertia of rectangle (1) about the horizontal axis and passing through its C.G.

I_{G_2} = Moment of inertia of rectangle (2) about the horizontal axis and passing through the C.G. of the rectangle (2)

h_1 = The distance between the C.G. of the given section and the C.G. of the rectangle (1)

$$= y_1 - \bar{y} = 9.0 - 6.777 = 2.223 \text{ cm}$$

h_2 = The distance between the C.G. of the given section and the C.G. of the rectangle (2)

$$= \bar{y} - y_2 = 6.777 - 4.0 = 2.777 \text{ cm.}$$

Now,

$$I_{G_1} = \frac{10 \times 2^3}{2} = 6.667 \text{ cm}^4$$

$$I_{G_2} = \frac{2 \times 8^3}{12} = 85.333 \text{ cm}^4.$$

From the theorem of parallel axes, the moment of inertia of the rectangle (1) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_1} + a_1 h_1^2 = 6.667 + 20 \times (2.223)^2$$

$$= 6.667 + 98.834 = 105.501 \text{ cm}^4.$$

Similarly, the moment of inertia of the rectangle (2) about the horizontal axis passing through the C.G. of the given section

$$= I_{G_2} + a_2 h_2^2 = 85.333 + 16 \times (2.777)^2$$

$$= 85.333 + 123.387 = 208.72 \text{ cm}^4.$$

$\therefore$ The moment of inertia of the given section about the horizontal axis passing through the C.G. of the given section is,

$$I_{xx} = 105.501 + 208.72 = \mathbf{314.221 \text{ cm}^4}. \text{ Ans.}$$

The moment of inertia of the given section about the vertical axis passing through the C.G. of the given section is,

$$I_{yy} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12}$$

$$= 166.67 + 5.33 = \mathbf{172 \text{ cm}^4}. \text{ Ans.}$$

Now the polar moment of inertia (I_{zz}) is obtained from equation (9.7) as

$$I_{zz} = I_{xx} + I_{yy}$$

$$= 314.221 + 221 = \mathbf{486.221 \text{ cm}^4}. \text{ Ans.}$$

Problem 9.15. Find the moment of inertia of the section shown in Fig. 9.24 about the centroidal axis $X-X$ perpendicular to the web.
(AMIE, Summer 1977)

Sol. First of all find the location of centre of gravity of the given figure. The given section is symmetrical about the axis $Y-Y$ and hence the C.G. of the section will lie on $Y-Y$ axis. The given section is split up into three rectangles $ABCD$, $EFGH$ and $JKLM$. The centre of gravity of the section is obtained by using

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} \quad \dots(i)$$

where $\bar{y}$ = Distance of the C.G. of the section from the bottom line ML

$$a_1 = \text{Area of rectangle } ABCD = 10 \times 2 = 20 \text{ cm}^2$$

$$y_1 = \text{Distance of the C.G. of the rectangle } ABCD \text{ from the bottom line } ML$$

$$= 2 + 10 + \frac{2}{2} = 12 + 1 = 13 \text{ cm}$$

$$a_2 = \text{Area of rectangle } EFGH = 10 \times 2 = 20 \text{ cm}^2$$

$$y_2 = \text{Distance of the C.G. of rectangle } EFGH \text{ from the bottom line } ML$$

$$= 2 + \frac{10}{2} = 2 + 5 = 7 \text{ cm}$$

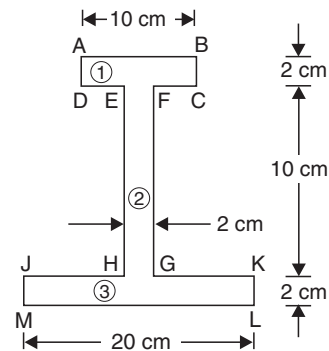


Fig. 9.24

$$a_3 = \text{Area of rectangle } JKLM = 20 \times 2 = 40 \text{ cm}^2$$

$$y_3 = \text{Distance of the C.G. of rectangle } JKLM \text{ from the bottom line } ML$$

$$= \frac{2}{2} = 1.0 \text{ cm.}$$

Substituting the above values in equation (i), we get

$$\begin{aligned}\bar{y} &= \frac{20 \times 13 + 20 \times 7 + 40 \times 1}{20 + 20 + 40} \\ &= \frac{260 + 140 + 40}{80} = \frac{440}{80} = 5.50 \text{ cm.}\end{aligned}$$

The C.G. of the given section lies at a distance of 5.50 cm from the bottom line ML . We want to find the moment of inertia of the given section about a horizontal axis passing through the C.G. of the given section.

Let I_{G_1} = Moment of inertia of rectangle (1) about the horizontal axis passing through its C.G.

I_{G_2} = Moment of inertia of rectangle (2) about the horizontal axis passing through the C.G. of rectangle (2)

I_{G_3} = Moment of inertia of rectangle (3) about the horizontal axis passing through the C.G. of rectangle (3)

h_1 = The distance between the C.G. of the rectangle (1) and the C.G. of the given section

$$= y_1 - \bar{y} = 13.0 - 5.50 = 7.50 \text{ cm}$$

h_2 = The distance between the C.G. of rectangle (2) and the C.G. of the given section

$$= y_2 - \bar{y} = 7.0 - 5.50 = 1.50 \text{ cm}$$

h_3 = The distance between the C.G. of the rectangle (3) and the C.G. of the given section

$$= \bar{y} - y_3 = 5.50 - 1.0 = 4.5 \text{ cm}$$

$$\text{Now, } I_{G_1} = \frac{10 \times 2^3}{12} = 6.667 \text{ cm}^4$$

$$I_{G_2} = \frac{2 \times 10^3}{12} = 166.667 \text{ cm}^4$$

$$I_{G_3} = \frac{20 \times 2^3}{12} = 13.333 \text{ cm}^4.$$

From the theorem of parallel axes, the moment of inertia of the rectangle (1) about the horizontal axis passing through the C.G. of the given section

$$\begin{aligned}&= I_{G_1} + a_1 h_1^2 = 6.667 + 20 \times (7.5)^2 \\ &= 6.667 + 1125 = 1131.667 \text{ cm}^4.\end{aligned}$$

Similarly, the moment of inertia of the rectangle (2) about the horizontal axis passing through the C.G. of the given section

$$\begin{aligned}&= I_{G_2} + a_2 h_2^2 = 166.667 + 20 \times 1.5^2 \\ &= 166.667 + 45 = 211.667 \text{ cm}^4.\end{aligned}$$

And moment of inertia of the rectangle (3) about the horizontal axis, passing through the C.G. of the given section

$$= I_{G_3} + a_3 h_3^2 = 13.333 + 40 \times 4.5^2$$

$$= 13.333 + 810 = 823.333 \text{ cm}^4$$

Now moment of inertia of the given section about the horizontal axis, passing through the C.G. of the given section

$$= \text{Sum of the moment of inertia of the rectangles (1), (2) and (3) about the horizontal axis, passing through the C.G. of the given section}$$

$$= 1131.667 + 211.667 + 823.333 = \mathbf{2166.667 \text{ cm}^4}. \text{ Ans.}$$

Problem 9.15 (A). Determine the polar moment of inertia of I-section shown in Fig. 9.24 (a). (All dimensions are in mm). (U.P. Tech. University, 2001–2002)

Sol. Let us first find the location of C.G. of the given section. It is symmetrical about the vertical axis, hence C.G. lies on this section.

Now,

$$A_1 = \text{Area of first rectangle}$$

$$= 80 \times 12 = 960 \text{ mm}^2$$

$$A_2 = \text{Area of second rectangle}$$

$$[(150 - 12 - 10) \times 12]$$

$$= 128 \times 12 = 1536 \text{ mm}^2$$

$$A_3 = \text{Area of third rectangle}$$

$$= 120 \times 10 = 1200 \text{ mm}^2$$

$$y_1 = \text{Distance of C.G. of area } A_1$$

$$\text{from bottom line}$$

$$= 150 - \frac{12}{2} = 144 \text{ mm}$$

$$y_2 = \text{Distance of C.G. of area } A_2 \text{ from bottom line}$$

$$= 10 + \frac{128}{2} = 74 \text{ mm}$$

$$y_3 = \text{Distance of C.G. of area } A_3 \text{ from bottom line} = \frac{10}{2} = 5 \text{ cm.}$$

$$\bar{y} = \text{Distance of C.G. of the given section}$$

from bottom line.

The C.G. of the section is obtained by using,

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$$= \frac{960 \times 144 + 1536 \times 74 + 1200 \times 5}{960 + 1536 + 1200}$$

$$= \frac{138240 + 113664 + 6000}{3696} = \frac{257904}{3696}$$

$$= 69.779 \approx 69.78 \text{ cm}$$

Location of centroidal axis is shown in Fig. 9.24 (b).

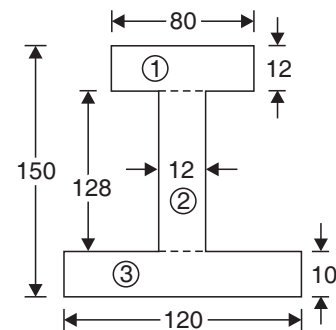


Fig. 9.24 (a)

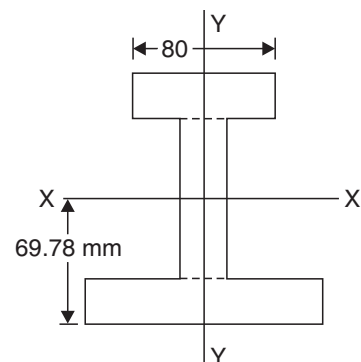


Fig. 9.24 (b)

(i) Moment of inertia of the given section about X-X

M.O.I. of the rectangle ① about centroid axis X-X is given by,

$$\begin{aligned} I_{XX1} &= (I_{G_1})_X + A_1 \times h_1^2, \text{ where } h_1 = (y_1 - \bar{y}) \\ &= \frac{80 \times 12^3}{12} + 960(144 - 69.78)^2 = 5.3 \times 10^6 \text{ mm}^4 \end{aligned}$$

M.O.I. of rectangle ② about centroid axis X-X is given by,

$$\begin{aligned} I_{XX2} &= (I_{G_2})_X + A_2 \times h_2^2, \text{ where } h_2 = (y_2 - \bar{y}) \\ &= \frac{12 \times 128^3}{12} + 1536 \times (74 - 69.78)^2 \\ &= 2.12 \times 10^6 \text{ mm}^4 \end{aligned}$$

and

$$\begin{aligned} I_{XX3} &= (I_{G_3})_X + A_3 \times h_3^2, \text{ where } h_3 = (y_3 - \bar{y}) \\ &= \frac{120 \times 10^3}{12} + 1200 \times (5 - 69.78)^2 = 5.04 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} \therefore I_{XX} &= I_{XX1} + I_{XX2} + I_{XX3} \\ &= 5.3 \times 10^6 + 2.12 \times 10^6 + 5.04 \times 10^6 \text{ mm}^4 \\ &= 12.46 \times 10^6 \text{ mm}^4 \end{aligned}$$

(ii) M.O.I. of the given section about Y-Y

$$I_{YY1} = (I_{G_1})_Y = \frac{12 \times 80^3}{12} = 521 \times 10^3 \text{ mm}^4 = 0.521 \times 10^6 \text{ mm}^4$$

$$I_{YY2} = (I_{G_2})_Y = \frac{128 \times 12^3}{12} = 18.432 \times 10^3 \text{ mm}^4 = 0.018432 \times 10^6 \text{ mm}^4$$

$$I_{YY3} = (I_{G_3})_Y = \frac{10 \times 120^3}{12} = 1.44 \times 10^6 \text{ mm}^4$$

$$\begin{aligned} \therefore I_{Y-Y} &= I_{YY1} + I_{YY2} + I_{YY3} \\ &= 0.521 \times 10^6 + 0.018432 \times 10^6 + 1.44 \times 10^6 \text{ mm}^4 = 1.979 \times 10^6 \text{ mm}^4 \end{aligned}$$

$\therefore$ Polar moment of inertia (I_{ZZ}) is given by,

$$\begin{aligned} I_{ZZ} &= I_{XX} + I_{YY} \\ &= 12.46 \times 10^6 + 1.979 \times 10^6 \text{ mm}^4 \\ &= 14.439 \times 10^6 \text{ mm}^4. \text{ Ans.} \end{aligned}$$

Problem 9.16. Find the moment of inertia of the area shown shaded in Fig. 9.25, about edge AB. (U.P. Tech. University, May 2009)

Sol. Given :

Radius of semi-circle, $R = 10 \text{ cm}$

Width of rectangle, $b = 20 \text{ cm}$

Depth of rectangle, $d = 25 \text{ cm}$

Moment of inertia of the shaded portion about AB

= M.O.I. of rectangle ABCD about AB

– M.O.I. of semi-circle on DC about AB

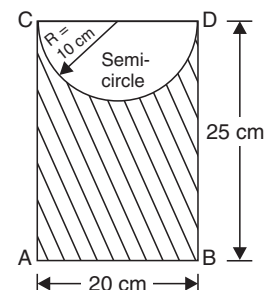


Fig. 9.25

M.O.I. of rectangle $ABCD$ about AB

$$= \frac{bd^3}{3} \quad [\text{see equation (9.11)}]$$

$$= \frac{20 \times 25^3}{12} = 104.167 \text{ cm}^4$$

M.O.I. of semi-circle about DC

$$= \frac{1}{2} \times [\text{M.O.I. of a circle of radius 10 cm about a diameter}]$$

$$= \frac{1}{2} \times \left[\frac{\pi}{64} d^4 \right] = \frac{1}{2} \times \frac{\pi}{64} \times 20^4 = 3.925 \text{ cm}^4$$

Distance of C.G. of semi-circle from DC

$$= \frac{4r}{3\pi} = \frac{4 \times 10}{3\pi} = 4.24 \text{ cm}$$

$$\text{Area of semi-circle, } A = \frac{\pi r^2}{2} = \frac{\pi \times 10^2}{2} = 157.1 \text{ cm}^2$$

M.O.I. of semi-circle about a line through its C.G. parallel to CD

$$= \text{M.O.I. of semi-circle about } CD - \text{Area} \times [\text{Distance of C.G. of semi-circle from } DC]^2$$

$$= 3925 - 157.1 \times 4.24^2$$

$$= 3925 - 2824.28 = 1100.72 \text{ cm}^4$$

Distance of C.G. of semi-circle from AB

$$= 25 - 4.24 = 20.76 \text{ cm}$$

M.O.I. of semi-circle about $AB = 1100.72 + 157.1 \times 20.76^2$

$$= 1100.72 + 67706.58 = 68807.30 \text{ cm}^4$$

$\therefore$ M.O.I. of shaded portion about AB

$$= 104.167 - 68807.30 = \mathbf{35359.7 \text{ cm}^4. \text{ Ans.}}$$

Problem 9.16 (A). Find the moments of inertia about the centroidal XX and YY axes of the section shown in Fig. 9.25 (a). (U.P. Tech. University, 2002–2003)

Sol. First find the location of the C.G. of the given figure.

Let $a_1 = \text{Area of complete rectangle}$
 $= B \times D$

$a_2 = \text{Area of removed rectangle portion}$

$$= \frac{B}{2} \times \frac{D}{2} = \frac{BD}{4}$$

$$x_1 = \frac{B}{2}, y_1 = \frac{D}{2}$$

and

$$x_2 = \frac{B}{2} + \frac{1}{2} \left(\frac{B}{2} \right) = \frac{3B}{4}, \quad y_2 = \frac{D}{2} + \frac{1}{2} \left(\frac{D}{2} \right) = \frac{3D}{4}$$

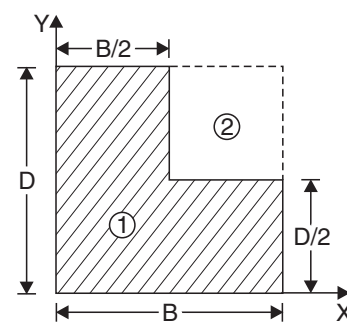


Fig. 9.25 (a)

where (x_1, y_1) and (x_2, y_2) are the co-ordinates of the C.G. of the complete rectangle and cut out rectangle respectively. Area a_2 is negative.

$$\begin{aligned} \text{Now, } \bar{x} &= \frac{a_1 x_1 - a_2 y_2}{a_1 - a_2} = \frac{BD \times \frac{B}{2} - \frac{B \times D}{4} \times \frac{3B}{4}}{\frac{3}{4}BD} \\ &= \frac{\frac{B^2 \times D}{2} - \frac{3}{16}B^2D}{\frac{3}{4}BD} = \frac{\frac{5}{16}BD^2}{\frac{3}{4}BD} = \frac{5}{12} \times D \end{aligned}$$

$$\begin{aligned} \text{Similarly, } \bar{y} &= \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} = \frac{BD \times \frac{D}{2} - \frac{BD}{4} \times \frac{3D}{4}}{\frac{3}{4}BD} \\ &= \frac{\frac{BD^2}{2} - \frac{3}{16}BD^2}{\frac{3}{4}BD} = \frac{\frac{5}{16}BD^2}{\frac{3}{4}BD} = \frac{5}{12} \times D \end{aligned}$$

Now draw the centroidal axes X-X and Y-Y as shown in Fig. 9.25 (b).

Let I_{XX1} = M.O.I. of complete rectangle ① about centroidal axis X-X

= M.O.I. of complete rectangle ① about horizontal axis passing through its C.G.

+ Area of complete rectangle ①

× Distance between X-X axis and horizontal axis passing through the C.G. of rectangle ① (By theorem of parallel axis) [$\because I_{XX1} = I_{G1XX} + A_1 h_1^2$]

$$= \frac{BD^3}{12} + (B \times D) [y_1 - \bar{y}]^2$$

$$= \frac{BD^3}{12} + BD \left[\frac{D}{2} - \frac{5D}{12} \right]^2$$

$$\left[\because y_1 = \frac{D}{2}, \bar{y} = \frac{5D}{12} \right] \quad \bar{y} = 5D/12$$

$$= \frac{BD^3}{12} + BD \left[\frac{D}{12} \right]^2$$

$$= \frac{BD^3}{12} + \frac{BD^3}{144} = \frac{13}{144} BD^3$$

Similarly, $I_{XX2} = (I_{G2X}) + A_2 \times h_2^2$

$$= \frac{\frac{B}{2} \times \left(\frac{D}{2} \right)^3}{12} + \frac{BD}{4} \times [y_2 - \bar{y}]^2$$

$$= \frac{BD^3}{192} + \frac{BD}{4} \left[\frac{3D}{4} - \frac{5D}{12} \right]^2$$

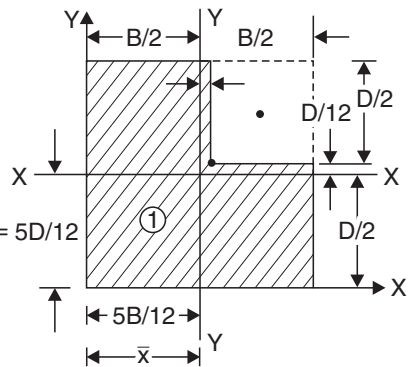


Fig. 9.25 (b)

$$\left[\because A_2 = \frac{BD}{4}, h_2 = (y_2 - \bar{y}) \right]$$

$$\left[\because y_2 = \frac{3D}{4}, \bar{y} = \frac{5D}{12} \right]$$

$$\begin{aligned}
 &= \frac{BD^3}{192} + \frac{BD}{4} \times \left(\frac{4D}{12}\right)^2 = \frac{BD^3}{192} + \frac{16BD^3}{4 \times 144} \\
 &= \frac{BD^3}{192} + \frac{BD^3}{36} = \frac{3BD^3 + 16BD^3}{576} = \frac{19BD^3}{576}
 \end{aligned}$$

Now,

$$\begin{aligned}
 I_{XX} &= \text{M.O.I. of given section about centroidal axis X-X} \\
 &= I_{XX1} - I_{XX2} \\
 &= \frac{13BD^3}{144} - \frac{19BD^3}{576} = \frac{52BD^3 - 19BD^3}{576} = \frac{33BD^3}{576} = 0.0573 BD^3. \quad \text{Ans.}
 \end{aligned}$$

Similarly, the M.O.I. of the given section about centroidal axis Y-Y is given by

$$\begin{aligned}
 I_{YY} &= I_{YY1} - I_{YY2} \\
 \text{where } I_{YY1} &= \text{M.O.I. of rectangle ① about centroidal axis Y-Y} \\
 &= I_{G1Y} + A_1 \times [x_1 - \bar{x}]^2 \\
 &= \frac{DB^3}{12} + BD \times \left[\frac{B}{2} - \frac{5B}{12}\right]^2 = \frac{DB^3}{12} + \frac{BD \times B^2}{144} = \frac{13}{144} DB^3
 \end{aligned}$$

and

$$\begin{aligned}
 I_{YY2} &= I_{G2Y} + A_2 [x_2 - \bar{x}]^2 \\
 &= \frac{\frac{D}{2} \times \left(\frac{B}{2}\right)^3}{12} + \frac{BD}{4} \left[\frac{3B}{4} - \frac{5B}{12}\right]^2 = \frac{DB^3}{192} + \frac{DB^3}{36} = \frac{19DB^3}{576}
 \end{aligned}$$

$$\therefore I_{YY} = \frac{13}{144} DB^3 - \frac{19DB^3}{576} = \frac{33}{576} DB^3 = 0.0573 DB^3. \quad \text{Ans.}$$

9.13. POLAR MOMENT OF INERTIA OF MASSES

Consider a body of mass M as shown in Fig. 9.26.

Let x = Distance of the centre of gravity of mass M from axis OY

y = Distance of the C.G. of mass M from axis OX

Then moment of the mass about the axis $OY = M \cdot x$

The above equation is known as *first moment of mass about the axis OY* .

If the moment of mass given by the above equation is again multiplied by the perpendicular distance between the C.G. of the mass and axis OY , then the quantity $(M \cdot x) \cdot x = M \cdot x^2$ is known as *second moment of mass about the axis OY* . This second moment of the mass (i.e., quantity $M \cdot x^2$) is known as *mass moment of inertia about the axis OY* .

Similarly, the second moment of mass or mass moment of inertia about the axis OX

$$= (M \cdot y) \cdot y = M \cdot y^2$$

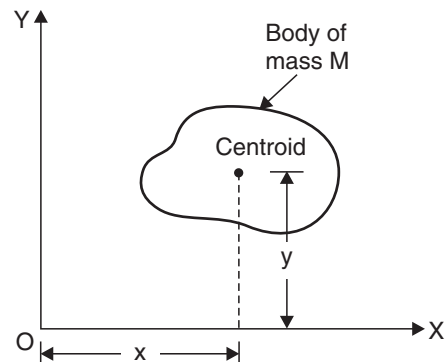


Fig. 9.26

Hence the product of the mass and the square of the distance of the centre of gravity of the mass from an axis is known as the mass moment of inertia about that axis. Mass moment

of inertia is represented by I_m . Hence mass moment of inertia about the axis OX is represented by $(I_m)_{XX}$ whereas about the axis OY by $(I_m)_{YY}$.

Consider a body which is split up into small masses $m_1, m_2, m_3, \dots$ etc. Let the C.G. of the small areas from a given axis be at a distance of $r_1, r_2, r_3, \dots$ etc. as shown in Fig. 9.27. Then mass moment of inertia of the body about the given axis is given by

$$I_m = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots \\ = \Sigma m r^2$$

If small masses are large in number then the summation in the above equation can be replaced by integration. Let the small masses are replaced by dm instead of ' m ', then the above equation can be written as

$$I_m = \int r^2 dm \quad \dots(9.16)$$

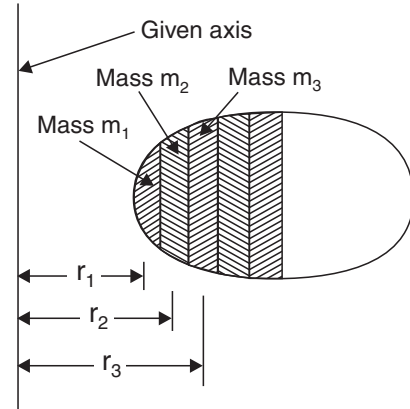


Fig. 9.27

9.14. PARALLEL AXIS THEOREM FOR MASS MOMENT OF INERTIA

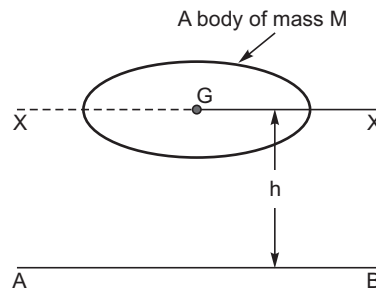


Fig. 9.27 (a)

Let $(I_m)_{XX}$ = Mass moment of inertia of a body of mass M about an axis $X-X$ passing through the C.G. of the body.

$(I_m)_{AB}$ = Mass moment of inertia of the body about axis AB which is parallel to axis $X-X$ and at a distance ' h ' from axis $X-X$.

Then according to parallel axis theorem,

$$(I_m)_{AB} = (I_m)_{XX} + M \times h^2 \quad \dots[9.16 (a)]$$

9.15. MASS MOMENT OF INERTIA OF SIMPLE BODIES BY INTEGRATION

The mass moment of inertia of the following bodies will be determined by the method of integration :

1. Mass moment of inertia of a rectangular plate,
2. Mass moment of inertia of a circular plate,
3. Mass moment of inertia of a hollow circular cylinder.

9.15.1. Mass Moment of Inertia of a Rectangular Plate

(a) Mass moment of inertia of a rectangular plate about $X-X$ axis passing through the C.G. of the plate.

Fig. 9.28 shows a rectangular plate of width b , depth ' d ' and uniform thickness ' t '. Consider a small element of width ' b ' at a distance ' y ' from $X-X$ axis as shown in Fig. 9.29.

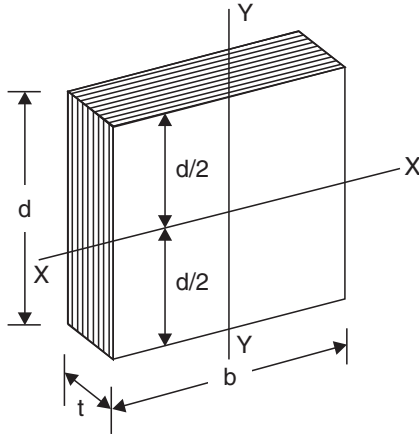


Fig. 9.28

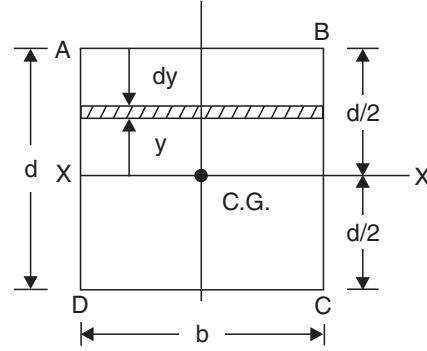


Fig. 9.29

Here $X-X$ axis is the horizontal line passing through the C.G. of the plate.

Area of the element $= b \times dy$

$\therefore$ Mass of the element = Density $\times$ Volume of element

$$= \rho \times [\text{Area} \times \text{thickness of element}]$$

$$= \rho \times [b \times dy \times t] \quad [\because \rho = \text{Density and } t = \text{thickness}]$$

$$= \rho b t \, dy$$

Mass moment of inertia of the element about $X-X$ axis

$$= \text{Mass of element} \times y^2$$

$$= (\rho b t \, dy) \times y^2 = \rho b t \, y^2 \, dy$$

Mass moment of inertia of the plate will be obtained by integrating the above equation between the limits $-\frac{d}{2}$ to $\frac{d}{2}$.

$$\therefore (I_m)_{XX} = \int_{-d/2}^{d/2} \rho b t \, y^2 \, dy = \rho b t \int_{-d/2}^{d/2} y^2 \, dy$$

$[\because \rho, b, t$ are constant and can be taken outside the integral sign]

$$= \rho b t \left[\frac{y^3}{3} \right]_{-d/2}^{d/2} = \frac{\rho b t}{3} \left[\left(\frac{d}{2} \right)^3 - \left(-\frac{d}{2} \right)^3 \right]$$

$$= \frac{\rho b t}{3} \left[\frac{d^3}{8} - \left(-\frac{d^3}{8} \right) \right] = \frac{\rho b t}{3} \left[\frac{d^3}{8} + \frac{d^3}{8} \right] = \frac{\rho b t}{3} \times \frac{2d^3}{8}$$

$$= \frac{\rho b t}{12} d^3 = \rho \times t \frac{b d^3}{12} \quad \dots(9.17)$$

But $\frac{b d^3}{12}$ is the moment of inertia of the area of the rectangular section about $X-X$ axis.

This moment of inertia of the area is represented by I_{XX} .

$$\therefore (I_m)_{XX} = \rho \times t \times I_{XX} \quad \dots(9.18)$$

where $(I_m)_{XX}$ = Mass moment of inertia of the plate about X-X axis passing through C.G. of the plate.

I_{XX} = Moment of inertia of the area of the plate about X-X axis.

Again from equation (9.5), we have

$$\begin{aligned} (I_m)_{XX} &= \rho \frac{bd^3}{12} \\ &= \rho b \times d \times t \times \frac{d^2}{12} \\ &= M \times \frac{d^2}{12} \\ (\because M &= \text{Mass of the plate} = \rho \times \text{Volume of the plate} = \rho \times [b \times d \times t]) \\ &= \frac{1}{12} Md^2 \quad \dots(9.19) \end{aligned}$$

Similarly, the mass moment of inertia of the rectangular plate about Y-Y axis passing through the C.G. of the plate is given by

$$(I_m)_{YY} = \frac{1}{12} Mb^2. \quad \dots(9.20)$$

(b) Mass moment of inertia of the rectangular plate about a line passing through the base

Fig. 9.30 shows a rectangular plate $ABCD$, having width = b , depth = d and uniform thickness = t . We want to find the mass moment of inertia of the rectangular plate about the line CD , which is the base of the plate. Consider a rectangular elementary strip of width b , thickness t and depth ' dy ' at a distance y from the line CD as shown in Fig. 9.30.

$$\begin{aligned} \text{Area of strip,} \quad dA &= b \cdot dy \\ \text{Volume of strip} &= dA \times t = b \cdot dy \cdot t = b \cdot t \cdot dy \\ \text{Mass of the strip,} \quad dm &= \text{Density} \times \text{Volume of strip} \\ &= \rho(b \cdot t \cdot dy) = \rho \cdot b \cdot t \cdot dy \end{aligned}$$

$$\begin{aligned} \text{Mass moment of inertia of the strip about the line } CD &= \text{Mass of strip} \cdot y^2 \\ &= dm \cdot y^2 = y^2 \cdot dm \end{aligned}$$

Mass moment of inertia of the whole rectangular plate about the line CD is obtained by integrating the above equation between the limits 0 to d .

$\therefore$ Mass moment of inertia the rectangular plate about the line CD

$$\begin{aligned} &= \int_0^d y^2 \cdot dm = \int_0^d y^2 (\rho \cdot b \cdot t \cdot dy) \quad [\because dm = \rho \cdot b \cdot t \cdot dy] \\ &= \rho \cdot b \cdot t \int_0^d y^2 dy \quad [\because \rho, b \text{ and } t \text{ are constant}] \end{aligned}$$

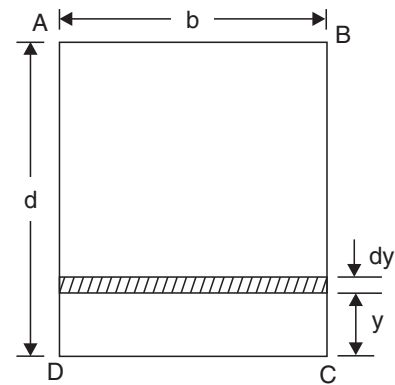


Fig. 9.30

$$\begin{aligned}
 &= \rho \cdot b \cdot t \cdot \left[\frac{y^3}{3} \right]_0^d = \rho \cdot b \cdot t \cdot \frac{d^3}{3} = \rho \cdot b \cdot t \cdot d \cdot \frac{d^2}{3} \\
 &= \frac{M \cdot d^2}{3} \quad [\because \rho \cdot b \cdot t \cdot d = \text{Mass of rectangular plate} = M] \quad \dots(9.21)
 \end{aligned}$$

(c) Mass moment of inertia of a hollow rectangular plate

Fig. 9.31 shows a hollow rectangular plate in which $ABCD$ is the main plate and $EFGH$ is the cut-out section.

The mass moment of inertia of the main plate $ABCD$ about $X-X$ is given by equation

$$= \frac{1}{12} M d^2$$

The mass moment of inertia of the cut-out section $EFGH$ about $X-X$ axis

$$= \frac{1}{12} m d_1^2$$

where M = Mass of main plate $ABCD$

$$= \rho \cdot b \cdot d \cdot t$$

m = Mass of the cut-out section $EFGH$

$$= \rho \cdot b_1 \cdot d_1 \cdot t$$

Then mass moment of inertia of hollow rectangular plate about $X-X$ axis is given by

$$(I_m)_{XX} = \frac{1}{12} M d^2 - \frac{1}{12} m d_1^2. \quad \dots(9.22)$$

9.15.2. Mass Moment of Inertia of a Circular Plate. Fig. 9.32 shows a circular plate of radius R and thickness t with O as centre. Consider an elementary circular ring of radius ' r ' and width dr as shown in Fig. 9.32 (a).

Area of ring, $dA = 2\pi r \cdot dr$

Volume of ring = Area of ring $\times t = dA \cdot t$
 $= 2\pi r \cdot dr \cdot t$

Mass of ring,

$$\begin{aligned}
 dm &= \text{Density} \times \text{Volume of ring} \\
 &= \rho(2\pi r \cdot dr \cdot t)
 \end{aligned}$$

In this case first find the mass moment of inertia about an axis passing through O and perpendicular to the plane of the paper *i.e.*, about axis $Z-Z$

$\therefore$ Mass moment of inertia of the circular ring about axis $Z-Z$

$$\begin{aligned}
 &= (\text{Mass of ring}) \times (\text{Radius of ring})^2 \\
 &= dm \times r^2 = (\rho \cdot 2\pi r \cdot dr \cdot t) \times r^2 = \rho \cdot t \cdot 2\pi r^3 dr
 \end{aligned}$$

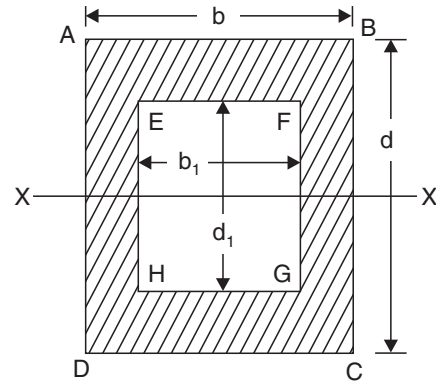


Fig. 9.31

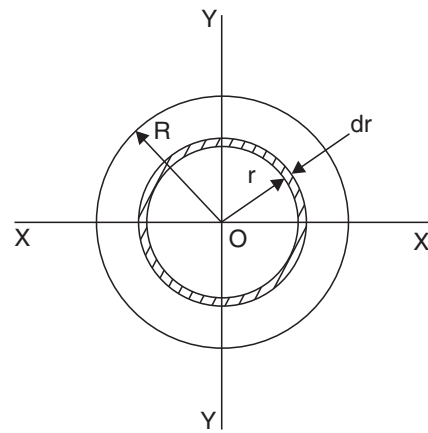


Fig. 9.32

The mass moment of inertia of the whole circular plate will be obtained by integrating the above equation between the limits O to R .

$\therefore$ Mass moment of inertia of circular plate about Z - Z axis is given by

$$\begin{aligned}(I_m)_{ZZ} &= \int_0^R \rho \cdot t \cdot 2\pi r^3 dr = 2\pi \cdot \rho \cdot t \int_0^R r^3 dr \\ &= 2\pi \rho \cdot t \left[\frac{r^4}{4} \right]_0^R \\ &= 2\pi \cdot \rho \cdot t \cdot \frac{R^4}{4} = \pi \cdot \rho \cdot t \cdot \frac{R^4}{2}\end{aligned}$$

Now mass of circular plate,

$$M = \rho \times \text{Volume of plate}$$

$$= \rho \times \pi R^2 \times t$$

$$[\text{Volume of plate} = \text{Area} \times t = \pi R^2 \times t]$$

Substituting this value in above equation, we get

$$\therefore (I_m)_{ZZ} = \rho \times \pi R^2 \times t \times \frac{R^2}{2} = \frac{MR^2}{2} \quad \dots(9.23)$$

But from the theorem of perpendicular axis given by equation (9.7), we have

$$I_{ZZ} = I_{XX} + I_{YY}$$

or

$$(I_m)_{ZZ} = (I_m)_{XX} + (I_m)_{YY}$$

And due to symmetry, we have $(I_m)_{XX} = (I_m)_{YY}$

$$\therefore (I_m)_{XX} = (I_m)_{YY} = (I_m)_{ZZ} / 2$$

$$= \left(\frac{MR^2}{2} \right) / 2 = \frac{MR^2}{4} \quad \dots(9.24)$$

9.15.3. Mass Moment of Inertia of a Hollow Circular Cylinder

Let R_o = Outer radius of the cylinder

R_i = Inner radius of the cylinder

L = Length of the cylinder

M = Mass of cylinder

= Density $\times$ Volume of cylinder

$$= \rho \times \pi [R_o^2 - R_i^2] \times L \quad \dots(i)$$

dm = Mass of a circular ring of radius ' r ' width ' dr ' and length L [Refer to Fig. 9.32]

= Density $\times$ Volume of ring = $\rho \times$ Area of ring $\times L$

$$= \rho \times 2\pi r dr \times L$$

Now mass moment of inertia of the circular ring about Z - Z axis

$$= \text{Mass of ring} \times (\text{radius})^2$$

$$= (\rho \times 2\pi r dr \times L) \times r^2$$

The mass moment of inertia of the hollow circular cylinder will be obtained by integrating the above equation between the limits R_i to R_o .

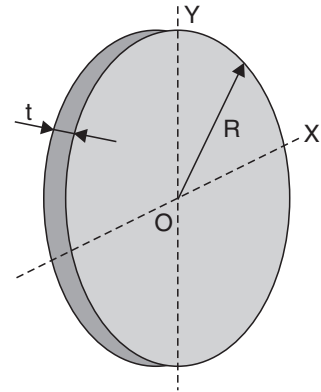


Fig. 9.32 (a)

∴ Mass moment of inertia of the hollow circular cylinder about Z-Z axis is given by,

$$\begin{aligned}
 (I_m)_{ZZ} &= \int_{R_i}^{R_o} (\rho \times 2\pi r \, dr \cdot L) r^2 \\
 &= \rho \times 2\pi \times L \int_{R_i}^{R_o} r^3 \, dr = \rho \times 2\pi \times L \left[\frac{r^4}{4} \right]_{R_i}^{R_o} \\
 &= \rho \times 2\pi \times L \times \left[\frac{R_o^4 - R_i^4}{4} \right] \\
 &= \rho \times 2\pi \times L \times \left[\frac{R_o^2 - R_i^2}{4} \right] [R_o^2 + R_i^2] \\
 &\quad [\because R_o^4 - R_i^4 = (R_o^2 - R_i^2)(R_o^2 + R_i^2)] \\
 &= \rho \times \pi [R_o^2 - R_i^2] \times L \times \frac{(R_o^2 + R_i^2)}{2} \\
 &= \frac{M(R_o^2 + R_i^2)}{2} \quad [\because \rho \times \pi \times (R_o^2 - R_i^2) = M]
 \end{aligned}$$

$$\text{Now, } (I_m)_{XX} = (I_m)_{YY} = \frac{(I_m)_{ZZ}}{2} = \frac{M(R_o^2 + R_i^2)}{4}.$$

9.15.4. Mass Moment of Inertia of a Right Circular Cone of Base Radius R, Height H and Mass M about its Axis

Let R = Radius of the base of the cone,
 H = Height of the cone,
 M = Mass of the cone

$$= \text{Density} \times \text{Volume of cone} = \rho \times \frac{1}{3} \pi R^2 \times H$$

Consider an elemental plate of thickness dy and of radius x at a distance y from the vertex (as shown in Fig. 9.32 (b)).

$$\text{We have, } \tan \alpha = \frac{x}{y} = \frac{R}{H} \quad \therefore x = \frac{R}{H} \times y$$

Mass of the elemental plate,

$$\begin{aligned}
 dm &= \rho \times \text{Volume} \\
 &= \rho \times (\pi x^2 \times dy) \\
 &= \rho \times \left[\pi \frac{R^2 \times y^2}{H^2} \times dy \right] \quad \left[\because x = \frac{R \times y}{H} \right]
 \end{aligned}$$

The mass moment of inertia of the circular elemental plate about the axis of the cone (here axis of the cone is Z-Z axis of the circular elemental plate) is given by equation (9.23) as

$$\begin{aligned}
 (I_m)_{ZZ} &= \frac{\text{Mass of plate} \times \text{Radius}^2}{2} \\
 &= \frac{(dm) \times r^2}{2} = \frac{dm \times x^2}{2} \quad (\because r = x)
 \end{aligned}$$

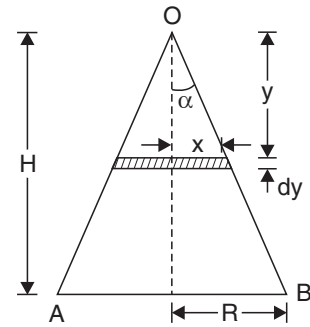


Fig. 9.32 (b)

$$\begin{aligned}
 &= \left[\rho \times \frac{\pi R^2 y^2}{H^2} \times dy \right] \times \frac{x^2}{2} && \left[\because dm = \rho \times \frac{\pi R y}{H^2} dy \right] \\
 &= \frac{\rho \times \pi R^2 y^2}{H^2} \times dy \times \left[\frac{R^2 y^2}{H^2} \right] \times \frac{1}{2} && \left[\because x = \frac{R y}{H} \right] \\
 &= \frac{\rho \times \pi R^4 \times y^4}{2H^4} dy
 \end{aligned}$$

Now the total mass moment of inertia of the circular cone will be obtained by integrating the above equation between the limits O to H .

$$\begin{aligned}
 \therefore (I_m)_{ZZ} &= \int_0^H \frac{\rho \pi R^4 \times y^4}{2H^4} dy = \frac{\rho \pi R^4}{2H^4} \times \left[\frac{y^5}{5} \right]_0^H \\
 &= \frac{\rho \pi R^4}{2H^4} \times \frac{H^5}{5} = \frac{\rho \pi R^4 \times H}{2 \times 5}
 \end{aligned}$$

But mass of cone, $M = \frac{\rho \pi R^4 \times H}{3}$

$$\therefore (I_m)_{ZZ} = \frac{\rho \pi R^4 \times H}{3} \times \frac{R^2 \times 3}{10} = M \times \frac{3}{10} R^2 = \frac{3}{10} MR^2 \quad \dots(9.25)$$

9.16. MASS MOMENT OF INERTIA OF COMPOSITE BODIES

Problem 9.17. Determine the mass moment of inertia of the composite body about Z-axis shown in Fig. 9.33. The mass density of the cylinder is 6000 kg/m^3 and the rectangular prism is 7000 kg/m^3 .

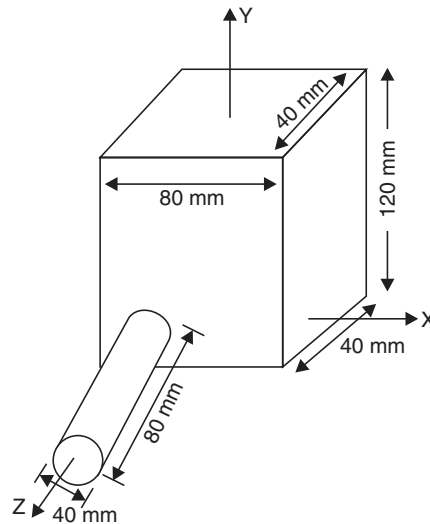


Fig. 9.33

Sol. Given :

Mass density of cylinder, $\rho_1 = 6000 \text{ kg/m}^3$

Mass density of rectangular prism, $\rho_2 = 7000 \text{ kg/m}^3$

Let us first find the mass of each element. We know mass,

$$M = \rho \times \text{Volume} = \rho \times A \times L$$

$$\begin{aligned} \text{Mass of cylinder, } M_1 &= \rho_1 \times A_1 \times L_1 \\ &= 6000 \times (\pi r^2) \times 0.08 \quad [\because L_1 = 0.08 \text{ m}] \\ &= 6000 \times (\pi \times 0.02^2) \times 0.08 \quad \left(\because r = \frac{40}{2} = 20 \text{ mm} = 0.02 \text{ m} \right) \\ &= 0.6 \text{ kg} \end{aligned}$$

Mass of rectangular prism,

$$\begin{aligned} M_2 &= \rho_2 \times \text{Volume of prism} \\ &= 7000 \times [0.08 \times 0.04 \times 0.12] \text{ kg} = 2.69 \text{ kg} \end{aligned}$$

Mass moment of inertia of the cylinder about Z-axis

$$\begin{aligned} &= \frac{1}{2} m r^2 = \frac{1}{2} M_1 \times 0.02^2 \quad (\because r = 0.02 \text{ m}) \\ &= \frac{1}{2} \times 0.6 \times 0.02^2 = 1.2 \times 10^{-4} \text{ kg m}^2 \end{aligned}$$

Mass moment of inertia of the rectangular prism about centroid axis parallel to Z-axis,

$$\begin{aligned} I_z &= \frac{1}{12} m (a^2 + b^2) = \frac{1}{12} \times M_2 \times (0.08^2 + 0.12^2) \quad (\because m = M_2) \\ &= \frac{1}{12} \times 2.69 \times (0.0064 + 0.0144) \text{ kg m}^2 \\ &= 4.66 \times 10^{-3} \text{ kg m}^2 \end{aligned}$$

Mass moment of inertia of the rectangular prism about the given Z-axis (Using parallel axis theorem for mass moment of inertia)

$$\begin{aligned} &= I_z + M_2 \times d^2, \text{ where } d = \frac{120}{2} - \frac{40}{2} = 60 - 20 \\ &= 40 \text{ mm} = 0.04 \text{ m} \\ &= 4.66 \times 10^{-3} + 2.69 \times 0.04^2 \\ &= 4.66 \times 10^{-3} + 4.3 \times 10^{-3} \\ &= 8.96 \times 10^{-3} \text{ kg m}^2 \end{aligned}$$

$\therefore$ Mass moment of inertia of the composite body about Z-axis

$$\begin{aligned} &= \text{M.O.I. of cylinder about Z-axis} + \text{M.O.I. of prism about given Z-axis} \\ &= 1.2 \times 10^{-4} + 8.96 \times 10^{-3} \\ &= 0.12 \times 10^{-3} + 8.96 \times 10^{-3} = \mathbf{9.08 \times 10^{-3} \text{ kg m}^2. \text{ Ans.}} \end{aligned}$$

HIGHLIGHTS

1. The point, through which the whole weight of the body acts, is known as centre of gravity.
2. The point, at which the total area of a plane figure is assumed to be concentrated, is known as centroid of that area. The centroid and centre of gravity are at the same point.
3. The centre of gravity of a uniform rod lies at its middle point.
4. The C.G. of a triangle lies at a point where the three medians of a triangle meet.
5. The C.G. of a parallelogram or a rectangle is at a point where its diagonal meet each other.
6. The C.G. of a circle lies at its centre.

7. The C.G. of a body consisting of different areas is given by

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + \dots}{a_1 + a_2 + a_3 + \dots} \quad \text{and} \quad \bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + \dots}{a_1 + a_2 + a_3 + \dots}$$

where $\bar{x}$ and $\bar{y}$ = Co-ordinates of the C.G. of the body from axis of reference

$a_1, a_2, a_3, \dots$ = Different areas of the sections of the body

$x_1, x_2, x_3, \dots$ = Distances of the C.G. of the areas $a_1, a_2, a_3, \dots$ from Y-axis

$y_1, y_2, y_3, \dots$ = Distances of the C.G. of the areas $a_1, a_2, a_3, \dots$ from X-axis.

8. If a given section is symmetrical about X-X axis or Y-Y axis, the C.G. of the section will lie on the axis symmetry.
9. The moment of inertia of an area (or mass) about an axis is the product of area (or mass) and square of the distance of the C.G. of the area (or mass) from that axis. It is represented by I .
10. Radius of gyration of a body (or a given lamina) is the distance from an axis of reference where the whole mass (or area) of the given body is assumed to be concentrated so as not to alter the

moment of inertia about the given axis. It is represented by k . Mathematically, $k = \sqrt{\frac{I}{A}}$.

11. According to theorem of perpendicular axis $I_{ZZ} = I_{XX} + I_{YY}$, where I_{XX} and I_{YY} = Moment of inertia of a plane section about two mutually perpendicular axes X-X and Y-Y in the plane of the section, I_{ZZ} = Moment of inertia of the section perpendicular to the plane and passing through the intersection of X-X and Y-Y axes.

12. According to the theorem of parallel axis $I_{AB} = I_G + Ah^2$, where

I_G = Moment of inertia of a given area about an axis passing through C.G. of the area

I_{AB} = Moment of inertia of the given area about an axis AB, which is parallel to the axis passing through G

h = Distance between the axis passing through G and axis AB

A = Area of the section.

13. Moment of inertia of a rectangular section :

(i) about an horizontal axis passing through C.G. = $\frac{bd^3}{12}$

(ii) about an horizontal axis passing through base = $\frac{bd^3}{3}$.

14. Moment of inertia of a circular section = $\frac{\pi D^4}{64}$.

15. Moment of inertia of a triangular section :

(i) about the base = $\frac{bh^3}{12}$

(ii) about an axis passing through C.G. and parallel to the base = $\frac{bh^3}{36}$.

where b = Base width, and h = Height of the triangle.

16. The C.G. of an area by integration method is given by

$$\bar{x} = \frac{\int x^* dA}{\int dA} \quad \text{and} \quad \bar{y} = \frac{\int y^* dA}{\int dA}$$

where x^* = Distance of C.G. of area dA from Y-axis

y^* = Distance of C.G. of area dA from X-axis.

17. The C.G. of a straight or curved line is given by

$$\bar{x} = \frac{\int x^* dL}{\int dL} \quad \text{and} \quad \bar{y} = \frac{\int y^* dL}{\int dL}.$$

EXERCISE 9

(A) Theoretical Questions

- Define centre of gravity and centroid.
- Derive an expression for the centre of gravity of a plane area using method of moments.
- What do you understand by axes of reference ?
- Define the terms : moment of inertia and radius of gyration.
- State the theorem of perpendicular axis. How will you prove this theorem ?
- State and prove the theorem of parallel axis.
- Find an expression for the moment of inertia of a rectangular section :
 - about an horizontal axis passing through the C.G. of the rectangular section, and
 - about an horizontal axis passing through the base of the rectangular section.

(AMIE Summer, 1985)
- Prove that the moment of inertia of a circular section about an horizontal axis (in the plane of the circular section) and passing through the C.G. of the section is given by $\frac{\pi D^4}{64}$.
- Prove that moment of inertia of a triangular section about the base of the section

$$= \frac{bh^3}{12}$$

where b = Base of triangular section, and
 h = Height of triangular section.
- Derive an expression for the moment of inertia of a triangular section about an axis passing through the C.G. of the section and parallel to the base.
- Show that $I_O = I_G + Ah^2$, where I_G is the moment of inertia of a lamina about an axis through its centroid and lying in its plane and h is the distance from the centroid to a parallel axis in the same plane about which its moment of inertia is I_O , A being the area of the lamina.
- State and prove the parallel axes theorem on moment of inertia for a plane area.
- Prove that the moment of area of any plane figure about a line passing through its centroid is zero.
- Show that the product of inertia of an area about two mutually perpendicular axis is zero, if the area is symmetrical about one of these axis. (U.P. Tech. University, 2002–2003)
- Determine an expression for mass moment of inertia of hollow steel cylinder of mass M , outer radius R_o , inner radius R_i and length L about its axis. The hole in the cylinder is concentric. (U.P. Tech. University, 2002–2003)
- Derive an expression for mass moment of inertia of a right circular cone of base radius R , height H and mass M about its axis. (U.P. Tech. University, 2001–2002)

(B) Numerical Problems

1. Find the centre of gravity of the *T*-section shown in Fig. 9.34.

[Ans. 8.272 cm]

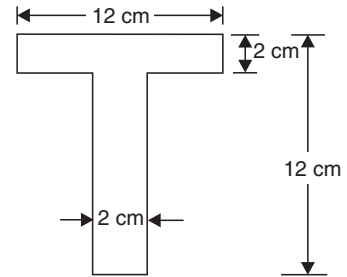


Fig. 9.34

2. Find the centre of gravity of the *I*-section shown in Fig. 9.35.

[Ans. 6.44 cm]

[Hint. $a_1 = 8 \times 2 = 16 \text{ cm}^2$, $a_2 = 12 \times 2 = 24 \text{ cm}^2$,
 $a_3 = 16 \times 2 = 32$; $y_1 = 2 + 12 + 1 = 15$,
 $y_2 = 2 + 6 = 8$, $y_3 = 1$

$$\begin{aligned} \therefore \bar{y} &= \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} \\ &= \frac{16 \times 15 + 24 \times 8 + 32 \times 1}{16 + 24 + 32} \\ &= \frac{240 + 192 + 32}{72} = \frac{464}{72} = 6.44 \text{ cm} \end{aligned}$$

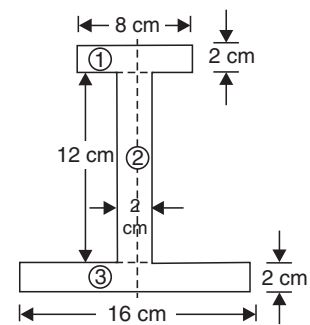


Fig. 9.35

3. (a) Find the centre of gravity of the *L*-section shown in Fig. 9.36.

[Ans. $\bar{x} = 1.857$, $\bar{y} = 3.857$]

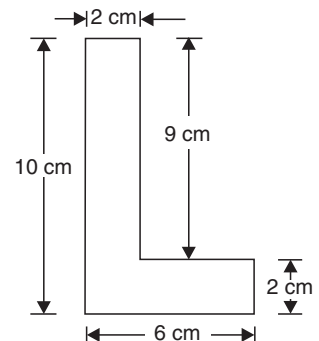


Fig. 9.36

- (b) Find the moment of inertia of *ISA* 100 × 75 × 6 about the centroidal *XX* and *YY*-axis, shown in Fig. 9.36 (a).

(U.P. Tech. University, 2001–2002)

[Hint. Locate first $\bar{x}$ and $\bar{y}$ first

$$a_1 = 100 \times 6 = 600 \text{ mm}^2, x_1 = 3 \text{ mm}, y_1 = 50$$

$$a_2 = 69 \times 6 = 414 \text{ mm}^2, x_2 = 6 + \frac{69}{2} = 40.5$$

$$y_2 = 3 \text{ mm}$$

$$\therefore \bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{600 \times 3 + 414 \times 40.5}{600 + 414} = 18.31 \text{ mm}$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{600 \times 50 + 414 \times 3}{(600 + 414)} = 30.81 \text{ mm}$$

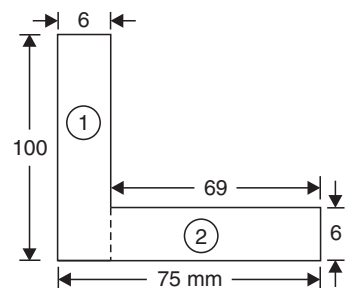


Fig. 9.36 (a)

Now find the moment of inertia about centroidal X-X axis :

$$\begin{aligned} I_{XX1} &= (I_{G_1})_x + a_1 h_1^2 \\ &= \frac{6 \times 100^3}{12} + 600 \times (y_1 - \bar{y})^2 = \frac{6 \times 100^3}{12} + 600(50 - 30.81)^2 \\ &= 720.95 \times 10^3 \text{ mm}^4. \end{aligned}$$

$$\begin{aligned} I_{XX2} &= (I_{G_2})_x + a_2 h_2^2 = \frac{69 \times 6^3}{12} + 414(y_2 - \bar{y})^2 \\ &= \frac{69 \times 6^3}{12} + 414(3 - 30.81)^2 = 321.428 \times 10^3 \end{aligned}$$

$$\therefore I_{XX} = I_{X1} + I_{XX2} = 720.95 \times 10^3 + 321.428 \times 10^3 = \mathbf{1042.378 \times 10^3 \text{ mm}^4. \text{ Ans.}}$$

To find M.O.I. about centroidal axis Y-Y

$$I_{YY1} = (I_{G_1})_y + a_1(x_1 - \bar{x})^2 = \frac{100 \times 6^3}{12} + 600(3 - 18.31)^2 = 142.437 \times 10^3 \text{ mm}^4$$

$$I_{YY2} = (I_{G_2})_y + a_2(x_2 - \bar{x})^2 = \frac{6 \times 69^3}{12} + 414(40.5 - 18.31)^2 = 368.1 \times 10^3 \text{ mm}^4$$

$$\therefore I_{YY} = I_{YY1} + I_{YY2} = (142.437 + 368.1) \times 10^3 \text{ mm}^4 = \mathbf{510.537 \times 10^3 \text{ mm}^4. \text{ Ans.}}$$

4. From a rectangular lamina $ABCD$ 10 cm $\times$ 14 cm a rectangular hole of 3 cm $\times$ 5 cm is cut as shown in Fig. 9.37. Find the centre of gravity of the remainder lamina.

[Ans. $\bar{x} = 4.7$ cm, $\bar{y} = 6.444$ cm]

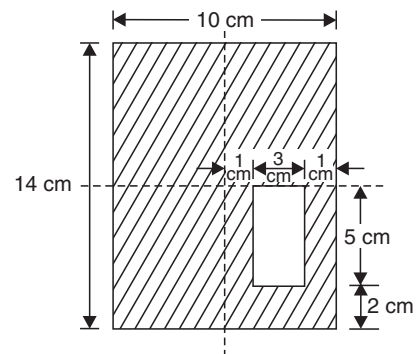


Fig. 9.37

5. For the T -section shown in Fig. 9.34, determine the moment of inertia of the section about the horizontal and vertical axes, passing through the centre of gravity of the section.
[Ans. 567.38 cm^4 , 294.67 cm^4]
6. For the I -section shown in Fig. 9.35, find the moment of inertia about the centroidal axis $X-X$ perpendicular to the web.
[Ans. 2481.76 cm^4]
7. Locate the C.G. of the area shown in Fig. 9.38 with respect to co-ordinate axes. All dimensions are in mm.

[Hint. $a_1 = 10 \times 30 = 300 \text{ mm}^2$, $x_1 = 5 \text{ mm}$, $y_1 = 15$
 $a_2 = 40 \times 10 = 400 \text{ mm}^2$, $x_2 = 10 + 20 = 30 \text{ mm}$,
 $y_2 = 5 \text{ mm}$
 $a_3 = 10 \times 20 = 200 \text{ mm}^2$, $x_3 = 5 \text{ mm}$, $y_3 = -10 \text{ mm}$
 $a_4 = 10 \times 10 = 100 \text{ mm}^2$, $x_4 = 45 \text{ mm}$,
 $y_4 = 10 + 5 = 15 \text{ mm}$
 $\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4}{(a_1 + a_2 + a_3 + a_4)}$
 $= \frac{1500 + 12000 + 1000 + 4500}{1000}$
 $= 1.5 + 12 + 1 + 4.5 = \mathbf{19 \text{ mm. Ans.}}$

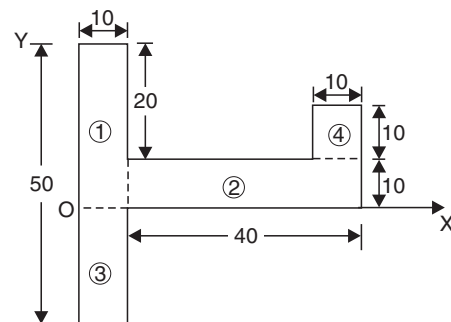


Fig. 9.38

$$\begin{aligned}\bar{y} &= \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4}{(a_1 + a_2 + a_3 + a_4)} \\ &= \frac{4500 + 2000 - 2000 + 1500}{1000} \\ &= 4.5 + 2 - 2 + 1.5 = 6 \text{ mm. Ans.}]\end{aligned}$$

8. A thin homogeneous wire is bent into a triangular shape ABC such that $AB = 240$ mm, $BC = 260$ mm and $AC = 100$ mm. Locate the C.G. of the wire with respect to co-ordinate axes. Angle at A is right angle.

[Hint. First determine angles α and β . Use sine rule]

$$\begin{aligned}\frac{BC}{\sin 90^\circ} &= \frac{AC}{\sin \alpha} = \frac{AB}{\sin \beta} \\ \therefore \sin \alpha &= \frac{AC \times \sin 90^\circ}{BC} = \frac{100}{260} \\ \therefore \alpha &= 22.62^\circ. \text{ Also } \sin \beta = \frac{AB}{BC} \times \sin 90^\circ = \frac{240}{260} \\ \therefore \beta &= 67.38^\circ\end{aligned}$$

Using equations [9.2 (C)] and [9.2 (D)]

$$\begin{aligned}\bar{x} &= \frac{L_1x_1 + L_2x_2 + L_3x_3}{(L_1 + L_2 + L_3)}, \text{ where } L_1 = AB = 240, \\ x_1 &= \text{distance of C.G. of } AB \text{ from } Y\text{-axis} \\ &= \frac{240}{2} \times \cos \alpha = 120 \times \cos 22.62^\circ = 110.77 \text{ mm} \\ L_2 &= BC = 260 \text{ mm}, x_2 = \text{Distance of C.G. of } BC \text{ from } Y\text{-axis} = 130 \\ L_3 &= AC = 100 \text{ mm}, x_3 = \text{Distance of C.G. of } AC \text{ from } Y\text{-axis} \\ &= BD + \frac{100}{2} \cos \beta = 240 \cos \alpha + 50 \cos \beta \\ &= 240 \times \cos 22.62^\circ + 50 \cos 67.38^\circ = 240.77 \\ \therefore \bar{x} &= \frac{240 \times 110.77 + 260 \times 130 + 100 \times 240.77}{240 + 260 + 100} = 140.77 \text{ mm. Ans.} \\ \bar{y} &= \frac{L_1y_1 + L_2y_2 + L_3y_3}{L_1 + L_2 + L_3}, \text{ where } y_1 = \frac{240}{2} \sin \alpha = 120 \times \sin 22.62^\circ = 46.154 \\ y_2 &= 0, y_3 = \frac{100}{2} \sin \beta = 50 \sin 67.38^\circ = 48.154 \\ \therefore \bar{y} &= \frac{240 \times 46.154 + 260 \times 0 + 100 \times 48.154}{600} = 26.154 \text{ mm. Ans.}]\end{aligned}$$

9. Determine the C.G. of the uniform plane lamina shown in Fig. 9.40. All dimensions are in cm.

[Hint. The figure is symmetrical about $Y-Y$ axis.]

$$\therefore \bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4}{a_1 + a_2 + a_3 + a_4}$$

$$\text{where } a_1 = 40 \times 30 = 1200 \text{ cm}^2, y_1 = \frac{30}{2} = 15 \text{ cm},$$

$$a_2 = 30 \times 20 = 600 \text{ cm}^2, y_2 = 30 + \frac{30}{2} = 45 \text{ cm}$$

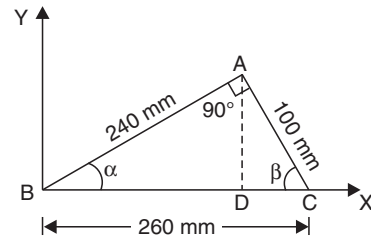


Fig. 9.39

$$\begin{aligned}
 a_3 &= -\frac{\pi \times 10^2}{2} = -50\pi, \quad y_3 = \frac{4r}{3\pi} = \frac{4 \times 10}{3\pi} = \frac{40}{3\pi} \\
 a_4 &= -\frac{20 \times 10}{2} = -100, \quad y_4 = 60 - \frac{10}{3} = \frac{170}{3} \\
 \therefore \bar{y} &= \frac{1200 \times 15 + 600 \times 45 - 50\pi \times \frac{40}{3\pi} - 100 \times \frac{170}{3}}{1200 + 600 - 50\pi - 100} \\
 &= \frac{18000 + 27000 - 666.7 - 5666.7}{1700 - 50\pi} = \frac{38666.6}{1542.92} \\
 &= 25.06 \text{ cm from origin } O. \quad \text{Ans.}
 \end{aligned}$$

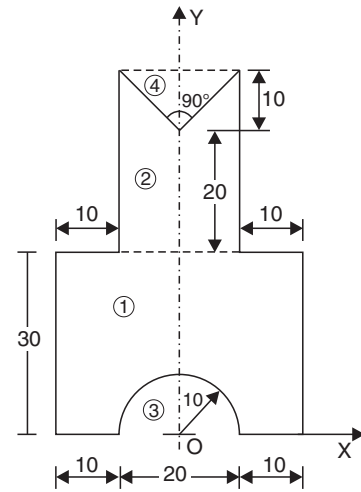


Fig. 9.40

10. From a circular plate of diameter 100 mm a circular part of diameter 50 mm is cut as shown in Fig. 9.41. Find the centroid of the remainder. (U.P. Tech. University, 2002–2003)

[Hint. Fig. 9.41 is symmetrical about X-axis. Hence centroid lies on X-axis.]

$$\therefore \bar{y} = 0.6. \text{ The value of } \bar{x} \text{ is given by } \bar{x} = \frac{a_1x_1 + a_2x_2}{a_1 - a_2}$$

$$\text{But } a_1 = \frac{\pi}{4} \times 100^2 = 7853.98 \text{ mm}^2, \quad x_1 = \frac{100}{2} = 50 \text{ mm}$$

$$a_2 = -\left(\frac{\pi}{4} \times 50^2\right) = -1963.5 \text{ mm}^2, \quad x_2 = 100 - 25 = 75 \text{ mm}$$

$$\therefore \bar{x} = \frac{7853.98 \times 50 - 1963.5 \times 75}{7853.98 - 1963.5} = 41.67 \text{ mm}$$

Hence centroid is at (41.67 mm, 0). Ans.]

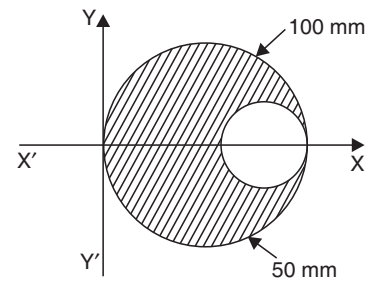


Fig. 9.41

UNIT IV

**KINEMATICS AND KINETICS
OF RIGID BODIES**

Chapters

- 10. Kinematics of Rigid Bodies
- 11. Kinetics of Rigid Bodies

10

Kinematics of Rigid Bodies

10.1. INTRODUCTION

Kinematics is that branch of Engineering Mechanics which deals with motion of particles and bodies without consideration of forces required to produce them. Hence, the term kinematic stands for the displacement, velocity and acceleration of a particle or of a body. Particle is a body whose size can be neglected whereas the rigid body is a body in which the distance between any two points remains fixed for all the time. The following are the important motion related with this chapter :

- (i) Absolute motion
- (ii) Relative motion
- (iii) Rectilinear motion (or linear motion)
- (iv) Curvilinear motion and
- (v) Uniform motion.

10.2. ABSOLUTE MOTION

Absolute motion is a motion in which quantities are expressed with reference to a system of reference axes fixed on earth.

Suppose a train is moving with a speed of 40 km/hr. A man in the train is also moving with a speed of 2 km/hr towards engine. Then the speed of man with respect to train is 2 km/hr whereas the speed of man with respect to earth (fixed axes) is $40 + 2 = 42$ km/hr. The speed of man with respect to earth is known as absolute motion.

10.2.1. Relative Motion. Relative motion is a motion in which quantities are measured with reference to moving axes.

10.2.2. Rectilinear Motion. It is motion of a body in a straight line.

10.2.3. Curvilinear Motion. It is motion of a body along a curved path.

10.2.4. Uniform Motion. It is the motion of a body in which a body travels equal distance in equal time how small the time interval may be.

10.3. PLANE RECTILINEAR MOTION OF RIGID BODY

The motion of a body in a straight line in a plane is known as plane rectilinear motion. The examples of the bodies moving in a straight line are : a car moving on the road ; a body projected vertically up/in the air with a certain velocity ; a body falling vertically downward

etc. We shall derive the equations of motion of a body moving in a straight line. These equations of motion are then used for solving problems for the bodies which are moving in straight lines.

10.3.1. Velocity. Velocity is defined as the rate of change of displacement of a body moving in a straight line. It is measured in metre per second. Velocity is a vector* quantity. It is denoted by v .

Let s = Distance travelled by a body in a straight line
 t = Time taken to travel the distance.

Then velocity of the body = $\frac{s}{t}$.

Mathematically, it can also be written as

$$\text{Velocity, } v = \frac{ds}{dt} \quad \dots(10.1)$$

10.3.2. Acceleration. Acceleration is defined as the rate of change of velocity of a body. It is measured in metre per second square and is written as m/sec^2 in M.K.S. system and m/s^2 in S.I. system.

$$\therefore \text{Acceleration} = \frac{\text{Change of velocity}}{\text{Time}}$$

Acceleration is denoted by ' a '. Acceleration is also a vector* quantity. Mathematically, it can also be written as

$$\text{Acceleration, } a = \frac{dv}{dt} \quad \dots(10.2)$$

$$= \frac{d}{dt} \left(\frac{ds}{dt} \right) \quad \left(\because v = \frac{ds}{dt} \text{ from equation (10.1)} \right)$$

$$= \frac{d^2s}{dt^2} \quad \dots(10.2A)$$

But $\frac{dv}{dt}$ can also be written as

$$\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v \quad \left(\because \frac{ds}{dt} = v \right)$$

But $\frac{dv}{dt}$ = Acceleration = a

$$\therefore a = \frac{dv}{ds} \cdot v \quad \dots(10.3)$$

10.3.3. Equation of Motion in a Straight Line. Let us consider a body, which is moving in a straight line.

Let u = Initial velocity of the body in m/s
 v = Final velocity of the body in m/s
 t = Time in seconds, during which velocity changes from u to v
 s = Distance travelled by the body in time t
 a = Acceleration of the body.

*A vector quantity is one which is represented in magnitude and direction.

10.3.4. Equation for Final Velocity

$$\begin{aligned}\text{Change of velocity} &= \text{Final velocity} - \text{Initial velocity} \\ &= (v - u)\end{aligned}$$

$$\text{Rate of change of velocity} = \frac{\text{Change of velocity}}{\text{Time}} = \frac{(v - u)}{t}.$$

$$\text{But rate of change of velocity} = \text{Acceleration} = a$$

$$\therefore a = \frac{(v - u)}{t} \quad \dots(10.4)$$

or

$$at = v - u$$

$$\therefore v = u + at \quad \dots(10.5)$$

10.3.5. Equation of Motion for Distance Covered (s)

$$\begin{aligned}\text{We know average velocity} &= \frac{\text{Initial velocity} + \text{Final velocity}}{2} \\ &= \frac{u + v}{2}.\end{aligned}$$

$$\begin{aligned}\text{Distance covered,} \quad s &= \text{Average velocity} \times \text{Time} \\ &= \frac{(u + v)}{2} \times t.\end{aligned}$$

Substituting the value of v from equation (10.5),

$$\begin{aligned}s &= \frac{(u + u + at)}{2} \times t = \frac{(2u + at)}{2} \times t = \left(u + \frac{1}{2}at\right) \times t \\ \therefore s &= ut + \frac{1}{2}at^2 \quad \dots(10.6)\end{aligned}$$

10.3.6. Derivation of $v^2 - u^2 = 2as$. The value of ' t ' from equation (10.5) is given as

$$t = \frac{(v - u)}{a}.$$

Substituting the value of ' t ' in equation (10.6), we get

$$\begin{aligned}s &= u \left(\frac{v - u}{a}\right) + \frac{1}{2}a \left(\frac{v - u}{a}\right)^2 \\ &= (v - u) \left[\frac{u}{a} + \frac{1}{2a}(v - u)\right] = (v - u) \left[\frac{2u + v - u}{2a}\right] \\ &= \frac{(v - u)(v + u)}{2a} = \frac{v^2 - u^2}{2a}\end{aligned}$$

or

$$v^2 - u^2 = 2as \quad \dots(10.7)$$

In equations (10.5), (10.6) and (10.7), a is the linear acceleration. If instead of acceleration the retardation is given, then the value of ' a ' in these equations should be taken negative. Retardation is -ve of acceleration.

2nd Method for deriving equations of motion for a body moving in a straight line

The equations of motion of a body moving in a straight line may also be derived by integration as given below :

1. Derivation of $s = ut + \frac{1}{2}at^2$

Let a body is moving with a uniform acceleration ' a '.

Then from equation (10.2), we have

$$\frac{d^2s}{dt^2} = a \quad \text{or} \quad \frac{d}{dt} \left(\frac{ds}{dt} \right) = a$$

or

$$d \left(\frac{ds}{dt} \right) = a \, dt.$$

Integrating the above equation,

$$\int d \left(\frac{ds}{dt} \right) = \int a \, dt \quad \text{or} \quad \frac{ds}{dt} = at + C_1 \quad \dots(i)$$

where C_1 is the constant of integration.

But $\frac{ds}{dt}$ = Velocity at any instant

when $t = 0$, the velocity is known as initial velocity which is represented by u .

$$\therefore \quad \text{at } t = 0, \quad \frac{ds}{dt} = \text{Initial velocity} = u$$

Substituting these values in equation (i).

$$u = a \times 0 + C_1$$

$$\therefore \quad C_1 = u.$$

Substituting the value of C_1 in equation (i),

$$\frac{ds}{dt} = at + u \quad \dots(ii)$$

Integrating the above equation, we get

$$s = \frac{at^2}{2} + ut + C_2 \quad \dots(iii)$$

when C_2 is another constant of integration.

When $t = 0$, then $s = 0$.

Substituting these values in equation (iii), we get

$$0 = \frac{a}{2} \times 0 + u \times 0 + C_2 = 0 + 0 + C_2$$

$$\therefore \quad C_2 = 0.$$

Substituting this value of C_2 in equation (iii), we get

$$s = \frac{1}{2} at^2 + ut$$

the above equation is the same as equation (10.6).

2. Derivation of $\mathbf{v} = \mathbf{u} + \mathbf{at}$.

From equation (ii), we have $\frac{ds}{dt} = at + u$.

But $\frac{ds}{dt}$ represents the velocity at any time. After the time ' t ' the velocity is known as final velocity, which is represented by v .

$$\therefore \quad \frac{ds}{dt} \text{ after time } 't' = \text{Final velocity} = v.$$

Substituting the value of $\frac{ds}{dt} = v$, we get

$$v = at + u.$$

3. Derivation of $v^2 = u^2 + 2as$.

From equation (10.3), acceleration 'a' is given by

$$a = \frac{v dv}{ds}$$

or $\frac{v dv}{ds} = a$ or $v dv = ads.$

Integrating, we get

$$\frac{v^2}{2} = as + C_3 \quad \dots(iv)$$

where C_3 is constant of integration.

When $s = 0$, the velocity is known as initial velocity.

$$\therefore \text{At } s = 0, \quad v = u.$$

Substituting these values in equation (iv), we get

$$\frac{u^2}{2} = a \times 0 + C_3$$

$$\therefore C_3 = \frac{u^2}{2}.$$

Substituting this value in equation (iv), we get

$$\frac{v^2}{2} = as + \frac{u^2}{2} \quad \text{or} \quad v^2 = 2as + u^2.$$

Problem 10.1. A bullet, moving at the rate of 250 m/s, is fired into a log of wood. The bullet penetrates to a depth of 40 cm. If the bullet moving with the same velocity is fired into a similar piece of wood 20 cm thick, with what velocity would it emerge. Take the resistance to be uniform in both the cases.

Sol. First Case. Given :

Initial velocity, $u = 250$ m/s

Final velocity, $v = 0$ (as bullet is not coming out of the wood)

Distance moved by bullet before coming to rest,

$$s = 40 = 0.40 \text{ m.}$$

First calculate the retardation of the bullet.

Let a = Acceleration of the bullet.

Using the equation (10.7) $v^2 - u^2 = 2as$

or $0 - 250^2 = 2a \times 0.40$

$$\therefore a = \frac{-250 \times 250}{2 \times 0.40} = -78125 \text{ m/s}^2.$$

Second Case

Initial velocity, $u = 250$ m/s (same as before)

Distance moved, $s = 20 \text{ cm} = 0.20 \text{ m.}$

As the wood is similar and also the resistance is uniform, the acceleration will be same as in the first case.

∴ Acceleration, $a = -78125 \text{ m/s}^2$.

Let v = Final velocity with which bullet would come out from the wood.

Using the equation (10.7),

$$v^2 - u^2 = 2as$$

$$\text{or } v^2 - 250^2 = 2 \times (-78125) \times .20 \quad (\because a = -78125 \text{ m/s}^2)$$

$$\text{or } v^2 = 250^2 - 2 \times 78125 \times .20 = 62500 - 31250 = 31250$$

$$\therefore v = \sqrt{31250} = 176.77 \text{ m/s. Ans.}$$

Problem 10.2. A motorist is driving his car at 90 km/hr. He observes red light 200 m ahead turns red. The traffic light is timed to remain red for 15 sec. If the motorist wishes to pass the light without stopping, find the required minimum deceleration.

(U.P. Tech. University, 2000–2001)

Sol. Given :

Speed of car = 90 km/hr

$$\therefore \text{Velocity of car} = \frac{90 \times 100}{60 \times 60} = 25 \text{ m/s}$$

This is the initial velocity of car.

$$\therefore u = 25 \text{ m/s}$$

Distance of car from red light, $s = 200 \text{ m}$

Time of traffic light to remain red = 15 sec.

Let a = Minimum deceleration so that the motorist can pass the light without stopping.

$$\text{Using, } s = ut + \frac{1}{2}at^2$$

$$200 = 25 \times 15 + \frac{1}{2} \times a \times 15^2 = 375 + 112.5 \times a$$

$$\text{or } a = \frac{200 - 375}{112.5} = -\frac{175}{112.5} = -1.555 \text{ m/s}^2$$

$$\approx -1.56 \text{ m/s}^2. \text{ Ans.}$$

–ve sign shows deceleration.

Now let us check that whether the motorist was in motion or at rest after 15 seconds. The final velocity of motorist after 15 seconds and having deceleration of 1.56 m/s² will be obtained by using

$$\begin{aligned} v &= u + at \\ &= 25 + (-1.56) \times 15 \quad (\because u = 25 \text{ m/s and } a = -1.56) \\ &= 25 - 23.4 = 1.6 \text{ m/s} \end{aligned}$$

As the final velocity is +ve, hence motorist was in motion. Hence he can pass the light without stopping.

10.3.7. Distance Travelled in the nth Second

Let u = Initial velocity of a body

a = Acceleration of the body

S_n = Distance covered in n seconds

S_{n-1} = Distance covered in $(n - 1)$ seconds

Then distance travelled in the n th seconds

$$\begin{aligned} &= \text{Distance travelled in } n \text{ seconds} - \text{Distance travelled in } (n - 1) \text{ seconds} \\ &= S_n - S_{n-1}. \end{aligned}$$

Distance travelled in n seconds is obtained by substituting $t = n$ in equation

$$s = ut + \frac{1}{2} at^2.$$

$$\therefore S_n = u \times n + \frac{1}{2} an^2.$$

$$\text{Similarly } S_{n-1} = u(n-1) + \frac{1}{2} a(n-1)^2 \quad (\text{Put } t = n-1 \text{ in } s = ut + \frac{1}{2} at^2)$$

$\therefore$ Distance travelled in the n th seconds

$$\begin{aligned} &= S_n - S_{n-1} \\ &= \left(un + \frac{1}{2} an^2 \right) - \left[u(n-1) + \frac{1}{2} a(n-1)^2 \right] \\ &= un + \frac{1}{2} an^2 - \left[un - u + \frac{1}{2} a(n^2 + 1 - 2n) \right] \\ &= un + \frac{1}{2} an^2 - un + u - \frac{1}{2} an^2 - \frac{1}{2} a + \frac{1}{2} a \times 2n \\ &= an + u - \frac{1}{2} a = u + \frac{a}{2} (2n - 1) \quad \dots(10.8) \end{aligned}$$

Problem 10.3. A body is moving with uniform acceleration and covers 15 m in fifth second and 25 m in 10th second. Determine :

(i) the initial velocity of the body, and

(ii) acceleration of the body.

Sol. Given :

Distance covered in 5th second = 15 m.

Distance covered in 10th second = 25 m.

Let u = Initial velocity, and

a = Acceleration of the body.

Using equation (10.8) for the distance covered in n th second $= u + \frac{a}{2} (2n - 1)$.

$\therefore$ Distance covered in 5th second

$$= u + \frac{a}{2} [2 \times 5 - 1] \quad (\because n = 5)$$

$$\text{or } 15 = u + \frac{9a}{2} \quad \dots(i)$$

Distance covered in 10th second

$$= u + \frac{a}{2} [2 \times 10 - 1] \quad (\because n = 10)$$

$$\text{or } 25 = u + \frac{19a}{2} \quad \dots(ii)$$

Subtracting equation (i) from (ii), we get

$$25 - 15 = \left(u + \frac{19a}{2} \right) - \left(u + \frac{9a}{2} \right)$$

or
$$10 = \frac{19a}{2} - \frac{9a}{2} = \frac{10a}{2}$$

$\therefore a = 2 \text{ m/s}^2. \quad \text{Ans.}$

Substituting the value of 'a' in equation (i),

$$15 = u + \frac{9}{2} \times 2 = u + 9$$

$\therefore u = 15 - 9 = 6 \text{ m/s.} \quad \text{Ans.}$

10.3.8. Equation of Motions Due to Gravity. The acceleration due to gravity is 'g'. Hence when a body falls, the equation of motions given by equations (10.5), (10.6) and (10.7) are modified by substituting 'g' in place of 'a'. But when the body is moving vertically up the acceleration due to gravity is acting in the opposite direction. In that case the equations are modified by substituting $(-g)$ in place of a . The value of g is taken as 981 cm/s^2 or 9.81 m/s^2 . The distance 'S' is replaced by 'h'. Hence the equations of motions due to gravity in the downward directions and upward directions becomes as :

1. For downward motion

$$a = +g$$

$$v = u + gt$$

$$S = h = ut + \frac{1}{2}gt^2$$

$$v^2 - u^2 = 2gh.$$

2. For upward motion

$$a = -g$$

$$v = u - gt$$

$$h = ut - \frac{1}{2}gt^2$$

$$v^2 - u^2 = -2gh.$$

Notes :

(i) If a body starts from rest, its initial velocity is zero, i.e., $u = 0$.

(ii) If a body comes to rest, its final velocity is zero, i.e., $v = 0$.

(iii) If a body is projected vertically upwards, the final velocity of the body at the highest point is zero, i.e., $v = 0$.

(iv) If a body starts moving vertically downwards, its initial velocity is zero, i.e., $u = 0$.

(v) Acceleration due to gravity is taken positive when a body is moving vertically downwards. But if the body is moving vertically upwards, the acceleration due to gravity is taken negative.

Problem 10.4. A stone dropped into a well is heard to strike the water after 4 seconds. Find the depth of the well, if the velocity of sound is 350 m/sec.

Sol. Given :

Initial velocity of stone, $u = 0$.

Time taken by the stone to reach the bottom of well and time taken by the sound to reach from the bottom of the well to the top of the well = 4 s.

Velocity of sound = 350 m/s.

Let h = Depth of well

t_1 = Time taken by the stone to reach the bottom of well

t_2 = Time taken by the sound to reach from bottom of the well to the top of the well.

Then $t_1 + t_2 = 4 \text{ s.} \quad \dots(i)$

Consider the motion of the stone

Using the equation, $h = ut + \frac{1}{2}gt^2$

where $u = 0$, $t = t_1$, $g = 9.81 \text{ m/s}^2$.

$$\therefore h = 0 \times t_1 + \frac{1}{2} \times 9.81 \times t_1^2 = 0 + 4.905t_1^2 = 4.905t_1^2. \quad \dots(ii)$$

Consider the motion of sound

When the stone strikes the water in the well the sound, produced by the stone, travels with uniform velocity of 350 m/s in all directions. Hence the time taken by the sound to reach the top of the well

$$= \frac{\text{Depth of well}}{\text{Velocity of sound}}$$

$$\text{or } t_2 = \frac{h}{350} = \frac{4.905t_1^2}{350} \quad (\because \text{From equation (ii), } h = 4.905t_1^2)$$

Substituting the value of t_2 in equation (i), we get

$$t_1 + \frac{4.905t_1^2}{350} = 4 \quad \text{or} \quad 350t_1 + 4.905t_1^2 = 350 \times 4 = 1400$$

$$\text{or } 4.905t_1^2 + 350t_1 - 1400 = 0 \quad \dots(iii)$$

The above equation is a quadratic equation, whose roots are given as,

$$\begin{aligned} t_1 &= \frac{-350 \pm \sqrt{350^2 - 4(4.905)(-1400)}}{2(4.905)} \\ &= \frac{-350 \pm \sqrt{122500 + 27468}}{9.81} \\ &= \frac{-350 \pm 387.26}{9.81} = \frac{-350 + 387.26}{9.81} \\ &= \frac{-350 - 387.26}{9.81} = 3.798 \text{ and } -75.154 \end{aligned}$$

$$\therefore t_1 = 3.798 \text{ s.} \quad (\text{--ve root is not possible})$$

Substituting this value in equation (ii), we get

$$h = 4.905 \times 3.798^2 = 70.75 \text{ m.}$$

$$\therefore \text{Depth of well} = \mathbf{70.75 \text{ m.}} \quad \mathbf{Ans.}$$

Problem 10.5. A stone is dropped from a height. After falling 5 seconds from rest, the stone breaks the glass pane and in breaking, the stone loses 20% of its velocity. Find the distance travelled by the stone in the next second. Take $g = 9.81 \text{ m/s}^2$.

Sol. Given :

Initial velocity, $u = 0$

Time taken by stone in striking the glass pane = 5 s.

Acceleration due to gravity, $g = 9.81 \text{ m/s}^2$.

Let v = the velocity with which stone strikes the pane.

Using $v = u + gt$, we get

$$v = 0 + 9.81 \times 5 = 49.05 \text{ m/s.}$$

Velocity lost in breaking the stone

= 20% of the velocity with which the stone strikes

$$= \frac{20}{100} \times 49.05 = 9.81 \text{ m/s}$$

∴ The velocity of the stone after breaking the glass pane

$$= 49.05 - 9.81 = 39.24 \text{ m/s.}$$

Now the distance travelled by the stone, after breaking the glass pane, in one second is given by

$$\begin{aligned} s &= ut + \frac{1}{2}gt^2 = 39.24 \times 1 + \frac{1}{2} \times 9.81 \times 1^2 \\ &\quad (\because t = 1 \text{ sec and initial velocity of stone after breaking} = 39.24 \text{ m/s}) \\ &= 29.24 + 4.905 = 44.145 \text{ m. Ans.} \end{aligned}$$

10.3.9. Velocity and Acceleration of a Body Moving in a Straight Line by Differentiation. Let the equation of motion of a body moving in a straight line is given in terms of displacement (s) and time (t) as $s = t^3 + 2t^2 + 5t + 6$...*(i)*

By differentiating the above equation with respect to time, we get

$$\begin{aligned} \text{Velocity, } \frac{ds}{dt} &= v \\ \therefore v &= \frac{ds}{dt} = \frac{d}{dt}(t^3 + 2t^2 + 5t + 6) \\ &= 3t^2 + 4t + 5 \end{aligned} \quad \dots(ii)$$

If the velocity at start is to be calculated, the value of $t = 0$ is to be substituted in equation (ii).

By differentiating the equation (ii) with respect to time, we get acceleration, i.e., $\frac{dv}{dt} = a$.

$$\begin{aligned} \therefore \text{Acceleration, } a &= \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 4t + 5) \\ &= 6t + 4 \end{aligned} \quad \dots(iii)$$

The acceleration at start will be obtained by substituting $t = 0$, in equation (iii).

Problem 10.6. A particle moves along a straight line so that its displacement in metre from a fixed point is given by, $s = t^3 + 3t^2 + 4t + 5$.

Find :

(i) Velocity at start and after 4 seconds

(ii) Acceleration at start and after 4 seconds.

Sol. Given :

$$s = t^3 + 3t^2 + 4t + 5$$

$$\begin{aligned} \text{Velocity, } v &= \frac{ds}{dt} = \frac{d}{dt}(t^3 + 3t^2 + 4t + 5) \\ &= 3t^2 + 6t + 4 \end{aligned} \quad \dots(i)$$

(i) Velocity at start will be obtained, if $t = 0$ is substituted in equation (i),

$$\begin{aligned} \therefore v(\text{at } t = 0) &= 3 \times 0^2 + 6 \times 0 + 4 = 0 + 0 + 4 \\ &= 4 \text{ m/s. Ans.} \end{aligned}$$

Velocity after four seconds. Substitute $t = 4$ in equation (i)

$$\begin{aligned} v(\text{at } t = 4) &= 3 \times 4^2 + 6 \times 4 + 4 \\ &= 48 + 24 + 4 = \mathbf{76 \text{ m/s.}} \quad \text{Ans.} \end{aligned}$$

(ii) Differentiating equation (i) with respect to time ' t ', we get acceleration.

$$\begin{aligned} \therefore \text{Acceleration, } a &= \frac{dv}{dt} = \frac{d}{dt} (3t^2 + 6t + 4) \\ &= 6t + 6. \end{aligned} \quad \dots(ii)$$

Acceleration at start will be obtained, if $t = 0$ is substituted in equation (ii).

$$\begin{aligned} \therefore a(\text{at } t = 0) &= 6 \times 0 + 6 = 0 + 6 \\ &= \mathbf{6 \text{ m/s}^2}. \quad \text{Ans.} \end{aligned}$$

Acceleration after four seconds. Substitute $t = 4$ in equation (ii),

$$\begin{aligned} \therefore a(\text{at } t = 4) &= 6 \times 4 + 6 = 24 + 6 \\ &= \mathbf{30 \text{ m/s}^2}. \quad \text{Ans.} \end{aligned}$$

Problem 10.7. The equation of motion of a particle moving in a straight line is given by

$$s = 18t + 3t^2 - 2t^3$$

where s is the total distance covered from the starting point is metres at the end of t seconds. Find :

- (i) the velocity and acceleration at start,
- (ii) the time, when the particle reaches its maximum velocity and
- (iii) the maximum velocity of the particle. (U.P. Tech. University, 2002–2003)

Sol. Given :

$$s = 18t + 3t^2 - 2t^3.$$

Differentiating both sides with respect to t ,

$$\frac{ds}{dt} = 18 + 3 \times 2t - 2 \times 3t^2 = 18 + 6t - 6t^2.$$

But from equation (10.1), we know $\frac{ds}{dt} = v$

$$\therefore \text{Velocity, } v = 18 + 6t - 6t^2. \quad \dots(i)$$

(i) At start, $t = 0$. Hence velocity at start will be obtained by substituting $t = 0$ in equation (i),

$$\begin{aligned} \therefore \text{Velocity at start,} \\ v &= 18 + 6 \times 0 - 6 \times 0^2 = 18 + 0 - 0 \\ &= \mathbf{18 \text{ m/s.}} \quad \text{Ans.} \end{aligned}$$

Differentiating both sides of equation (i) with respect to t ,

$$\frac{dv}{dt} = 6 - 6 \times 2t = 6 - 12t \quad \dots(ii)$$

But from equation (10.2), we know that $\frac{dv}{dt} = \text{Acceleration} = a$

$$\therefore a = 6 - 12t \quad \dots(iii)$$

At start, $t = 0$. Hence acceleration at start will be obtained by substituting $t = 0$ in equation (iii).

$\therefore$ Acceleration at start $= 6 - 12 \times 0 = 6 \text{ m/s}^2$. **Ans.**

(ii) The velocity will be maximum, when the differentiation of the velocity with respect to time is equal to zero or $\frac{dv}{dt} = 0$. But $\frac{dv}{dt}$ is given by equation (ii). Hence equating the equation (ii) to zero for maximum velocity, we get

$$6 - 12t = 0 \quad \text{or} \quad 12t = 6$$

or
$$t = \frac{6}{12} = \frac{1}{2} \text{ s. Ans.}$$

(iii) From equation (i), velocity is given as

$$v = 18 + 6t - 6t^2$$

For maximum velocity, $t = \frac{1}{2}$

Hence substituting $t = \frac{1}{2}$ in the above equation, we get maximum velocity as

$$\begin{aligned} v_{\max} &= 18 + 6 \times \frac{1}{2} - 6 \times \left(\frac{1}{2}\right)^2 = 18 + 3 - \frac{6}{4} \\ &= 21 - 1.5 = 19.5 \text{ m/s. Ans.} \end{aligned}$$

10.3.10. Velocity and Displacement of a Body Moving in a Straight Line by Integration. Let the equation of motion of a body moving in a straight line is given in terms of acceleration (a) and time (t) as

$$a = t^3 + 2t^2 + 4t + 5 \quad \dots(i)$$

But
$$a = \frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = t^3 + 2t^2 + 4t + 5$$

Integrating the above equation, we get

$$\int dv = \int (t^3 + 2t^2 + 4t + 5)dt$$

or
$$v = \left(\frac{t^4}{4} + \frac{2t^3}{3} + \frac{4t^2}{2} + 5t \right) + C_1 \quad \dots(ii)$$

where C_1 is the constant of integration and its value is obtained from given condition. Let the condition is given that at $t = 1$, the velocity is 10 m/s.

Substituting this condition in equation (ii), we get

$$\begin{aligned} 10 &= \frac{1^4}{4} + \frac{2 \times 1^3}{3} + \frac{4 \times 1^2}{2} + 5 \times 1 + C_1 \\ &= \frac{1}{4} + \frac{2}{3} + 2 + 5 + C_1 \\ &= 0.25 + 0.67 + 7 + C_1 = 7.92 + C_1 \end{aligned}$$

$$\therefore C_1 = 10 - 7.92 = 2.08.$$

Substituting the value of C_1 in equation (ii), we get the velocity at any time t ,

$$v = \frac{t^4}{4} + \frac{2t^3}{3} + \frac{4t^2}{2} + 5t + 2.08 \quad \dots(iii)$$

But

$$v = \frac{ds}{dt}$$

$$\therefore \frac{ds}{dt} = \frac{t^4}{4} + \frac{2t^3}{3} + 2t^2 + 5t + 2.08$$

Integrating the above equation, we get

$$\int ds = \int \left(\frac{t^4}{4} + \frac{2t^3}{3} + 2t^2 + 5t + 2.08 \right) dt$$

or

$$s = \frac{t^5}{4 \times 5} + \frac{2t^4}{3 \times 4} + \frac{2t^3}{3} + \frac{5t^2}{2} + 2.08t + C_2 \quad \dots(iv)$$

where C_2 is another constant of integration. The value of C_2 is also obtained from given condition. If at $t = 1$, the displacement (s) is given as equal to 10 m. Substituting this condition in equation (iv), we get

$$10 = \frac{1^5}{20} + \frac{2 \times 1^4}{12} + \frac{2 \times 1^3}{3} + \frac{5 \times 1^2}{2} + 2.08 \times 1 + C_2$$

or

$$\begin{aligned} 10 &= \frac{1}{20} + \frac{1}{6} + \frac{2}{3} + 2.5 + 2.08 + C_2 \\ &= .05 + .1667 + .667 + 2.5 + 2.08 + C_2 = 5.4637 + C_2 \end{aligned}$$

$$\therefore C_2 = 10 - 5.4637 = 4.5363.$$

Substituting the value of C_2 in equation (iv), we get the displacement at any time t as

$$s = \frac{t^5}{20} + \frac{2t^4}{12} + \frac{2t^3}{3} + \frac{5t^2}{2} + 2.08t + 4.5363 \quad \dots(v)$$

If it is required to find the velocity and displacement at any time ' t ', then equation (iii) and (v) are used.

Problem 10.8. A particle moves along a straight line with an acceleration described by the equation $a = -8s^{-2}$ where a is in m/s^2 and s in m . When $t = 1$ s, $s = 4$ m and $v = 2$ m/s. Determine acceleration when $t = 2$ s. (Osmania, 1989)

Sol. Given :

$$\text{Acceleration,} \quad a = -8s^{-2}$$

$$\text{When } t = 1 \text{ s, } s = 4 \text{ m and } v = 2 \text{ m/s}$$

$$\text{Find acceleration when } t = 2 \text{ s.}$$

The acceleration in terms of velocity and displacement is given by equation (10.3),

$$a = \frac{v dv}{ds}$$

$$\text{But} \quad a = -8s^{-2} \quad \text{(given)}$$

$$\therefore -8s^{-2} = \frac{v dv}{ds} \quad \text{or} \quad v dv = -8s^{-2} ds$$

Integrating the above equation, we get

$$\int v \, dv = \int -8s^{-2} \, ds \quad \frac{v^2}{2} = -8 \frac{s^{-2+1}}{-2+1} + C_1$$

or
$$\frac{v^2}{2} = \frac{8}{s} + C_1 \quad \dots(i)$$

where C_1 is the constant of integration and its value is obtained from given condition *i.e.*, $s = 4$ m and $v = 2$ m/s when $t = 1$ s.

Substituting $v = 2$ m/s and $s = 4$ m in equation (i),

$$\frac{2^2}{2} = \frac{8}{4} + C_1 \quad \text{or} \quad 2 = 2 + C_1$$

or $C_1 = 0$

Substituting $C_1 = 0$ in equation (i), we get

$$\frac{v^2}{2} = \frac{8}{s} \quad \text{or} \quad v^2 = \frac{4}{s} \quad \text{or} \quad v = \frac{4}{\sqrt{s}} \quad \dots(ii)$$

But $v = \frac{ds}{dt}$

Substituting the value of v in equation (ii), we get

$$\frac{ds}{dt} = \frac{4}{\sqrt{s}} \quad \text{or} \quad \sqrt{s} \, ds = 4 \, dt$$

Integrating, we get

$$\int \sqrt{s} \, ds = \int 4 \, dt \quad \text{or} \quad \frac{s^{\frac{1}{2}+1}}{\frac{1}{2}+1} = 4t + C_2$$

or
$$\frac{2}{3} s^{3/2} = 4t + C_2 \quad \dots(iii)$$

when $t = 1$ s, $s = 4$ m.

Substituting these values in equation (iii), we get

$$\frac{2}{3} \times 4^{3/2} = 4 \times 1 + C_2 \quad \text{or} \quad \frac{16}{3} = 4 + C_2$$

or
$$C_2 = \frac{16}{3} - 4 = \frac{16 - 12}{3} = \frac{4}{3}$$

Substituting the value of C_2 in equation (iii), we get

$$\frac{2}{3} s^{3/2} = 4t + \frac{4}{3}$$

When $t = 2$ s, the above equation becomes as

$$\frac{2}{3} s^{3/2} = 4 \times 2 + \frac{4}{3} = 8 + \frac{4}{3} = \frac{28}{3}$$

or
$$s^{3/2} = \frac{28}{3} \times \frac{3}{2} = 14 \quad \text{or} \quad s = 14^{2/3} = 5.81$$

Now acceleration, $a = -8s^{-2} = -8 \times 5.81^{-2} = 0.2373 \text{ m/s}^2$. **Ans.**

Problem 10.9. A particle moves along a straight line with an acceleration described by the equation $a = (4t^2 - 2)$, where a is in m/s^2 and t is in seconds. When $t = 0$, the particle is located 2 m to the left of origin and when $t = 2$ seconds, the particle is 20 m to left of origin. Determine the position of the particle, when $t = 4$ seconds.

Sol. Given :

Acceleration, $a = 4t^2 - 2$

(If left of origin is considered -ve)

When $t = 0$, distance $s = -2$ m

When $t = 2$ seconds, distance $s = -20$ m

Find distance s when $t = 4$ seconds

Acceleration in terms of distance is given by,

$$a = \frac{d^2s}{dt^2}$$

But $a = 4t^2 - 2$

Equating the two values of acceleration, we get

$$\frac{d^2s}{dt^2} = 4t^2 - 2$$

Integrating the above equation, we get

$$\frac{ds}{dt} = 4 \times \frac{t^3}{3} - 2t + C_1 \quad \dots(i)$$

Integrating the above equation again,

$$s = \frac{4}{3} \times \frac{t^4}{4} - \frac{2t^2}{2} + C_1t + C_2$$

$$\text{or} \quad s = \frac{t^4}{3} - t^2 + C_1t + C_2 \quad \dots(ii)$$

where C_1 and C_2 are constant of integrations and their values are obtained from given conditions

(i) When $t = 0$, $s = -2$ m.

Substituting these values in equation (ii), we get

$$-2 = \frac{1}{3} \times 0 - 0 + C_1 \times 0 + C_2 = 0 - 0 + 0 + C_2$$

$$\therefore C_2 = -2$$

Substituting the value of C_2 in equation (ii), we get

$$s = \frac{t^4}{3} - t^2 + C_1t - 2 \quad \dots(iii)$$

(ii) When $t = 2$ seconds, $s = -20$ m.

Substituting these values in equation (iii), we get

$$-20 = \frac{1}{3} \times 2^4 - 2^2 + C_1 \times 2 - 2 = \frac{16}{3} - 4 + 2C_1 - 2$$

or
$$2C_1 = -20 - \frac{16}{3} + 4 + 2 = \frac{-60 - 16 + 12 + 6}{3} = \frac{-58}{3} = -19.33$$

$$\therefore C_1 = -\frac{19.33}{2} = -9.665$$

Substituting this value in equation (iii),

$$s = \frac{t^4}{3} - t^2 - 9.665t - 2$$

When $t = 4$ seconds, then s becomes as

$$\begin{aligned} s &= \frac{1}{3} \times 4^4 - 4^2 - 9.665 \times 4 - 2 = \frac{256}{3} - 16 - 38.66 - 2 \\ &= 85.33 - 16 - 38.66 - 2 = 85.33 - 56.66 \\ &= \mathbf{28.67} \text{ (Right of origin). } \mathbf{Ans.} \end{aligned}$$

(+ve sign shows that particle will be right of origin.)

Problem 10.10. A particle moves along a straight line with a velocity given by the equation

$$v = 2t^3 - t^2 - 2t + 4$$

where v is the velocity in m/s and t is time in seconds. When $t = 2$ seconds, the particle is found to be at a distance of 10 m from a station A. Determine :

(i) the acceleration and (ii) displacement of the particle after 6 seconds.

(Osmania, 1998)

Sol. Given :

Velocity, $v = 2t^3 - t^2 - 2t + 4$

When $t = 2$ seconds, distance $s = 10$ m

Find : (i) acceleration and (ii) displacement when $t = 6$ seconds $v = 2t^3 - t^2 - 2t + 4$

(i) Acceleration when $t = 6$ seconds

Acceleration,
$$\begin{aligned} a &= \frac{dv}{dt} = \frac{d}{dt} (2t^3 - t^2 - 2t + 4) \\ &= 6t^2 - 2t - 2 = 6 \times 6^2 - 2 \times 6 - 2 \quad (\because t = 6 \text{ seconds}) \\ &= 216 - 12 - 2 = \mathbf{202 \text{ m/s}^2}. \mathbf{Ans.} \end{aligned}$$

(ii) Displacement when $t = 6$ seconds

The velocity in terms of displacement is given by

or
$$v = \frac{ds}{dt} \quad \therefore 2t^3 - t^2 - 2t + 4 = \frac{ds}{dt}$$

Integrating,
$$\int ds = \int (2t^3 - t^2 - 2t + 4) dt$$

or
$$s = \frac{2t^4}{4} - \frac{t^3}{3} - \frac{2t^2}{2} + 4t + C \quad \dots(i)$$

where C is a constant of integration and its value is obtained from the given condition i.e., $s = 10$ m when $t = 2$ seconds.

Substituting these values in equation (i), we get

$$10 = \frac{2}{4} \times 2^4 - \frac{2^3}{3} - \frac{2}{2} \times 2^2 + 4 \times 2 + C = 8 - \frac{8}{3} - 4 + 8 + C$$

or
$$C = 10 - 8 + \frac{8}{3} + 4 - 8 \text{ or } = -2 + \frac{8}{3} = \frac{-6+8}{3} = \frac{2}{3}$$

Substituting the value of C in equation (i), we get

$$s = \frac{2t^4}{4} - \frac{t^3}{3} - \frac{2t^2}{2} + 4t + \frac{2}{3}$$

To find the displacement, when $t = 6$ s, substitute $t = 6$ s in the above equation.

$$\begin{aligned} \therefore s &= \frac{2}{4} \times 6^4 - \frac{6^3}{3} - \frac{2}{2} \times 6^2 + 4 \times 6 + \frac{2}{3} \\ &= 648 - 72 - 36 + 24 + \frac{2}{3} = \mathbf{564.67 \text{ m. Ans.}} \end{aligned}$$

Problem 10.11. A body moves along a straight line and its acceleration a which varies with t is given by $a = 2 - 3t$. After 5 seconds from start of observations its velocity is observed to be 20 m/s. After 10 seconds from start of observations the body was 85 metres from the origin.

(a) Determine its acceleration, velocity and distance from the origin at the start of observations.

(b) Determine the time after start of observation in which the velocity becomes zero and its distance from the origin.

Sol. Given :

Acceleration, $a = 2 - 3t$...(i)

At $t = 5$ seconds, $v = 20$ m/s

At $t = 10$ seconds, $s = 85$ m.

(a) (i) Acceleration at start will be obtained by substituting $t = 0$, in equation (i) as,

$$a = 2 - 3 \times 0 = 2 - 0 = \mathbf{2 \text{ m/s}^2. \text{ Ans.}}$$

(ii) The equation (i) can be written as

$$\frac{dv}{dt} = 2 - 3t. \quad \left(\because a = \frac{dv}{dt} \right)$$

Integrating the above equation,

$$\int dv = \int (2 - 3t) dt \quad \text{or} \quad v = 2t - \frac{3t^2}{2} + C_1 \quad \text{...(ii)}$$

where C_1 is the constant of integration.

At $t = 5$ seconds, velocity $v = 20$ m/s. Substituting these values in equation (ii), we get

$$20 = 2 \times 5 - \frac{3 \times 5^2}{2} + C_1$$

or $20 = 10 - 37.5 + C_1 = -27.5 + C_1$

or $C_1 = 20 + 27.5 = 47.5$

Substituting the value of C_1 in equation (ii), we get

$$v = 2t - \frac{3t^2}{2} + 47.5 \quad \text{...(iii)}$$

The velocity at start will be obtained by substituting $t = 0$, in equation (iii) as

$$v = 2 \times 0 - \frac{3}{2} \times 0^2 + 47.5 = 0 - 0 + 47.5 = \mathbf{47.5 \text{ m/s. Ans.}}$$

(iii) As $v = \frac{ds}{dt}$, the equation (iii) can be written as

$$\frac{ds}{dt} = 2t - \frac{3t^2}{2} + 47.5$$

Integrating the above equation, we get

$$\int ds = \int \left(2t - \frac{3t^2}{2} + 47.5 \right) dt$$

$$\text{or} \quad s = \frac{2t^2}{2} - \frac{3t^3}{3 \times 2} + 47.5t + C_2 \quad \dots(iv)$$

where C_2 is another constant of integration.

At $t = 10$ seconds, distance $s = 85$ m is given. Substituting these values in equation (iv), we get

$$\begin{aligned} 85 &= \frac{2 \times 10^2}{2} - \frac{3 \times 10^3}{6} + 47.5 \times 10 + C_2 \\ &= 100 - 500 + 47.5 \times 10 + C_2 = 75 + C_2 \end{aligned}$$

$$\therefore C_2 = 85 - 75 = 10.$$

Substituting the value of C_2 in equation (iv), we get

$$s = t^2 - \frac{t^3}{2} + 47.5t + 10 \quad \dots(v)$$

The distance at start will be obtained by substituting $t = 0$, in equation (v) as

$$\begin{aligned} s &= 0^2 - \frac{0^3}{2} + 47.5 \times 0 + 10 \\ &= 0 - 0 + 0 + 10 = \mathbf{10 \text{ m. Ans.}} \end{aligned}$$

(b) The equation (iii) gives the velocity at any time t . To find the time after start of observation in which velocity becomes zero, substitute $v = 0$ in equation (iii).

$$\therefore 0 = 2t - \frac{3t^2}{2} + 47.5 \quad \text{or} \quad 0 = 4t - 3t^2 + 95$$

$$\text{or} \quad 3t^2 - 4t - 95 = 0$$

The above equation is a quadratic equation. Its solution is given by

$$\begin{aligned} t &= \frac{4 \pm \sqrt{4^2 + 4 \times 3 \times 95}}{2 \times 3} = \frac{4 \pm \sqrt{16 + 1140}}{6} = \frac{4 \pm \sqrt{1156}}{6} \\ &= \frac{4 \pm 34}{6} = \frac{38}{6} \text{ and } \frac{-30}{6} = \mathbf{6.33 \text{ (Neglecting -ve root). Ans.}} \end{aligned}$$

The distance from origin after $t = 6.33$ second is obtained by substituting this value of t in equation (v).

$$\begin{aligned}
 \therefore s &= 6.33^2 - \frac{6.33^3}{2} + 47.5 \times 6.33 + 10 \\
 &= 40.069 - 126.818 + 300.675 + 10 = \mathbf{223.926 \text{ m.}} \quad \text{Ans.}
 \end{aligned}$$

10.3.11. To find Distance Travelled when Velocity-time Graph is given.

Sometimes, the velocity-time graph of a body moving in a straight line is given, then it is required to find the distance travelled by the body. The distance travelled is equal to area of velocity-time graph.

$$\begin{aligned}
 \text{Hence distance} &= \text{Area of velocity-time graph} = \text{Velocity} \times \text{Time} \\
 &= (m/s) \times s = m.
 \end{aligned}$$

Problem 10.12. A vehicle starts from rest with a linear motion and comes to rest after 25 sec. The velocity-time graph is shown in Fig. 10.1. Calculate total distance travelled by the vehicle. (U.P. Tech. University, 2000–2001)

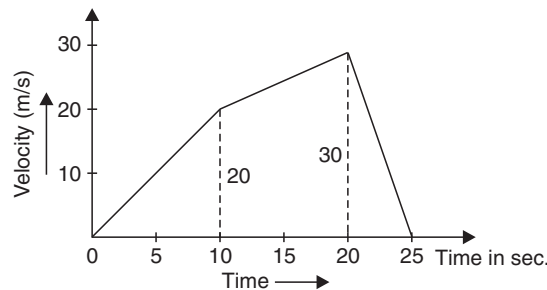


Fig. 10.1

Sol. The area of velocity-time graph gives the distance travelled by the vehicle, as

$$\text{Area} = \text{Velocity} \times \text{time} = \frac{m}{s} \times s = m$$

$$\begin{aligned}
 \therefore \text{Total distance travelled} &= \text{Area under the graph} \\
 &= \text{Area of triangle} + \text{area of trapezium} + \text{area of triangle} \\
 &= \frac{10 \times 20}{2} + \frac{20 + 30}{2} \times 10 + \frac{30 \times 5}{2} \\
 &= 100 + 250 + 75 = \mathbf{425 \text{ m.}} \quad \text{Ans.}
 \end{aligned}$$

10.4. PLANE CURVILINEAR MOTION OF RIGID BODY

The motion of a body in a plane along a circular path is known as plane curvilinear motion. In circular motion, the centre of rotation remains fixed. The examples of the bodies moving in a circular path are : shafts, flywheels, pulleys etc., rotating about their geometric axis. Here the equations of circular motion are derived. These equations of circular motions are then used for solving problems for the bodies which are having circular motion.

10.4.1. Angular Velocity. It is defined as the rate of change of angular displacement of a body. Angular displacement is always measured in terms of angle covered by the body from the initial position.

Let a body is moving along a circular path as shown in Fig. 10.2. Let initially the body is at A and after time 't', the body is at B. Let $\angle AOB = \theta$.

Then Angular displacement = $\angle AOB = \theta$.

Time taken = t .

$\therefore$ Angular velocity

$$= \frac{\text{Angular displacement}}{\text{Time}} = \frac{\theta}{t}$$

Mathematically, angular velocity = $\frac{d\theta}{dt}$.

Angular velocity is denoted by the symbol, ω .

$$\therefore \omega = \frac{d\theta}{dt} \quad \dots(10.9)$$

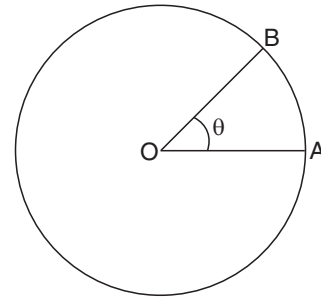


Fig. 10.2. Body moving in a circle.

It is measured in radians per second and is written as rad/s.

10.4.2. Relation Between Linear Velocity and Angular Velocity. Consider the body moving in a circle as shown in Fig. 10.2. The initial position of the body is at A and after time ' t ' the body is at B. The angle AOB is equal to θ .

$$\text{Angular velocity} = \frac{\theta}{t}$$

Let $V = \text{linear velocity} = \frac{\text{Linear displacement}}{\text{Time}}$

But linear displacement = Arc AB = $OA \times \theta = r \times \theta$ ($\because$ OA = radius of circle = r)

$$\therefore V = \frac{r \times \theta}{t} = r \times \text{Angular velocity} \quad \left(\because \frac{\theta}{t} = \text{Angular velocity} \right)$$

$$= r \times \omega \quad \dots(10.10)$$

where $\omega = \text{Angular velocity}$.

10.4.3. Angular Acceleration. It is defined as the rate of change of angular velocity. It is measured in radians per second per second and written as rad/s^2 . Mathematically, angular acceleration (α) is given as

$\alpha = \text{Rate of change of angular velocity}$

$$= \frac{d\omega}{dt} = \frac{d}{dt} \left(\frac{d\theta}{dt} \right) \quad \left(\because \omega = \frac{d\theta}{dt} \right)$$

$$= \frac{d^2\theta}{dt^2} \quad \dots(10.11)$$

$$\text{Also} \quad \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \frac{d\omega}{d\theta} \times \omega \quad \left(\because \frac{d\theta}{dt} = \omega \right)$$

But $\frac{d\omega}{dt} = \text{Angular acceleration} = \alpha$

$$\therefore \alpha = \frac{d\omega}{dt} = \omega \frac{d\omega}{d\theta} \quad \dots(10.12)$$

10.4.4. Relation Between Linear Acceleration and Angular Acceleration. From equation (10.10), we have $V = \omega \times r$.

Differentiating the above equation w.r.t. ' t '

$$\frac{dV}{dt} = \frac{d}{dt} (\omega r)$$

$$\text{or} \quad = r \frac{d\omega}{dt} \quad (\because r = \text{constant})$$

But $\frac{dV}{dt}$ = linear acceleration = a
 $\frac{d\omega}{dt}$ = angular acceleration = α

Substituting these values in the above equation, we get $a = r\alpha$... (10.13)

$\therefore$ Linear acceleration is equal to 'r' times the angular acceleration.

10.4.5. Equations of Motions along a Circular Path. Consider a body, which is moving along a circular path.

Let ω_0 = Final angular velocity of the body in radians per second

ω = Final angular velocity in radians per second

t = Time in second during which angular velocity changes from ω_0 to ω

θ = Angular displacement or angle traversed in radians

α = Angular acceleration in radians/s².

From Art. 10.4.3, we know that angular acceleration (α) is the rate of change of angular velocity. Hence

$$\begin{aligned}\alpha &= \text{Rate of change of angular velocity} = \frac{\text{Change of velocity}}{\text{Time}} \\ &= \frac{(\text{Final angular velocity} - \text{Initial angular velocity})}{\text{Time}} \\ &= \frac{(\omega - \omega_0)}{t}\end{aligned}\quad \dots(10.14)$$

or

$$\alpha t = \omega - \omega_0$$

$$\therefore \omega = \omega_0 + \alpha t \quad \dots(10.15)$$

10.4.6. Equation for Angular Displacement (θ). We know average angular velocity

$$= \frac{(\text{Initial} + \text{Final angular velocity})}{2} = \frac{\omega_0 + \omega}{2}.$$

$\therefore$ Angular displacement, θ = (Average angular velocity) $\times$ Time

$$= \frac{(\omega_0 + \omega)}{2} \times t.$$

Substituting the value of ω from equation (10.15) in the above equation, we get

$$\begin{aligned}\theta &= \frac{(\omega_0 + \omega_0 + \alpha t)}{2} \times t = \frac{(2\omega_0 + \alpha t)}{2} \times t \\ &= (\omega_0 + \frac{1}{2}\alpha t)t = \omega_0 t + \frac{1}{2}\alpha t^2 \\ \theta &= \omega_0 t + \frac{1}{2}\alpha t^2\end{aligned}\quad \dots(10.16)$$

10.4.7. Derivation of the Angular Displacement in Terms of Initial and Final Angular Velocities. The final angular velocity is given by the equation (10.15)

$$\omega = \omega_0 + \alpha t \quad \text{or} \quad \frac{\omega - \omega_0}{\alpha} = t$$

Substituting the above value of t in the equation (10.16), we get

$$\begin{aligned}\theta &= \omega_0 \times \left(\frac{\omega - \omega_0}{\alpha} \right) + \frac{1}{2} \alpha \left(\frac{\omega - \omega_0}{\alpha} \right)^2 = (\omega - \omega_0) \left[\frac{\omega_0}{\alpha} + \frac{1}{2\alpha} (\omega - \omega_0) \right] \\ &= (\omega - \omega_0) \left[\frac{2\omega_0 + \omega - \omega_0}{2\alpha} \right] = (\omega - \omega_0) \left(\frac{\omega + \omega_0}{2\alpha} \right) = \frac{\omega^2 - \omega_0^2}{2\alpha}\end{aligned}$$

$$\text{or } \omega^2 - \omega_0^2 = 2\alpha\theta \quad \dots(10.17)$$

$$\text{Angular displacement in the } n\text{th second} = \omega_0 + \left(\frac{2n-1}{2}\right)\alpha.$$

10.4.8. Relationship between r.p.m. (N) and Angular Velocity (ω).

Let N = r.p.m. of a body

ω = angular velocity of the body.

Now the number of revolution in one minute or 60 seconds = N .

$$\therefore \text{Number of revolution in one second} = \frac{N}{60}.$$

But in one revolution, the body covers an angle equal to 360° or 2π radians.

$\therefore$ Angle covered by the body in one second

$$= \text{Angle covered in one revolution} \times \text{Number of revolution per second}$$

$$= 2\pi \times \frac{N}{60} = \frac{2\pi N}{60}.$$

But angle covered per second = Angular velocity = ω .

$$\therefore \omega = \frac{2\pi N}{60} \quad \dots(10.18)$$

10.5. ROTATION OF RIGID BODIES

When a rigid body is rotating about a fixed axis, the following relations as explained earlier are used.

$$\omega = \frac{d\theta}{dt}, \alpha = \frac{d^2\theta}{dt^2}$$

$$V = \omega \times r$$

$$a = \alpha \times r$$

$$\omega = \frac{2\pi N}{60}.$$

The following solved problems are based on rotation of rigid bodies.

Problem 10.13. A body is rotating with an angular velocity of 5 radians/s. After 4 seconds, the angular velocity of the body becomes 13 radians/s. Determine the angular acceleration of the body.

Sol. Given :

Initial angular velocity, $\omega_0 = 5 \text{ rad/s}$

Final angular velocity, $\omega = 13 \text{ rad/s}$

Time, $t = 4 \text{ s}$

Let α = Angular acceleration of the body.

Using equation (10.15),

$$\omega = \omega_0 + \alpha t \quad \text{or} \quad 13 = 5 + \alpha \times 4$$

$$\therefore \alpha = \frac{(13-5)}{4} = \frac{8}{4} = 2.0 \text{ rad/s}^2. \quad \text{Ans.}$$

Problem 10.14. A wheel, rotating about a fixed axis at 20 r.p.m., is uniformly accelerated for 70 seconds, during which time it makes 50 revolution. Find : (i) Angular velocity at the end of this interval, and (ii) Time required for the speed to reach 100 revolutions per minute.

Sol. Given :

Initial r.p.m. of wheel, $N_0 = 20$

$\therefore$ Initial angular velocity,

$$\omega_0 = \frac{2\pi N_0}{60} = \frac{2\pi \times 20}{60} = \frac{2\pi}{3} \text{ rad/s.}$$

Time, $t = 70 \text{ s}$

Number of revolution in 70 s = 50

$\therefore$ Angular displacement, $\theta = \text{Angular displacement in one revolution} \times \text{Number of revolution}$

$$= 2\pi \times 50 = 100\pi \text{ radians.}$$

(i) Angular velocity at the end of 70 s.

Let $\omega = \text{Angular velocity at the end of 70 s}$

$\alpha = \text{Angular acceleration.}$

Using equation (10.16),

$$\begin{aligned} \theta &= \omega_0 t + \frac{1}{2} \alpha t^2 \\ 100\pi &= \frac{2\pi}{3} \times 70 + \frac{1}{2} \alpha \times (70)^2 = 46.67\pi + 2450\alpha \end{aligned}$$

$$\therefore \alpha = \frac{100\pi - 46.67\pi}{2450} = 0.0684 \text{ rad/s}^2.$$

From equation (10.17), we have

$$\begin{aligned} \omega &= \omega_0 + \alpha t = \frac{2\pi}{3} + 0.0684 \times 70 = 2.0944 + 4.788 \\ &= \mathbf{6.8824 \text{ rad/s.} \quad \text{Ans.}} \end{aligned}$$

(ii) Time required for the speed to reach 100 r.p.m.

Final speed, $N = 100 \text{ r.p.m.}$

$$\therefore \text{ Final angular velocity, } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 100}{60} = 10.472 \text{ rad/s.}$$

Using equation (10.15), we have $\omega = \omega_0 + \alpha \times t$

$$\therefore 10.472 = \frac{2\pi}{3} + 0.0684 \times t = 2.0944 + .0684t$$

$$\therefore t = \frac{(10.472 - 2.0944)}{.0684} = 122.48 \text{ s} = \mathbf{2 \text{ min } 2.48 \text{ s.} \quad \text{Ans.}}$$

Problem 10.15. A wheel is rotating about its axis with a constant angular acceleration of 1 rad/s^2 . If the initial and final angular velocities are 5.25 rad/s and 10.5 rad/s , determine the total angle turned through during the time interval this change of angular velocity took place.

Sol. Given :

Constant angular acceleration, $\alpha = 1 \text{ rad/s}^2$

Initial angular velocity, $\omega_0 = 5.25 \text{ rad/s}$

Final angular velocity, $\omega = 10.50 \text{ rad/s}$

Let the total angle turned $= \theta$

Using equation (10.17),

$$\omega^2 - \omega_0^2 = 2\alpha\theta \quad \text{or} \quad 13.50^2 - 5.25^2 = 2 \times 1 \times \theta$$

or $110.25 - 27.56 = 2\theta$

$$\therefore \theta = \frac{(110.25 - 27.56)}{2} = \frac{82.69}{2} = 41.35 \text{ rad. Ans.}$$

Problem 10.16. (a) A flywheel starts rotating from rest and is given an acceleration of 1 rad/s^2 . Find the angular velocity and speed in r.p.m. after 1.5 minutes.

(b) If the flywheel is brought to rest with a uniform angular retardation of 0.5 rad/s^2 , determine the time taken by the flywheel in seconds to come to rest.

Sol. Given :

(a) Initial angular velocity, $\omega_0 = 0$

Angular acceleration, $\alpha = 1 \text{ rad/s}^2$

Time, $t = 1.5 \text{ min} = 1.5 \times 60 = 90 \text{ s}$

Let final angular velocity, $= \omega$.

Using equation (10.15), $\omega = \omega_0 + \alpha t$
 $= 0 + 1 \times 90 = 90 \text{ rad/s. Ans.}$

Speed in r.p.m. is given by equation (10.18) as, $\omega = \frac{2\pi N}{60}$

$$\therefore N = \frac{60\omega}{2\pi} = \frac{60 \times 90}{2\pi} = 859.44 \text{ r.p.m. Ans.}$$

(b) In this case, we have

Initial angular velocity, $\omega_0 = 90 \text{ rad/s}$

Final angular velocity, $\omega = 0$

Angular retardation, $\alpha = 0.5 \text{ rad/s}^2$

$$\alpha = -0.5 \text{ rad/s}^2$$

Let the time taken by the flywheel in coming to rest $= t_1$.

Using equation (10.15), $\omega = \omega_0 + \alpha t$

$$\text{or} \quad 0 = 90 + (-0.5) \times t_1 \quad (\because \alpha = -0.5 \text{ rad/s}^2)$$

$$= 90 - 0.5 \times t_1$$

$$\therefore t_1 = \frac{90}{0.5} = 180 \text{ s} = 3 \text{ min. Ans.}$$

Problem 10.17. A wheel rotates for 5 seconds with constant angular acceleration and describes 100 radians during this time. It then rotates with constant angular velocity and during the next 5 seconds describes 80 radians. Find the initial angular velocity and the angular acceleration.
 (U.P. Tech. University, May 2009)

Sol. Given :

Time, $t_1 = 5 \text{ seconds}$

Angular displacement, $\theta_1 = 100 \text{ radians}$

Let $\omega_0 =$ Initial angular velocity and

$\alpha =$ Angular acceleration

Using $\theta = \omega_0 \times t + \frac{1}{2} \alpha t^2$, we get

$$100 = \omega_0 \times 5 + \frac{1}{2} \alpha \times 5^2 \quad (\because t = t_1 = 5 \text{ sec and } \theta = \theta_1 = 100 \text{ radians})$$

or $100 = 5\omega_0 + 12.5 \times \alpha$ [Dividing by 5]

or $20 = \omega_0 + 2.5 \times \alpha$... (i)

Let ω = Angular velocity after 5 seconds

The wheel rotates with constant angular velocity (*i.e.*, with angular velocity ω) for next 5 seconds and describes 80 radians.

Now using $\theta = \omega \times t$ [Angular velocity is constant]

$\therefore 80 = \omega \times 5$

$\therefore \omega = \frac{80}{5} = 16 \text{ rad/s}$

For the first five seconds, $\omega = \omega_0 + \alpha t$

or $16 = \omega_0 + \alpha \times 5$ [$\because \omega = 16, t = 5 \text{ sec}$]

or $16 = \omega_0 + 5\alpha$... (ii)

Subtracting equation (ii) from (i), we get

$$20 - 16 = (\omega_0 + 2.5\alpha) - (\omega_0 + 5\alpha)$$

or $4 = -2.5\alpha$

or $\alpha = \frac{4}{-2.5} = -1.6 \text{ rad/s}^2$. **Ans.**

Substituting the value of α in equation (ii), we get

$$16 = \omega_0 + 5(-1.6)$$

$$= \omega_0 - 8$$

$\therefore \omega_0 = 16 + 8 = 24 \text{ rad/s}$. **Ans.**

Problem 10.18. The rotor of an electric motor uniformly accelerates to a speed of 1800 rpm from rest in 5 seconds and then the rotor decelerates uniformly to stop. If the total time elapsed from start to stop is 12.3 seconds, determine the number of revolutions made while acceleration and deceleration. What would be the reason for the longer time taken during stopping. (U.P. Tech. University, 2002–2003)

Sol. Given :

1st Case. Initial velocity, $\omega_0 = 0$

Final speed, $N = 1800 \text{ r.p.m.}$

Time = 5 seconds

$\therefore$ Final angular velocity $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60} = 60\pi \text{ rad/s}$.

Let us first find the angular acceleration (α).

Using relation, $\omega = \omega_0 + \alpha \times t$

or $60\pi = 0 + \alpha \times 5$

$\therefore \alpha = \frac{60\pi}{5} = 12\pi \text{ rad/s}^2$

To find the number of revolutions made by motor during acceleration find the angular displacement (θ).

$$\begin{aligned}\text{Using equation (10.16), } \theta &= \omega_0 \times t + \frac{1}{2} \alpha t^2 \\ &= 0 \times 5 + \frac{1}{2} \times 12\pi \times 5^2\end{aligned}$$

or

$$\theta = 150\pi$$

$\therefore$ Number of revolution during acceleration

$$\begin{aligned}&= \frac{\text{Total angular displacement}}{\text{Angle covered in one revolution}} \\ &= \frac{150\pi}{2\pi} = 75 \quad \dots(i)\end{aligned}$$

2nd Case. Total time from start to stop = 12.3 seconds

Initial angular velocity, $\omega_0 = 60\pi$ rad/s

Final angular velocity, $\omega = 0$

Time required during deceleration

$$\begin{aligned}&= \text{Total time} - 5 \\ &= 12.3 - 5 = 7.3 \text{ seconds}\end{aligned}$$

Let us find the angular deceleration.

Using, $\omega = \omega_0 + \alpha t$

or

$$0 = 60\pi + \alpha \times 7.3$$

$$\therefore \alpha = -\frac{60\pi}{2\pi} \text{ rad/s}^2 = -25.82 \text{ rad/s}^2$$

To find the number of revolutions made by motor during deceleration, find the angular displacement (θ).

$$\begin{aligned}\text{Using } \omega^2 - \omega_0^2 &= 2\alpha\theta & [\text{Eqn. (10.17)}] \\ 0^2 - (60\pi)^2 &= 2(-25.82) \times \theta\end{aligned}$$

or

$$\theta = \frac{-(60\pi)^2}{-2 \times 25.82} = 688 \text{ rads}$$

$\therefore$ Number of revolutions during deceleration

$$\begin{aligned}&= \frac{\text{Angular displacement}}{\text{Angular displacement in one revolution}} \\ &= \frac{688}{2\pi} = 109.5\end{aligned}$$

$\therefore$ Total number of revolutions

$$= 75 + 109.5 = \mathbf{184.5. \text{ Ans.}}$$

Due to inertia force, the rotor continues to rotate and hence the rotor takes longer time during stopping.

Problem 10.19. A wheel rotating about a fixed axis at 20 r.p.m. is uniformly accelerated for 70 second during which time it makes 50 revolutions. Find :

(i) angular velocity at the end of this interval, and

(ii) time required for speed to reach 100 revolutions per minute.

(U.P. Tech. University, 2000–2001)

Sol. Given :

Initial speed, $N_0 = 20$ r.p.m.

$$\therefore \text{Initial angular velocity, } \omega_0 = \frac{2\pi N}{60} = \frac{2\pi \times 20}{60} = 0.67\pi \text{ rad/s}$$

$$\text{Time, } t = 70 \text{ seconds}$$

$$\text{Angular displacement, } \theta = 50 \text{ revolution} = 50 \times 2\pi = 100\pi \text{ radians}$$

$$(\because \text{One revolution} = 2\pi \text{ radians})$$

(i) *Angular velocity at the end of 70 seconds*

Let ω = Angular velocity after 70 seconds

α = Angular acceleration.

$$\text{We know, } \theta = \omega_0 \times t + \frac{1}{2} \alpha t^2$$

$$\therefore 100\pi = 0.67\pi \times 70 + \frac{1}{2} \alpha \times 70^2 \quad (\because \theta = 100\pi, \omega_0 = 0.67\pi, t = 70)$$

$$\therefore 100\pi = 46.9\pi + 2450\alpha$$

$$\therefore \alpha = \frac{100\pi - 46.9\pi}{2450} = \frac{53.1 \times \pi}{2450} = 0.068 \text{ rad/s}^2$$

$$\text{Now, using relation, } \omega = \omega_0 + \alpha t$$

$$= 0.67\pi + 0.068 \times 70 = \mathbf{6.86 \text{ rad/s. Ans.}}$$

(ii) *Time required for speed to reach 100 revolution per minute.*

$$\text{Let } t = \text{time required}$$

$$\text{Final speed, } N = 100 \text{ r.p.m.}$$

$$\text{Final angular velocity, } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 100}{60} = \frac{10\pi}{3} \text{ rad/s}$$

$$\text{Using, } \omega = \omega_0 + \alpha t$$

$$\text{or } \frac{10\pi}{3} = 0.67\pi + 0.068 \times t$$

$$\therefore t = \frac{\left(\frac{10\pi}{3} - 0.67\pi\right)}{0.068} = \mathbf{123 \text{ seconds. Ans.}}$$

Problem 10.20. *The angle of rotation of a body is given by the equation*

$$\theta = 2t^3 - 5t^2 + 8t + 6$$

where θ is expressed in radians and t in seconds. Determine :

(i) *angular velocity and*

(ii) *angular acceleration of the body*

when $t = 0$ and $t = 4$ seconds.

Sol. Given :

$$\text{Angular displacement } \theta = 2t^3 - 5t^2 + 8t + 6.$$

Angular velocity (ω) is obtained differentiating the above equation with respect to time (t).

$$\therefore \omega = \frac{d\theta}{dt} = \frac{d}{dt} (2t^3 - 5t^2 + 8t + 6) = 6t^2 - 10t + 8 \quad \dots(i)$$

Angular acceleration (α) is obtained by differentiating the equation (i) with respect to time (t).

$$\therefore \alpha = \frac{d\omega}{dt} = \frac{d}{dt} (6t^2 - 10t + 8) = 12t - 10 \quad \dots(ii)$$

When $t = 0$

Angular velocity (ω) and angular acceleration (α), when $t = 0$, are obtained by substituting $t = 0$ in equations (i) and (ii) respectively.

$$\therefore \quad \omega = 6 \times 0 - 10 \times 0 + 8 = 0 - 0 + 8 = \mathbf{8 \text{ radian/s.} \quad \text{Ans.}}$$

and $\alpha = 12 \times 0 - 10 = -\mathbf{10 \text{ radian/s}^2. \quad \text{Ans.}}$

When $t = 4$ seconds

Substituting $t = 4$ seconds in equations (i) and (ii), we get

$$\omega = 6 \times 4^2 - 10 \times 4 + 8 = 96 - 40 + 8 = \mathbf{64 \text{ radian/s.} \quad \text{Ans.}}$$

and $\alpha = 12 \times 4 - 10 = 48 - 10 = \mathbf{38 \text{ radian/s}^2. \quad \text{Ans.}}$

Problem 10.21. *The initial angular velocity of a rotating body is 2 rad/s and initial angular acceleration is zero. The rotation of the body is according to the relation $\alpha = 3t^2 - 3$. Find : (i) angular velocity and (ii) angular displacement when $t = 5$ seconds. Consider the angular displacement in radians and time in seconds.*

Sol. Given :

Initial angular velocity, $\omega_0 = 2 \text{ rad/s}$

This means that when $t = 0$, the angular velocity is 2 rad/s

Initial angular acceleration, $\alpha = 0$

The law of rotation, $\alpha = 3t^2 - 3$.

Find : (i) angular velocity (ω) and (ii) angular displacement θ when $t = 5$ s.

(i) *Angular velocity* (ω)

We know, $\alpha = \frac{d\omega}{dt} \quad \text{or} \quad d\omega = \alpha dt$
 $= (3t^2 - 3) dt \quad (\because \alpha = 3t^2 - 3)$

Integrating, we get

$$\int d\omega = \int (3t^2 - 3) dt \quad \text{or} \quad \omega = \frac{3t^3}{3} - 3t + C \quad \dots(i)$$

where C is a constant of integration. Its value is obtained from the given condition i.e., $\omega = 2 \text{ rad/s}$ when $t = 0$.

Substituting these values in equation (i), we get

$$2 = 0 - 0 + C \quad \text{or} \quad C = 2$$

Substituting the value of C in equation (i),

$$\omega = t^3 - 3t + 2$$

When $t = 5$ s, angular velocity becomes

$$\omega = 5^3 - 3 \times 5 + 2 = 125 - 15 + 2 = \mathbf{112 \text{ rad/s.} \quad \text{Ans.}}$$

(ii) *Angular displacement* (θ)

We know that $\omega = \frac{d\theta}{dt} \quad \text{or} \quad d\theta = \omega dt$
 $= (t^3 - 3t + 2) dt \quad (\because \omega = t^3 - 3t + 2)$

Integrating, we get

$$\int d\theta = \int (t^3 - 3t + 2) dt$$

$$\text{or} \quad \theta = \frac{t^4}{4} - \frac{3t^2}{2} + 2t + C_1 \quad \dots(ii)$$

where C_1 is a constant of integration. Its value is obtained from the condition, $\theta = 0$ when $t = 0$.

Substituting these values in equation (ii), we get

$$0 = 0 - 0 + 0 + C_1 \quad \text{or} \quad C_1 = 0$$

Now equation (ii) becomes as

$$\theta = \frac{t^4}{4} - \frac{3t^2}{2} + 2t$$

When $t = 5$ seconds, the angular displacement becomes as

$$\begin{aligned} \theta &= \frac{5^4}{4} - \frac{3 \times 5^2}{2} + 2 \times 5 \\ &= 156.25 - 37.5 + 10 = \mathbf{128.75 \text{ radians.} \quad \text{Ans.}} \end{aligned}$$

Problem 10.22. The angle of rotation of a body is given as a function of time by the equation,

$$\theta = \theta_0 + at + bt^2$$

where θ_0 initial angular displacement, a and b are constants.

Obtain general expressions for : (a) the angular velocity and (b) the angular acceleration of the body.

If the initial angular velocity be 3π radian per second and after two seconds the angular velocity is 8π radian per second, determine the constants a and b .

Sol. Given :

$$\theta = \theta_0 + at + bt^2.$$

Angular velocity (ω) is obtained by differentiating the above equation with respect to time (t).

$$\begin{aligned} \therefore \quad \omega &= \frac{d\theta}{dt} = \frac{d}{dt} (\theta_0 + at + bt^2) = a + 2bt \quad \dots(i) \\ &\left(\because \theta_0 \text{ is constant and hence } \frac{d\theta_0}{dt} = 0 \right) \end{aligned}$$

(i) When $t = 0$, $\omega = 3\pi$

Substituting these values in equation (i), we get

$$3\pi = a + 2b \times 0 = a$$

$$\therefore \quad a = 3\pi. \quad \text{Ans.}$$

(ii) When $t = 2$ seconds, $\omega = 8\pi$ radians.

Substituting these values in $\omega = a + 2bt$, we get

$$8\pi = a + 2b \times 2 = a + 4b = 3\pi + 4b \quad (\because a = 3\pi)$$

$$\therefore \quad 4b = 8\pi - 3\pi = 5\pi$$

$$\text{or} \quad b = \frac{5\pi}{4} = \mathbf{1.25\pi. \quad \text{Ans.}}$$

(a) **General expression for angular velocity**

General expression for angular velocity is obtained by substituting the values of a and b in equation (i).

$$\begin{aligned} \therefore \quad \omega &= a + 2bt \\ &= 3\pi + 2 \times 1.25\pi \times t \quad (\because a = 3\pi, b = 1.25\pi) \\ &= \mathbf{3\pi + 2.5\pi t. \quad \text{Ans.}} \end{aligned}$$

(b) **General expression for angular acceleration**

Angular acceleration (α) is obtained by differentiating the equation (i) with respect to time (t).

$$\begin{aligned}
 \therefore \alpha &= \frac{d\omega}{dt} = \frac{d}{dt} (a + 2bt) = 2b & \left(\because a \text{ is constant, hence } \frac{da}{dt} = 0 \right) \\
 &= 2 \times 1.25\pi & (\because b = 1.25\pi) \\
 &= 2.50\pi. \quad \text{Ans.}
 \end{aligned}$$

10.6. x – y AND n – t COMPONENTS

Fig. 10.3 shows a particle P moving along a curvilinear path having acceleration ' a '. The acceleration of the particle can be resolved into two components mutually perpendicular to each other. The components of the acceleration along x -axis and y -axis are known as x - y components of acceleration.

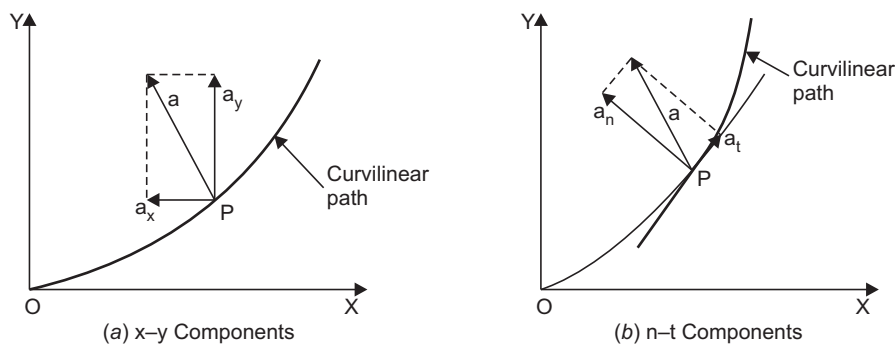


Fig. 10.3

Here a_x is component along x -axis and a_y along y -axis. Fig. 10.3 (a) shows x - y components. x - y components are also known as rectangular components.

Similarly the acceleration a can be resolved along the direction of tangent and normal to the curve as shown in Fig. 10.3 (b). The component along tangent direction is a_t whereas normal to the curve the component is a_n . n - t components are also known as normal and tangential components.

The equations of motion for x - y components are :

Net force in x -direction = Mass $\times$ acceleration in x -direction

$$\text{or} \quad \Sigma F_x = m \times a_x$$

Similarly net force in y -direction = Mass $\times$ Acceleration in y -direction

$$\text{or} \quad \Sigma F_y = m \times a_y$$

Equations of motion for n - t components are :

Net force in tangent direction = Mass $\times$ Acceleration in tangent direction

$$\text{or} \quad \Sigma F_t = m \times a_t$$

Similarly, net force in normal direction = Mass $\times$ Acceleration in normal direction

$$\text{or} \quad \Sigma F_n = m \times a_n$$

For a particle moving with a velocity v in a curved path of radius r , the components of acceleration along tangent direction and normal to the curve are given as

$$a_t = \frac{dv}{dt} \quad \text{and} \quad a_n = \frac{v^2}{r} \quad [\text{Refer to Art. 10.6.1}]$$

10.6.1. Normal Acceleration and Tangential Acceleration. Consider a particle moving along a circular path as shown in Fig. 10.4 (a). The velocity of the particle will be in the direction of displacement. As the displacement at any instant is in tangential direction, hence the velocity will also be tangential to the path of the particle. In Fig. 10.4 (a) the displacement takes place along a circular path XYZ, hence the velocity of the particle at the instant it occupies positions X, Y and Z will be in the directions given by the tangents XA, YB and ZC respectively.

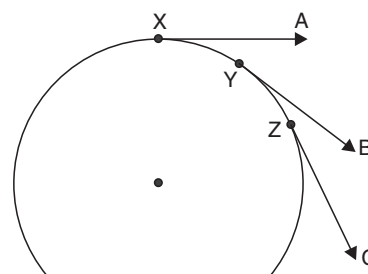
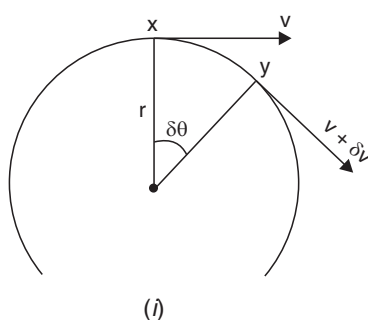


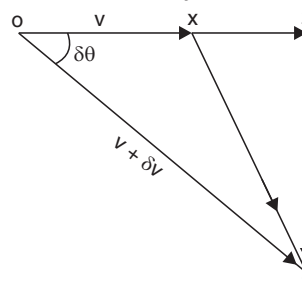
Fig. 10.4 (a)

Again consider another particle moving along a circular path as shown in Fig. 10.4 (b)(i). Let x and y are the two positions of the particle along the circular path.

Let v = Velocity of particle at position x ,
 $v + \delta v$ = Velocity of particle at position y ,
 r = Radius of circular path, and
 δt = Time required by the particle to move from x to y .



(i)



(ii)

Fig. 10.4 (b)

To find the change in velocity, as the particle moves from x to y , a vector triangle oxy is drawn as shown in Fig. 10.4 (b) (ii) in which ox is equal to v in magnitude and direction (*i.e.* velocity at x) and oy is equal to $v + \delta v$ in magnitude and direction (*i.e.*, velocity at y). In Fig. 10.4 (b)(ii) the vector xy represents the change of velocity in time δt . The vector xy is resolved in two components xz and zy . The component xz is parallel to ox whereas the component zy is $\perp$ to oz .

Now $xz = oz - ox = oy \cos \delta\theta - v = (v + \delta v) \cos \delta\theta - v$

and $zy = oy \sin \delta\theta = (v + \delta v) \sin \delta\theta$.

Now, the components acceleration of the particle in tangential and normal direction can be obtained.

Let a_t = Component acceleration in tangential direction *i.e.*, in the direction parallel to ox .

a_n = Component acceleration in normal direction *i.e.*, in the direction normal to ox .

This component is also known as centripetal component of acceleration.

a_t = Rate of change of velocity in tangential direction

$$= \frac{\text{Change of velocity in tangential direction}}{\text{Time}}$$

$$= \frac{xz}{\delta t} = \frac{(v + \delta v) \cos \delta\theta - v}{\delta t}$$

In the limit as δt approaches zero, $\cos \delta\theta \rightarrow 1$ and the above equation becomes as

$$a_t = \frac{dv}{dt} = \frac{d}{dt}(\omega \times r) = r \frac{d\omega}{dt} = r \times \alpha \quad \dots(10.18A) \quad \left(\because \frac{d\omega}{dt} = \alpha \right)$$

Similarly, the component acceleration in normal direction (*i.e.*, centripetal acceleration) is obtained as

$$\begin{aligned} a_n &= \text{Rate of change of velocity in normal direction} \\ &= \frac{\text{Change of velocity in normal direction}}{\text{Time}} \\ &= \frac{zy}{\delta t} = \frac{(v + \delta v) \sin \delta\theta}{\delta t} \end{aligned}$$

In the limit as δt approaches zero, $\sin \delta\theta \rightarrow \delta\theta$ and the above equation becomes as

$$\begin{aligned} a_n &= \frac{(v + dv) d\theta}{dt} = v \frac{d\theta}{dt} + \frac{dv \times d\theta}{dt} \\ &= v \frac{d\theta}{dt} \quad (\text{the product of two small terms i.e., } dv \text{ and } d\theta \text{ are negligible}) \\ &= v \times \omega \quad \left(\because \frac{d\theta}{dt} = \omega \right) \\ &= \frac{v^2}{r} \quad \text{or} \quad \omega^2 r \quad \dots(10.18B) \quad (\because v = \omega \times r) \end{aligned}$$

From equation (10.18A), it is clear that tangential component of acceleration is equal to the rate of change of the magnitude of the velocity of the particle whereas from equation (10.18B), it is clear that the normal acceleration (or radial acceleration or centripetal acceleration) depends upon the velocity and radius of the circular path.

After knowing the tangential acceleration and normal acceleration, the total acceleration (a) is obtained as

$$\text{Total acceleration, } a = \sqrt{a_t^2 + a_n^2} \quad \dots(10.18C)$$

Note. (i) For the displacement along a straight path, the radius of the straight path is infinitely great and hence the normal component $\left(a_n = \frac{v^2}{r} \right)$ will be zero. There will be only tangential component of acceleration $\left(i.e., a_t = \frac{dv}{dt} \right)$. The direction of tangential component will be same as the direction of displacement and of the velocity.

(ii) For the displacement along a circular path with constant speed, the tangential component of acceleration $\left(i.e., a_t = \frac{dv}{dt} \right)$ will be zero. There will be only normal component of acceleration $\left(i.e., a_n = \frac{v^2}{r} \right)$. The direction of the centripetal acceleration will be normal to the velocity and the displacement. As velocity and displacement are normal to the radius of the circular path, hence direction of centripetal acceleration will be along the radius towards the centre.

Problem 10.23. A train starts from rest and moves along a curved track of radius 800 m with uniform acceleration until it attains a velocity of 72 km/hr at the end of third minute. Determine the tangential, normal and total acceleration in m/s^2 of the train at the end of second minute. (U.P. Tech. University, 2000–2001)

Sol. Given :

Train starts from rest, $\therefore u = \omega_0 = 0$

Radius, $r = 800$ m

Speed after 3 minutes = 72 km/hr

$\therefore$ Velocity after 3 minutes $= \frac{72 \times 1000}{60 \times 60}$ m/s = 20 m/s.

Find : Tangential acceleration, normal acceleration and total acceleration after 2 minutes (*i.e.*, 120 seconds).

The train is moving with uniform acceleration, hence the relation $\omega = \omega_0 + \alpha t$ can be used.

Here ω = Angular velocity after 3 minutes (*i.e.*, after $3 \times 60 = 180$ seconds)

α = Angular acceleration

ω_0 = Initial angular velocity = 0

t = Time in seconds *i.e.*, 180 seconds

But angular velocity after 3 minutes

$$= \frac{\text{Velocity after 3 minutes}}{\text{Radius}} \quad \left(\because \omega = \frac{V}{r} \right)$$

$$\therefore \omega = \frac{20}{800} \quad (\because V = 20 \text{ m/s}, r = 800 \text{ m})$$

$$= \frac{1}{40} \text{ rad/s}$$

Now using, $\omega = \omega_0 + \alpha t$

$$\frac{1}{40} = 0 + \alpha \times 180 \quad (\because t = 180 \text{ seconds})$$

$$\therefore \alpha = \frac{1}{40} \times \frac{1}{180} = \frac{1}{40 \times 180} \text{ rad/s}^2$$

As the train is moving with uniform acceleration, the angular acceleration will be same. To find the tangential acceleration, normal acceleration and total acceleration after 2 minutes (*i.e.*, after 120 seconds), let us first find the angular velocity after 120 seconds.

Again using $\omega = \omega_0 + \alpha t$

$$= 0 + \frac{1}{40 \times 180} \times 120 \quad \left(\because \alpha = \frac{1}{40 \times 180} \right)$$

$$= \frac{1}{60} \text{ rad/s.}$$

(i) Now tangential acceleration (a_t) is given by equation (10.18A) as

$$a_t = r \times \alpha = 800 \times \frac{1}{40 \times 180} = \mathbf{0.111 \text{ m/s}^2. \text{ Ans.}}$$

(ii) Normal acceleration (a_n) is given by equation (10.18B) as

$$a_n = \omega^2 \times r$$

$$= \left(\frac{1}{60} \right)^2 \times 800 = \frac{1}{60 \times 60} \times 800 = \mathbf{0.222 \text{ m/s}^2. \text{ Ans.}}$$

(iii) Total acceleration (a) is given by equation (10.18C) as

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0.111^2 + 0.222^2} = \mathbf{0.246 \text{ m/s}^2. \text{ Ans.}}$$

10.7. RELATIVE MOTION

The motion between two moving bodies is known as relative motion.

The motion of a moving body with respect to another moving body, is known as the relative motion of the first body with respect to second body.

10.7.1. Relative Motion between Two Bodies Moving in Straight Lines

Fig. 10.5 (a) shows the two bodies A and B which are moving along parallel lines in the same direction.

Let v_A = Absolute velocity of A

v_B = Absolute velocity of B

Let $v_A > v_B$. To find the relative velocity of A with respect to B, make the body B stationary. This is possible if a velocity equal to v_B is applied on the body B in the opposite direction as shown in Fig. 10.5 (b). Then we shall have to apply a velocity equal to v_B on the body A also in the opposite direction. The resultant of these two velocities acting on A will represent the relative velocity of A with respect to B. As $v_A > v_B$, hence the resultant velocity will be in the direction of v_A and equal to $(v_A - v_B)$. This resultant velocity is the relative velocity of A with respect to B.

The relative velocity of A with respect to B in vector form is also given by,

$$\begin{aligned} v_{AB} &= \text{Vector difference of } v_A \text{ and } v_B \\ &= \vec{v}_A - \vec{v}_B \quad \dots(i) \end{aligned}$$

Graphically, the relative velocity of A with respect to B can also be determined as [Refer to Fig. 10.5 (c)] given below :

(i) Take any point o . From o , draw oa parallel to velocity v_A and take $oa = v_A$ to some suitable scale.

(ii) From o , also draw ob parallel to velocity v_B . Take $ob = v_B$. Join b to a .

Then ba represents the relative velocity of A with respect to body B in magnitude and direction. Hence relative velocity of A with respect to B may also be written in the vector form. [Refer to Fig. 10.5 (c)].

$$\vec{ba} = \vec{oa} - \vec{ob}.$$

Please note that the velocity of A with respect to B (i.e., v_{AB}) in vector form is written as $\vec{ba}$ and not $\vec{ab}$. And mathematically it is written as v_{AB} . Hence to find v_{AB} in magnitude and direction, start from point b and reach towards point a .

Similarly the relative velocity of B with respect to A is given by

$$\begin{aligned} v_{BA} &= \text{Vector difference of } v_B \text{ and } v_A \\ &= \vec{v}_B - \vec{v}_A \quad \dots(ii) \end{aligned}$$

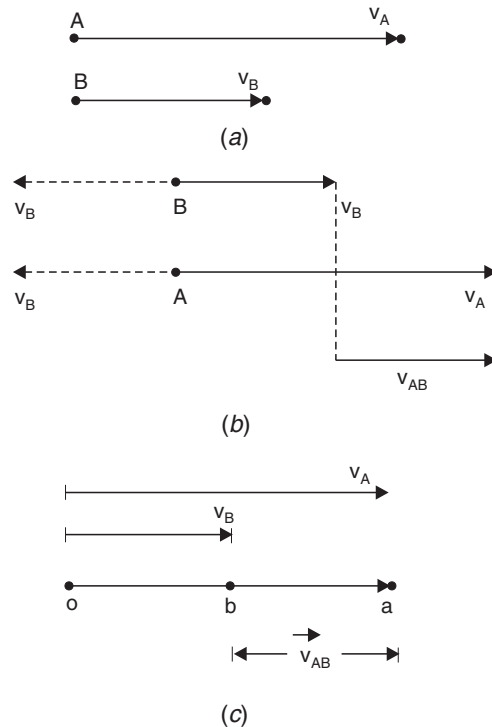


Fig. 10.5

The relative velocity of B with respect to A in vector form is written as

$$\vec{ab} = \vec{ob} - \vec{oa}.$$

From equations (i) and (ii), it is clear that the relative velocity of point A with respect to B (v_{AB}) and the relative velocity of point B with respect to A (v_{BA}) are equal in magnitude but opposite in direction *i.e.*,

$$v_{AB} = -v_{BA} \quad \text{or} \quad \vec{ba} = -\vec{ab}.$$

10.7.2. Relative Motion between Two Bodies Moving Along Inclined Lines

Fig. 10.6 (a) shows the two bodies A and B moving along inclined lines with absolute velocities v_A and v_B respectively. To find the relative velocity of A with respect to B , make the body B stationary. This is possible if a velocity equal to v_B is applied on the body B in the opposite direction as shown in Fig. 10.6 (b). Then we shall have to apply a velocity equal to v_B on the body A also in opposite direction.

The resultant gives the relative velocity of A with respect to B as shown in Fig. 10.6 (b).

The vector oa in Fig. 10.6 (c) represents the velocity v_A in magnitude and direction, whereas the vector ob represents the velocity v_B in magnitude and direction. Join points b and a . Then relative velocity of A with respect to B is written as v_{AB} and is given by

$$v_{AB} = \text{Vector } ba = \vec{v_A} - \vec{v_B}$$

and relative velocity of B with respect to A is given by,

$$v_{BA} = \text{Vector } ab.$$

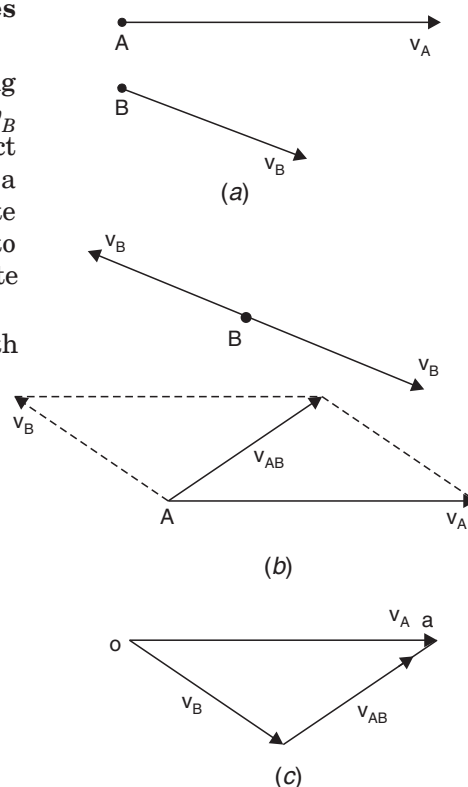


Fig. 10.6

Problem 10.24. Two roads which are at an angle of 60° , intersect at a point O . A car 1 is moving with a velocity of 72 km/hr away from point O along x -axis at an instant of time. At the same time another car 2 is moving away from point O with a velocity of 48 km/hr along the road 2.

Determine the velocity of car 1 with respect to car 2.

Sol. Given :

Velocity of car 1 along x -axis = 72 km/hr

$$= \frac{72 \times 1000}{60 \times 60} = 20 \text{ m/s}$$

The car 1 is moving along x -axis, there is no component of velocity along y -axis.

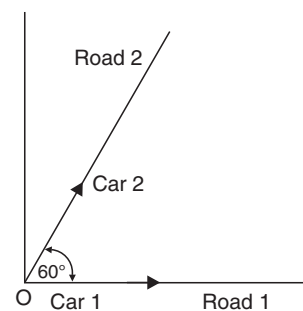


Fig. 10.7

Hence, the velocity of car 1 in vector form is written as

$$\mathbf{V}_1 = 20\mathbf{i} \quad (\text{As velocity is along } x\text{-axis only, hence only } i \text{ component exists})$$

Velocity of car 2 along road 2, which is at an angle of 60° with x -axis = 48 km/hr.

$$= \frac{48 \times 1000}{60 \times 60} = 13.33 \text{ m/s}$$

The components of this velocity along x -axis = $13.33 \cos 60^\circ = 6.66 \text{ m/s}$ and along y -axis = $13.33 \sin 60^\circ = 11.54 \text{ m/s}$.

Hence, the velocity of car 2 in vector form is written as

$$\begin{aligned} \mathbf{V}_2 &= (13.33 \cos 60^\circ)\mathbf{i} + (13.33 \sin 60^\circ)\mathbf{j} \\ &= 6.66\mathbf{i} + 11.54\mathbf{j} \end{aligned}$$

Now the velocity of car 1 with respect to car 2 is given by

$$\begin{aligned} \mathbf{V}_{12} &= \mathbf{V}_1 - \mathbf{V}_2 = (20\mathbf{i}) - (6.66\mathbf{i} + 11.54\mathbf{j}) \\ &= (20 - 6.66)\mathbf{i} - 11.54\mathbf{j} = \mathbf{13.34i} - \mathbf{11.54j}. \quad \text{Ans.} \end{aligned}$$

The velocity of car 1 with respect to car 2 is also known as the relative velocity of car 1 with respect to car 2.

$$\begin{aligned} \text{The magnitude of this relative velocity} &= \sqrt{13.34^2 + 11.54^2} = \sqrt{177.95 + 133.17} \\ &= \mathbf{17.64 \text{ m/s.} \quad \text{Ans.}} \end{aligned}$$

The direction of this relative velocity is given by,

$$\tan \theta = \frac{-11.54 \text{ (y-axis component)}}{13.34 \text{ (x-axis component)}} = -0.865$$

As $\tan \theta$ is negative, hence θ should be in second or fourth quadrant. If θ is in second quadrant, then x -component should be $-ve$ and y -component should be positive. But here y -component is negative, this means θ lies in fourth quadrant.

$$\therefore \theta = \tan^{-1}(-0.865) = 40.86^\circ \quad \text{or} \quad (360 - 40.86) = \mathbf{319.14^\circ. \quad \text{Ans.}}$$

Alternate method (Graphical Method)

(i) Take any point o . From o draw a line oa parallel to the velocity of car 1 or V_1 in magnitude and direction. Cut $oa = 20 \text{ m/s}$.

(ii) From o , also draw line ob parallel to the velocity of car 2 or V_2 in magnitude and direction. Cut $ob = 13.33 \text{ m/s}$. Join b to a . Then vector ba is the relative velocity of car 1 with respect to car 2.

By measurement vector $ba = 17.6 \text{ m/s}$ and angle $\alpha = 40.86^\circ$.

$$\therefore \theta = 360 - 40.86 = \mathbf{319.14^\circ. \quad \text{Ans.}}$$

Note. For graphical method both the velocities should be directed either away from the origin (point o) or towards the origin.

Problem 10.25. If in the previous problem, the car 1 is at a distance of 100 m from O along x -axis and is moving with a speed of 72 km/hr towards O at the time of observation and car 2 is moving with a speed of 48 km/hr. along road 2 and away from the origin as shown in Fig. 10.8, then find the velocity of car 1 with respect to car 2.

Sol. Given :

Distance of car 1 from O along x -axis = 100 m

$$\text{Speed of car 1} = 72 \text{ km/hr} = \frac{72 \times 1000}{60 \times 60} = 20 \text{ m/s}$$

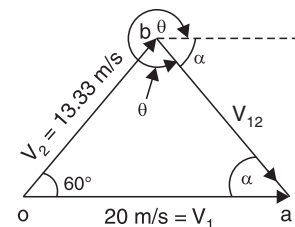


Fig. 10.7 (a)

∴ Velocity of car 1 = 20 m/s

The car 1 is moving along x -axis and towards origin.

Hence, the velocity of car 1 in vector form is written as

$$\mathbf{V}_1 = -20\mathbf{i} \quad [\text{-ve sign is due to opposite direction of car 1}]$$

$$\text{Velocity of car 2} = 48 \text{ km/hr} = \frac{48 \times 1000}{60 \times 60} = 13.33 \text{ m/s}$$

The velocity of car 2 in vector form is written as

$$\mathbf{V}_2 = 13.33 [\cos 60^\circ \mathbf{i} + \sin 60^\circ \mathbf{j}] = 6.66\mathbf{i} + 11.54\mathbf{j}$$

The velocity of car 1 with respect to car 2 is given by,

$$\begin{aligned} \mathbf{V}_{12} &= \mathbf{V}_1 - \mathbf{V}_2 = -20\mathbf{i} - [6.66\mathbf{i} + 11.54\mathbf{j}] \\ &= -26.66\mathbf{i} - 11.54\mathbf{j}. \quad \text{Ans.} \end{aligned}$$

Graphical Method

(i) Take any point o . From o draw a line oa parallel to velocity V_1 in magnitude and direction. Take $oa = 20$ m/s.

(ii) From o , also draw a line ob parallel to velocity V_2 in magnitude and direction. Take $ob = 13.33$ m/s. Now, vector ba represents the velocity of 1 with respect to 2 in magnitude and direction. Measure vector ba .

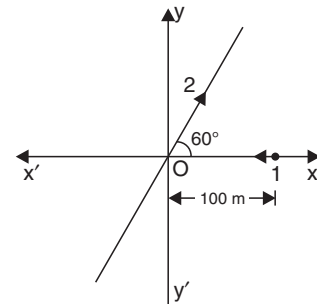


Fig. 10.8

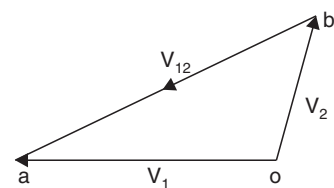


Fig. 10.8 (a)

Problem 10.26. At an instant of time, a car 1 at a distance of 120 m from point o along x -axis moving at a velocity of 72 km/hr decelerates at 1.5 m/s^2 as it approaches the point o . At the same instant, a car 2, is moving along a road which is at an angle of 60° with the road on which car 1 is moving. This car is moving away from o with a velocity of 48 km/hr and accelerates at 2 m/s^2 as shown in Fig. 10.9. Find the velocity of car 1 with respect to car 2 after three seconds.

Sol. Given :

$$\text{Car 1. Initial velocity, } u_1 = 72 \text{ km/hr} = \frac{72 \times 1000}{60 \times 60} = 20 \text{ m/s}$$

$$= -20\mathbf{i} \text{ m/s (in vector notation)}$$

(-ve sign is due to car 1 is moving towards o along x -axis)

$$\text{Deceleration} = 1.5 \text{ m/s}^2 \quad (\text{towards } o)$$

$$\text{or Acceleration, } a = 1.5 \text{ m/s}^2 \quad (\text{away from } o \text{ along } x\text{-axis})$$

$$= 1.5\mathbf{i} \text{ m/s}^2 \quad (\text{in vector notation})$$

(+ve sign is as the acceleration is acting along x -axis away from o)

$$\text{Time, } t = 3 \text{ seconds}$$

$$\text{Let } v_1 = \text{Final velocity of car 1 after 3 seconds}$$

$$\begin{aligned} \text{Then } v_1 &= u_1 + at = (-20\mathbf{i}) + (1.5\mathbf{i}) \times 3 \\ & \quad (\because u_1 = -20\mathbf{i}; a_1 = 1.5\mathbf{i}) \\ &= -20\mathbf{i} + 4.5\mathbf{i} = -15.5\mathbf{i} \end{aligned}$$

$$\begin{aligned} \therefore \mathbf{V}_1 &= -15.5\mathbf{i} \\ & \quad [\text{Final velocity of car 1, in vector notation}] \end{aligned}$$

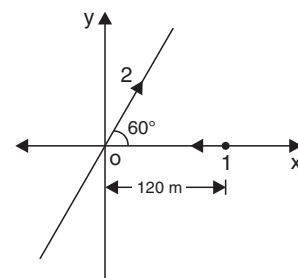


Fig. 10.9

Car 2. Initial velocity, $u_2 = 48 \text{ km/hr} = \frac{48 \times 1000}{60 \times 60} = 13.33 \text{ m/s}$

This velocity is at an angle of 60° with x -axis. Hence in vector notation, it is given by

$$\begin{aligned} u_2 &= 13.33 (\cos 60^\circ)i + 13.33 (\sin 60^\circ)j \\ &= 6.67i + 11.54j \end{aligned}$$

Acceleration, $a = 2 \text{ m/s}^2$. This acceleration is at an angle of 60° with x -axis. Hence, this acceleration in vector form is given by,

$$\mathbf{a} = 2(\cos 60^\circ)i + 2 \sin (60^\circ)j = 1.0i + 1.732j$$

Time $t = 3 \text{ s}$

Let $v_2 =$ Final velocity of car 2 after 3 seconds.

$$\begin{aligned} \text{Then } v_2 &= u_2 + at \\ &= (6.67i + 11.54j) + (1.0i + 1.732j) \times 3 \\ &= 6.67i + 11.54j + 3i + 5.196j = 9.67i + 16.736j \end{aligned}$$

or $\mathbf{V}_2 = 9.67i + 16.736j$

Now the velocity of car 1 with respect to the velocity of car 2 after 3 seconds

$$\begin{aligned} V_{12} &= \mathbf{V}_1 - \mathbf{V}_2 = -15.5i - [9.67i + 16.736j] \\ &= -25.17i - 16.736j \text{ m/s. Ans.} \end{aligned}$$

10.8. PLANE MOTION OF RIGID BODIES

Plane motion of a rigid body is the sum of the translation of the body and rotation of the body. The examples of general plane motion are rolling wheels, connecting rod of a reciprocating engine, links joining rotating elements of a machine. Hence

Plane Motion = Motion of translation + Motion of rotation.

Consider a link AB , which moves from its initial position AB to A_1B_1 in a short interval of time as shown in Fig. 10.10. The link has neither entire motion of translation nor entire rotation, but a combination of the two. The motion of the link from the position AB to the position A_1B_1 may be regarded as to consist of:

1st Case

(i) A motion of entire translation from the position AB to the position A_1B^* so that A_1B^* is parallel to AB is as shown in Fig. 10.10 (a), and

(ii) A motion of entire rotation about A_1 from the position A_1B^* to the position A_1B_1 .

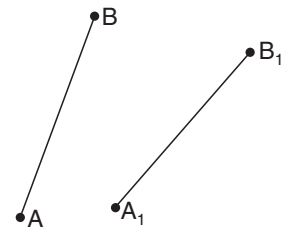


Fig. 10.10

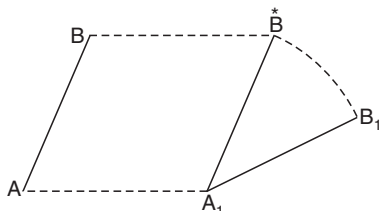


Fig. 10.10 (a)

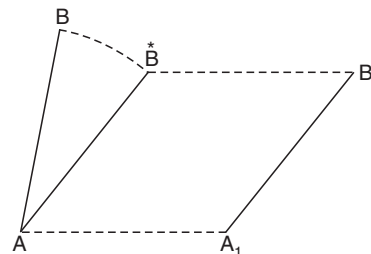


Fig. 10.10 (b)

2nd Case

(i) A motion of entire rotation about A from the position AB to the position AB^* as shown in Fig. 10.10 (b) and

(ii) A motion of entire translation from the position AB^* to the position A_1B_1 .

3rd Case

The combined motion of translation and rotation of the link from its initial position AB to the position A_1B_1 is known as **plane motion**.

Problem 10.27. Prove that the motion of a wheel rolling on a horizontal surface without slipping and the motion of a rod sliding against a wall at one end and the floor at the other end are the cases of plane motion.

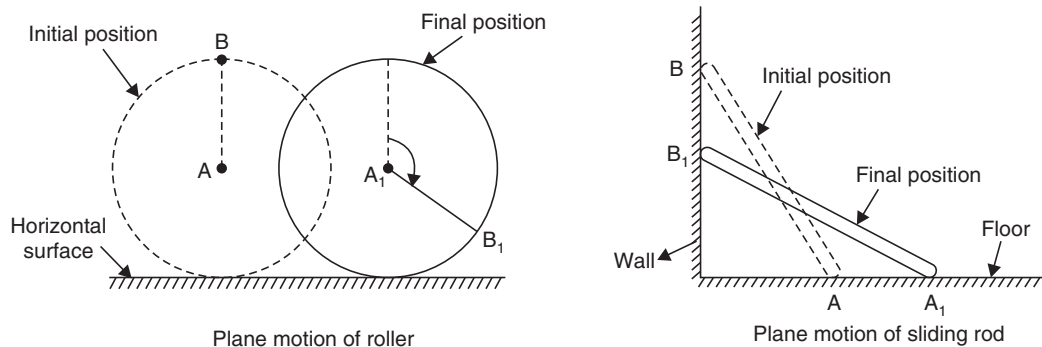


Fig. 10.11

Sol. Case I. A wheel rolling on a horizontal surface without slipping. [Refer to Fig. 10.12]

Plane motion means the motion of translation and motion of rotation simultaneously of a body at any particular instant.

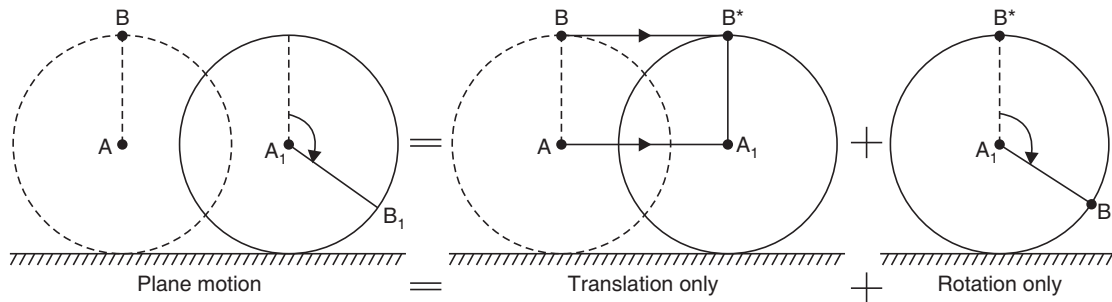


Fig. 10.12

The Fig. 10.12 shows a wheel rolling on a horizontal surface without slipping. The line AB moves from its initial position AB to A_1B_1 in a short interval of time. The motion of line AB from the position AB to A_1B_1 may be regarded as to consist of:

(i) A motion of entire translation from the position AB to the position A_1B^* so that A_1B^* is parallel to AB .

(ii) A motion of entire rotation about A_1 from the position A_1B^* to the position A_1B_1 as shown in Fig. 10.12.

Case II. A rod sliding against wall at one end and the floor at the other end. [Refer to Fig. 10.13]

Fig. 10.13 shows a rod AB sliding against wall at one end and floor at other end. The rod AB moves from its initial position AB to final position A_1B_1 in a short interval of time. The motion of the rod from the position AB to A_1B_1 may be regarded as to consist of :

(i) A motion of entire translation from AB to the position A_1B^* so that A_1B^* is parallel to AB .

(ii) A motion of entire rotation about A_1 from the position A_1B^* to the position A_1B_1 as shown in Fig. 10.13.

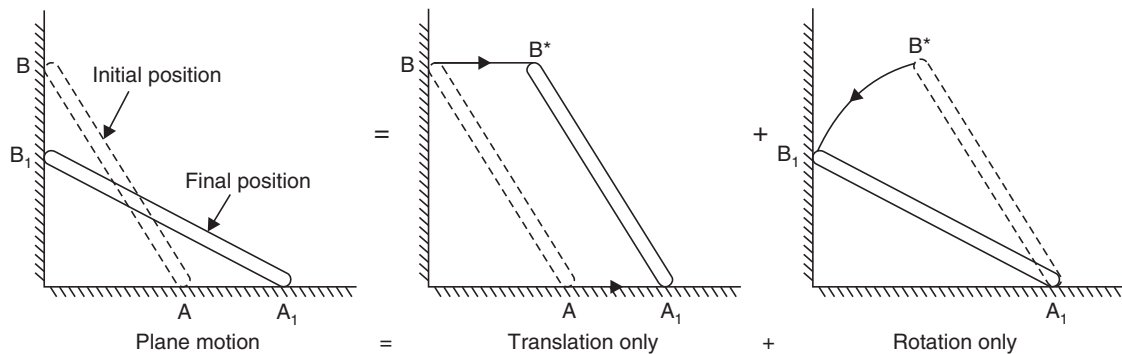


Fig. 10.13

10.8.1. Kinematics of Plane Motion. Kinematics of plane motion means to find velocity and acceleration of different points of a body having plane motion.

Fig. 10.14 shows a rigid body which is having plane motion. The body occupies position 1, at a particular time. This means that the body is having translation motion and also rotary motion. After a short interval of time the body occupies position 2. The distance between any two points is fixed. Let the distance between points A and B is r . This is constant in both positions. The inclination of line AB with x -axis is changing. This is not same for both positions. Thus the body is having translation motion and also rotational motion about point A . The point, about which the body is supposed to rotate, is known as pole or a reference point. Hence point A is a pole. Consider the body again in position 1, as shown again in Fig. 10.14 (a) in a larger view.

Let x_A, y_A = Co-ordinates of point A
 x_B, y_B = Co-ordinates of point B
 r = Distance between AB ,
 this is the distance of B
 with respect to A .
 θ = Inclination of AB with
 x -axis

Let us find the velocity and acceleration of point B .

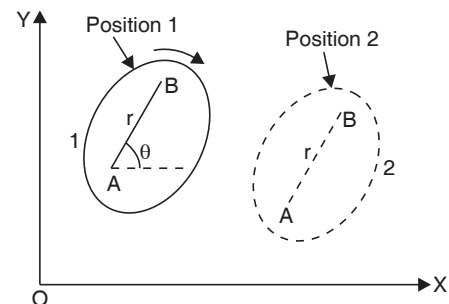


Fig. 10.14. Plane motion of a rigid body.

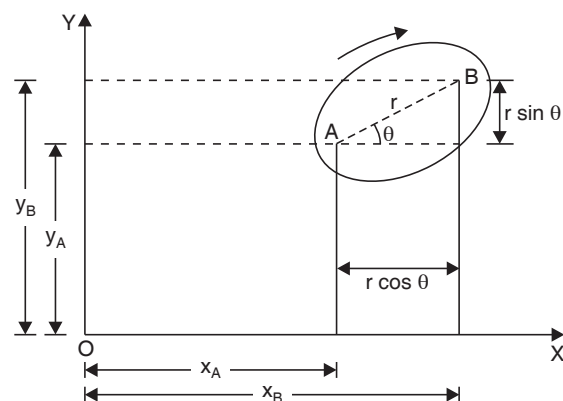


Fig. 10.14 (a)

Velocity of point B

The x_B, y_B co-ordinates of point B in terms of co-ordinates of point A are

$$x_B = x_A + r \cos \theta \text{ and } y_B = y_A + r \sin \theta \quad \dots(i)$$

Here, $r \cos \theta$ and $r \sin \theta$ are the distances of point B with respect to A in x -direction and y -directions.

Differentiating the above equations with respect to time, we get the velocity of point B. In the above equations r is constant but x_A, y_A and θ are changing with time (as body is having translation motion and rotary motion).

$$\therefore \frac{dx_B}{dt} = (v_B)_x = \frac{dx_A}{dt} + r(-\sin \theta) \frac{d\theta}{dt} = \frac{dx_A}{dt} - \omega \times r \sin \theta \quad \dots(ii)$$

$$\left(\because \frac{d\theta}{dt} = \omega \text{ i.e., Angular velocity of B about A} \right)$$

$$= (v_A)_x + (-\omega r \sin \theta) = (v_A)_x + (v_{B/A})_x \quad \dots(iii)$$

$$\frac{dy_B}{dt} = (v_B)_y = \frac{dy_A}{dt} + r \cos \theta \times \frac{d\theta}{dt} = \frac{dy_A}{dt} + \omega r \cos \theta \quad \dots(iv)$$

$$= (v_A)_y + (v_{B/A})_y \quad \dots(v)$$

Here $(-\omega r \sin \theta)$ is the velocity of B with respect to A in x -direction and $\omega r \cos \theta$ is the velocity of B with respect to A in y -direction.

$$\therefore (-\omega r \sin \theta) = (v_{B/A})_x \text{ and } \omega r \cos \theta = (v_{B/A})_y$$

Vector sum of $(v_B)_x$ and $(v_B)_y$ will give the velocity of B i.e., v_B whereas the vector sum of $(v_A)_x$ and $(v_A)_y$ will be equal to v_A . Similarly vector sum of $(v_{B/A})_x$ and $(v_{B/A})_y$ will be equal to $v_{B/A}$.

$$[\because v_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2} = \sqrt{(-\omega r \sin \theta)^2 + (\omega r \cos \theta)^2} = \omega \times r.]$$

$\therefore$ Vector sum of the above two equations (iii) and (v) gives

$$\text{Velocity of B} = \text{Velocity of A} + \text{Velocity of B with respect to A (vector sum)} \quad \dots(10.19)$$

$$\text{or } v_B = v_A + \rightarrow v_{B/A} (= \omega \times r) \text{ (vector sum)} \quad \dots(10.20)$$

where sign $+\rightarrow$ means that the vector v_A and $v_{B/A}$ are to be added geometrically. Also $v_{B/A}$ is equal to $\omega \times r$, where r is the distance between A and B and ω is the rotational velocity of B about point A.

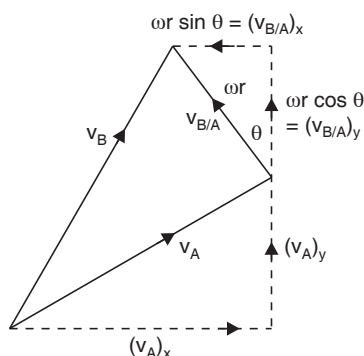


Fig. 10.14 (b)

Acceleration of point B

The acceleration of point B is obtained by differentiating equations (ii) and (iv) with respect to time.

$$\therefore \frac{dx_B^2}{dt^2} = (a_B)_x = \frac{d^2x_A}{dt^2} - r \left(\omega \cos \theta \times \frac{d\theta}{dt} + \sin \theta \times \frac{d\omega}{dt} \right)$$

(r is constant but θ and ω are variable)

$$\begin{aligned} &= (a_A)_x - r (\omega \times \cos \theta \times \omega + \sin \theta \times \alpha) \quad \left[\because \frac{d\theta}{dt} = \omega \text{ and } \frac{d\omega}{dt} = \alpha \right] \\ &= (a_A)_x - r (\omega^2 \cos \theta + \alpha \sin \theta) \\ &= (a_A)_x - \omega^2 r \cos \theta - r \times \alpha \sin \theta \quad \dots(vi) \end{aligned}$$

and

$$\begin{aligned} \frac{dy_B^2}{dt^2} &= (a_B)_y = \frac{d^2y_A}{dt^2} + r \left[\omega (-\sin \theta) \times \frac{d\theta}{dt} + \cos \theta \times \frac{d\omega}{dt} \right] \\ &= (a_A)_y + r [\omega (-\sin \theta) \times \omega + \cos \theta \times \alpha] \quad \left[\because \frac{d\theta}{dt} = \omega \text{ and } \frac{d\omega}{dt} = \alpha \right] \\ &= (a_A)_y - \omega^2 r \sin \theta + r \times \alpha \cos \theta \quad \dots(vii) \end{aligned}$$

In equations (vi) and (vii), the vector sum of $(a_A)_x$ and $(a_A)_y$ will give acceleration of A i.e., a_A . Vector sum of $\omega^2 r \cos \theta$ and $\omega^2 r \sin \theta$ will give the normal acceleration of B with respect to A . Similarly, vector sum of $r \times \alpha \cos \theta$ and $r \times \alpha \sin \theta$ will give the tangential acceleration of B with respect to A .

$\therefore$ Vector sum of the above two equations (vi) and (vii) gives

Acceleration of B = Acceleration of A $\rightarrow$ Acceleration of B with respect to A

or $a_B = a_A \rightarrow a_{B/A}$

where $a_{B/A}$ consists of normal acceleration $\omega^2 r$ and tangential acceleration $\alpha \times r$.

$$\therefore a_B = a_A \rightarrow a_{B/A} (= \omega^2 r + \alpha \times r) \quad \dots(10.21)$$

Problem 10.28. A link AB is moving in a vertical plane. At a certain instant, when the link is inclined at 30° to the horizontal, the point A is moving horizontally at 4 m/s , while B is moving vertically upwards. Find the velocity of B .

Sol. Given :

Inclination of link AB with horizontal = 30°

Horizontal velocity of A , $V_A = 4 \text{ m/s}$

Let V_B = Velocity of B in the vertically upward direction.

The link AB is having plane motion.

The velocity v_A at point A is known. Hence, this point can be chosen as the pole (or reference point). The velocity of point B with respect to A can be obtained by using equation (10.20).

$$\therefore v_B = v_A \rightarrow v_{B/A} (= \omega \times r)$$

where ω = angular rotational velocity of B about point A and r = length AB .

The velocity v_B is vertically upwards, velocity v_A is in horizontal direction towards right of A and velocity $v_{B/A}$ is perpendicular to link AB at point B . The directions and magnitude of these velocities are shown below :

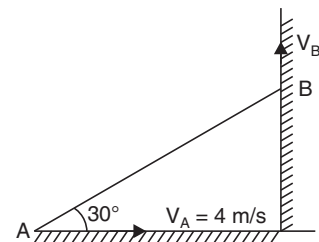


Fig. 10.15

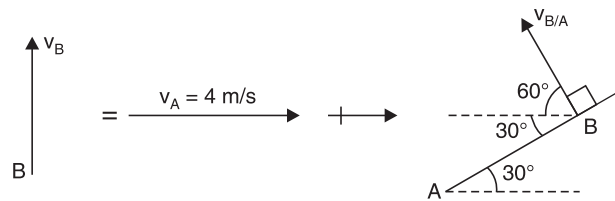


Fig. 10.15 (a)

Resolving the velocities in x and y directions, we get

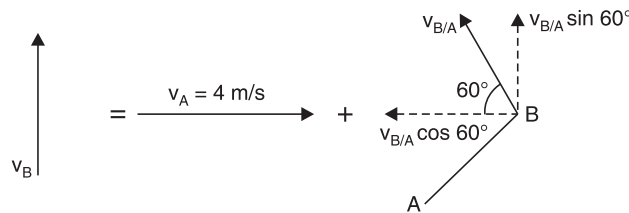


Fig. 10.15 (b)

Equating the velocity components in x and y direction, we get

$$\begin{aligned} \text{For } x\text{-direction : } 0 &= v_A - v_{B/A} \cos 60^\circ \\ &= 4 - v_{B/A} \times 0.5 \end{aligned}$$

$$\therefore v_{B/A} = \frac{4}{0.5} = 8 \text{ m/s}$$

$$\begin{aligned} \text{For } y\text{-direction : } v_B &= 0 + v_{B/A} \sin 60^\circ \\ &= 0 + 8 \times 0.866 = \mathbf{6.928 \text{ m/s.}} \quad \text{Ans.} \end{aligned}$$

Problem 10.29. A cylinder of radius 1 m rolls without slipping along a horizontal plane AB . Its centre has uniform velocity of 20 m/s. Find the velocity of point E and F on the circumference of the cylinder shown in Fig. 10.16.

Sol. Given : $r = 1 \text{ m}$, $v_C = 20 \text{ m/s}$

As the cylinder rolls without slipping, hence the point of contact D with the horizontal plane AB at any instant will have zero velocity.

The rolling cylinder without slipping is having plane motion. The velocity v_C at point C is known. Hence, point C can be taken as the pole (or the reference point). The velocity of point D with respect to point C can be written as [Use equation (10.20)]

$v_D = v_C \rightarrow v_{D/C} (= \omega \times r)$, where ω = angular rotational velocity of DC about point C and $r = CD = 1 \text{ m}$)

The velocity v_D at point D is zero. The velocity v_C at point C is 20 m/s and is in horizontal direction towards right of C and velocity $v_{D/C}$ is perpendicular to line CD at point D . The line CD is vertical and hence velocity $v_{D/C}$ will be in horizontal direction towards left of D . These velocities are shown in Fig. 10.16(a) below :

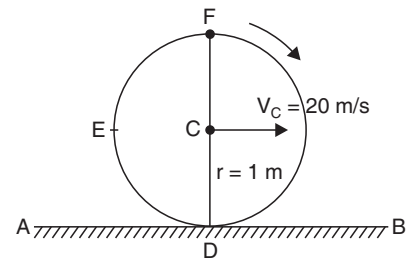


Fig. 10.16

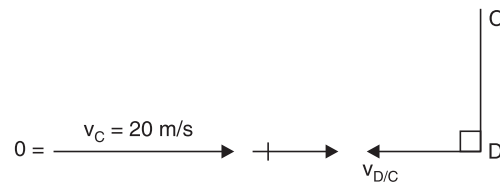


Fig. 10.16 (a)

Resolving all velocities in x -direction, we get

$$0 = 20 + (-v_{D/C})$$

or $v_{D/C} = 20$ But $v_{D/C} = \omega \times r = \omega \times 1$

$$\therefore \omega \times r = 20 \quad \text{or} \quad \omega = \frac{20}{r} = \frac{20}{1} = 20 \text{ rad/s}$$

Velocity of E

The velocity v_E of point E with respect to pole (*i.e.*, point C) can be obtained from equation (10.20) as

$$v_E = v_C + v_{E/C}$$

($= \omega \times r$, where ω = rotational angular velocity of E about point $C = 20$ rad/s and $r = EC = 1$ m)

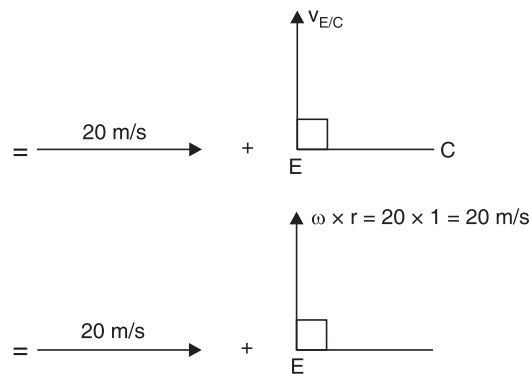


Fig. 10.16 (b)

Here, $(v_E)_x = 20$ m/s (Equating velocities in x -direction)

$$(v_E)_y = 20 \text{ m/s}$$

$$\therefore v_E = \sqrt{(v_E)_x^2 + (v_E)_y^2}$$

$$= \sqrt{20^2 + 20^2} = 20 \times \sqrt{2} = 28.28 \text{ m/s}$$

The angle made by v_E with EC (or x -axis) is given by

$$\tan \theta = \frac{20}{20} = 1$$

$$\therefore \theta = \tan^{-1} 1 = 45^\circ. \text{ Ans.}$$

Velocity of F

The velocity v_F of point F with respect to pole (*i.e.*, point C) is written as

$$v_F = v_C + v_{F/C} \quad (= \omega \times r, \text{ where } \omega = 20 \text{ rad/s and } r = CF = 1 \text{ m and } v_{F/C} = 20 \times 1 = 20 \text{ m/s. This is } \perp \text{ to line } CF \text{ at point } F).$$

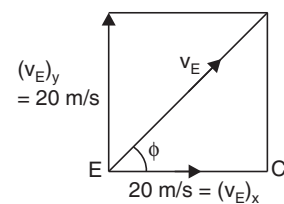
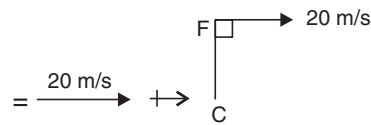


Fig. 10.16 (c)

(parallel to x -axis at point F)

Problem 10.30. The crank of a reciprocating engine is rotating at 210 r.p.m. The lengths of the crank and connecting rod are 20 cm and 100 cm respectively. Find the velocity of the point A (i.e., velocity of piston), when crank has turned through an angle of 45° with the horizontal as shown in Fig. 10.17.

Sol. Given :

Speed of crank, $N = 210 \text{ r.p.m.}$

$\therefore$ Angular velocity of crank, $\omega = \frac{2\pi N}{60} = 2 \times \frac{22}{7} \times \frac{210}{60} = 22 \text{ rad/s}$

Crank length or radius, $r = 20 \text{ cm} = 0.20 \text{ m}$

Length of connecting rod, $L = 100 \text{ cm} = 1.0 \text{ m}$

Angle turned by crank, $\theta = 45^\circ$.

Let us first calculate the angle ϕ .

Applying the sine rule for the triangle ABC shown in Fig. 10.17.

$$\frac{BC}{\sin \phi} = \frac{AB}{\sin \theta}$$

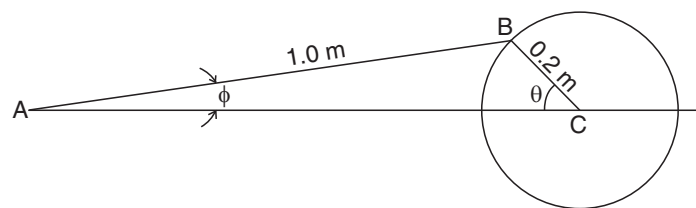


Fig. 10.17

$$\therefore \sin \phi = \frac{BC \sin \theta}{AB} = \frac{0.2}{1.0} \times \sin 45^\circ = \frac{0.2}{1.0} \times 0.7071 = 0.1414$$

$$\therefore \phi = \sin^{-1} 0.1414 = 8.13^\circ$$

The velocity v_B of point B is known $v_B = \omega \times r = 22 \times 0.2 = 4.4 \text{ m/s}$. Hence the point B can be chosen as reference point or pole. The connecting rod can be assumed to be rotating about point B .

The velocity v_A of point A with respect to point B (i.e., with respect to pole) can be obtained from equation (10.20) as

$$\therefore v_A = v_B + \rightarrow v_{A/B} \text{ (vector sum)} \quad [\text{Equation (10.20)}] \quad \dots(i)$$

The velocity of point A (i.e., piston) v_A is in horizontal direction towards right of A , when crank BC is moving clockwise and has turned an angle 45° as shown in figure. The velocity of point B , v_B is perpendicular to BC . The magnitude of v_B is 4.4 m/s. The velocity $v_{A/B}$ (Rotational velocity of connecting rod AB about point B is perpendicular to AB at point A and it is equal to $\omega_{AB} \times L$.

Now, substituting in magnitude and direction of all velocities in equation (i), we get

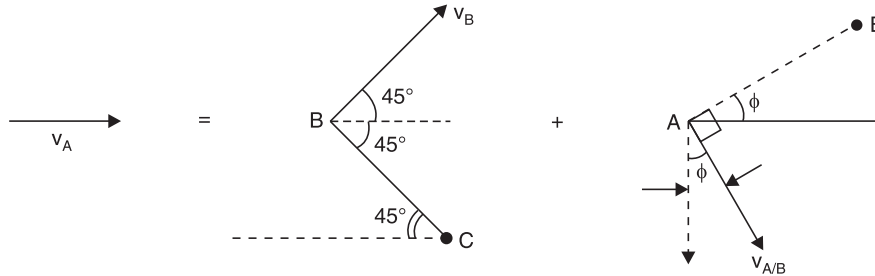


Fig. 10.17 (a)

$$\therefore \cos \phi = 0.9899$$

Resolving all velocities in x - and y -direction, we get

Equating the velocities in x - and y -directions,

$$\begin{aligned} \text{For } x\text{-axis : } v_A &= v_B \cos 45^\circ + v_{A/B} \sin \phi \\ &= 4.4 \cos 45^\circ + v_{A/B} \times 0.1414 \quad (\because v_B = 4.4 \text{ m/s } \phi \text{ and } \sin \phi = 0.1414) \\ &= 3.11 + 0.1414 v_{A/B} \quad \dots(ii) \end{aligned}$$

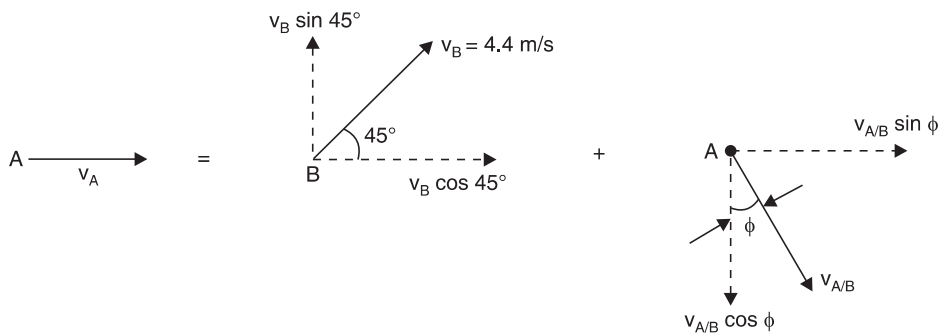


Fig. 10.17 (b)

$$\begin{aligned} \text{For } y\text{-axis : } 0 &= v_B \sin 45^\circ - v_{A/B} \cos \phi \\ &= 4.4 \sin 45^\circ - v_{A/B} \times 0.9899 \\ &= 3.11 - 0.9899 v_{A/B} \end{aligned}$$

$$\therefore v_{A/B} = \frac{3.11}{0.9899} = 3.143 \text{ m/s}$$

Substituting this value in equation (ii), we get

$$\begin{aligned} v_A &= 3.11 + 0.1414 \times 3.143 \\ &= 3.55 \text{ m/s. Ans.} \end{aligned}$$

Problem 10.31. A steam engine has a crank of radius 15 cm and connecting rod of length 75 cm as shown in Fig. 10.18. The crank CQ rotates in a clockwise direction with a constant speed of 300 r.p.m. Calculate the velocity and acceleration of the piston P at the instant when the angle $\theta = 30^\circ$.

Sol. Given :

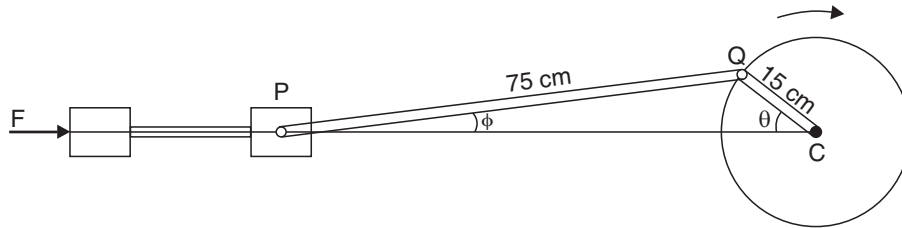


Fig. 10.18

Crank radius, $r = 15 \text{ cm} = 0.15 \text{ m}$

Length of connecting rod, $L = 75 \text{ cm} = 0.75 \text{ m}$

Speed of crank, $N = 300 \text{ r.p.m.}$

$\therefore$ Angular velocity of crank, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 10\pi \text{ radian/s.}$

Angle $\theta = 30^\circ$.

(i) **Velocity of Piston P.** The velocity of piston P can be obtained either by analytical method or by graphical method.

First find the value of angle ϕ . In triangle PCQ , using sine rule, we have

$$\frac{CQ}{\sin \phi} = \frac{PQ}{\sin \theta}$$

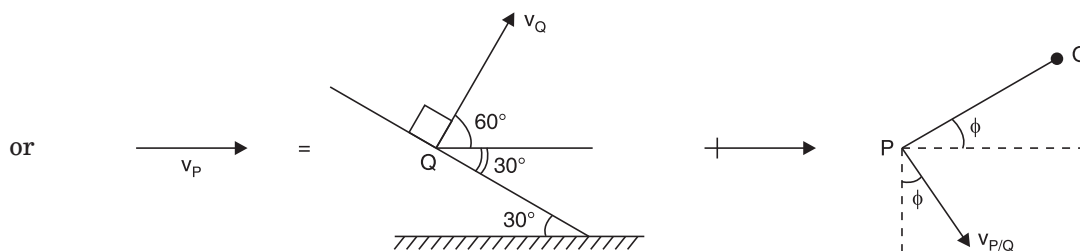


Fig. 10.18 (a)

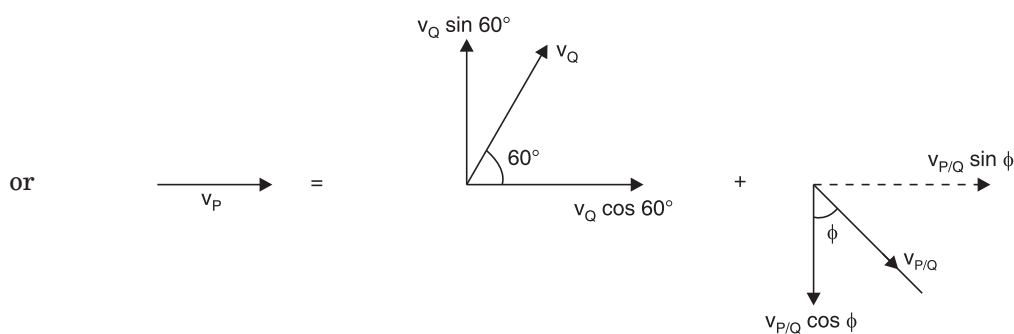


Fig. 10.18 (b)

$$\therefore \sin \phi = \frac{CQ}{PQ} \sin \theta = \frac{0.15}{0.75} \sin 30^\circ = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10} = 0.1$$

$$\therefore \phi = \sin^{-1} 0.1 = 5.739^\circ.$$

$$v_Q = \omega \times r = 10\pi \times 15 = 150\pi \text{ cm/s}$$

$$\sin \phi = 0.1, \phi = 5.739^\circ, \cos \phi = 0.995$$

Using equation for velocity for plane motion and taking point Q as the pole, we get

$$v_P = V_Q + \rightarrow v_{P/Q} \quad \dots[\text{Equation (10.20)}]$$

Equating the velocities in x and y directions,

$$\begin{aligned} \text{Along } x\text{-axis :} \quad v_P &= v_Q \cos 60^\circ + v_{P/Q} \sin \phi \\ &= 150\pi \times 0.5 + v_{P/Q} \times 0.1 \\ &= 235.6 + 0.1 v_{P/Q} \end{aligned} \quad \dots(1)$$

$$\begin{aligned} \text{Along } y\text{-axis :} \quad 0 &= v_Q \sin 60^\circ - v_{P/Q} \cos \phi \\ &= 150\pi \times 0.866 - v_{P/Q} \times 0.995 \\ &= 408.9 - v_{P/Q} \times 0.995 \end{aligned}$$

$$\text{or} \quad v_{P/Q} = \frac{408.9}{0.995} = 410.95 \quad \dots(2)$$

Substitute this value in (1), we get

$$v_P = 235.6 + 0.1 \times 410.95 = \mathbf{277.6 \text{ cm/s. Ans.}}$$

Let us also find the angular velocity (ω_{PQ}) of connecting rod PQ . We know

$$v_{P/Q} = \omega_{PQ} \times PQ \quad \text{But } v_{P/Q} = 410.95 \text{ m/s (calculated above)}$$

$$\therefore 410.95 = \omega_{PQ} \times 75 \quad \therefore \omega_{PQ} = \frac{410.95}{75} = 5.48 \text{ rad/s}$$

(ii) **Acceleration of Piston P.** Using equation of plane motion [equation (10.21)] for acceleration and taking Q as the pole, we get

$$a_P = a_Q + \rightarrow a_{P/Q} \text{ (which is } = r\omega^2 + r\alpha \text{ acting at } P)$$

$$\begin{aligned} \text{[Here } r = PQ \text{ and } \alpha = \omega \times PQ. \text{ Radial acceleration towards } Q \text{ from } P &= \omega_{PQ}^2 \times PQ = (5.48)^2 \\ &\times 75 = 2252 \text{ cm/s}^2 \text{ and tangential acceleration at } P = PQ \times \alpha_{PQ} = 75 \alpha_{PQ}] \end{aligned}$$

Vectorially the above equation becomes as

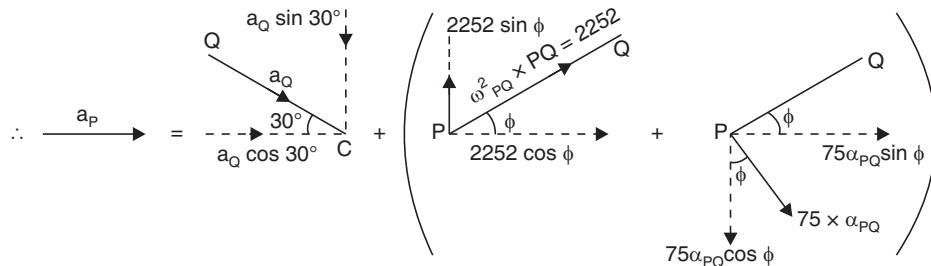


Fig. 10.18 (c)

$$\text{But} \quad a_Q = \omega^2 \times r = (10\pi)^2 \times 15 = 14804 \text{ m/s}^2 \quad [\text{here } r = 15]$$

Resolving all accelerations in x - and y -directions.

$$\begin{aligned} \text{Along } x\text{-axis :} \quad a_P &= +a_Q \cos 30^\circ + 2252 \cos \phi + 75 \alpha_{PQ} \sin \phi \\ &= 14804 \times 0.866 + 2252 \times 0.995 + 75 \times \alpha_{PQ} \times 0.1 \\ &= 12820 + 2240 + 75 \alpha_{PQ} \end{aligned} \quad \dots(3)$$

$$\begin{aligned}
 \text{Along } y\text{-axis : } 0 &= -a_Q \sin 30^\circ + 2252 \sin \phi - 75 \times \alpha_{PQ} \times \cos \phi \\
 &= -14804 \times 0.5 + 2252 \times 0.1 - 75 \times \alpha_{PQ} \times .995 \\
 &= -7402 + 225.2 - 77.62 \alpha_{PQ} \\
 \therefore \alpha_{PQ} &= \frac{-7402 + 225.2}{77.62} = -92.46
 \end{aligned}$$

Substituting this value in (3), we get

$$\begin{aligned}
 a_P &= 12820 + 2240 + 7.5 \times (-92.46) \\
 &= 15060 - 693.45 = \mathbf{14366.55 \text{ cm/s}^2}. \text{ Ans.}
 \end{aligned}$$

Problem 10.32. The lengths of connecting rod (BC) and crank (AB) in a reciprocating pump are 1125 mm and 250 mm respectively. The crank is rotating at 420 r.p.m. Find the velocity with which the piston will move, when the crank has turned through an angle of 40° from the inner dead centre. Also find the angular velocity of the connected rod.

(U.P. Tech. University, 2002–2003)

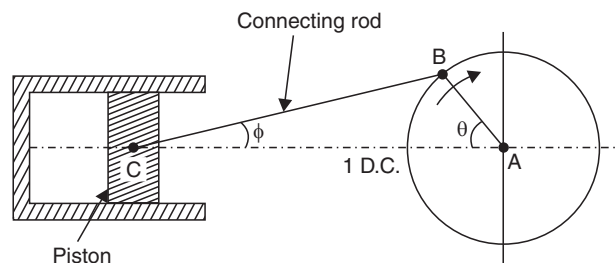


Fig. 10.19

Sol. Given :

Length of connecting rod, $L = BC = 1125 \text{ mm} = 1.125 \text{ m}$

Crank radius, $r = AB = 250 \text{ mm} = 0.25 \text{ m}$

Speed of crank, $N = 420 \text{ r.p.m.}$

$$\therefore \text{Angular velocity of crank, } \omega = \frac{2\pi N}{60} = \frac{2\pi \times 420}{60} = 14\pi \text{ rad/s} \quad \dots(i)$$

Angle turned by crank, $\theta = 40^\circ$

Find : (i) Velocity of piston and

(ii) Angular velocity of the connecting rod.

(i) *Velocity of piston*

Let us find the value of ϕ in terms of θ .

From Fig. 10.19 (a), we have

$$BD = L \sin \phi = r \sin \theta$$

$$\therefore \sin \phi = \frac{r}{L} \sin \theta \quad \dots(ii)$$

$$\begin{aligned}
 \text{or } \sin \phi &= \frac{0.25}{1.125} \sin 40^\circ \\
 [\because r = 0.25, L = 1.125, \theta = 40^\circ] \\
 &= 0.1428 \quad \dots(iii)
 \end{aligned}$$

$$\therefore \phi = \sin^{-1} 0.1428 = 8.2^\circ \quad \dots(iv)$$

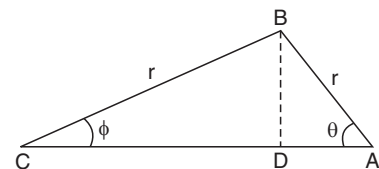


Fig. 10.19 (a)

and

$$\cos \phi = \cos 8.2^\circ = 0.9897$$

$$v_B = \omega \times r = 14\pi \times 0.25 = 10.995 \approx 11 \text{ m/s}$$

The link BC is having plane motion.

As velocity of point B is known. The link BC may be assumed as rotating about point B . Hence take point B as reference point. Then from the equation of plane motion for link BC , we get [equation (10.20)]

$$v_C = v_B + v_{C/B}$$

(where $v_{C/B}$ = rotational velocity of line BC about point

$$B = \omega \times r, \text{ where } \omega = \omega_{BC} \text{ and } r = BC)$$

The velocity v_C is along x -axis, velocity v_B is perpendicular to crank AB and $v_{C/B}$ is perpendicular to link CB acting at point C as shown in Fig. 10.19 (b). The angle ϕ can be

obtained by applying sine rule to triangle ABC as $\frac{AB}{\sin \phi} = \frac{BC}{\sin \theta}$ or $\sin \phi = \frac{AB}{BC} \times \sin \theta$

The above equation can be written in vector form (*i.e.*, the magnitude and direction of all velocities), we get

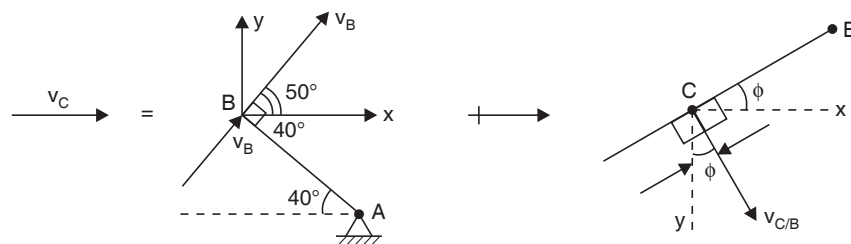


Fig. 10.19 (b)

Resolving all velocities in x and y -directions, we get

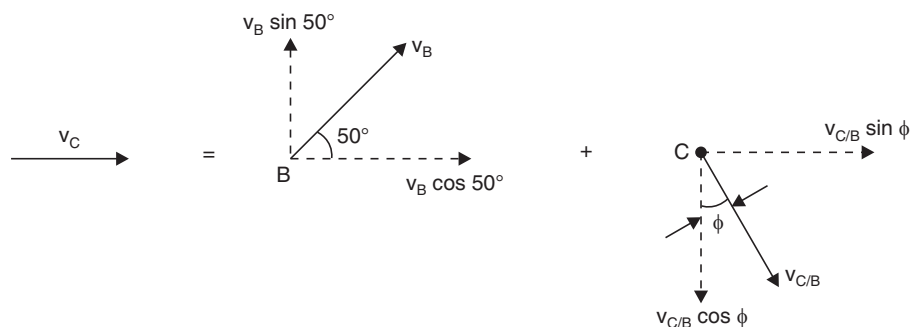


Fig. 10.19 (c)

Equating the velocities in x and y -directions,

$$\begin{aligned} \text{Along } x\text{-axis : } v_C &= v_B \cos 50^\circ + v_{C/B} \sin \phi \\ &= 11 \cos 50^\circ + v_{C/B} \sin \phi \\ &= 11 \cos 50^\circ + v_{C/B} \times 0.1428 \\ &= 7.07 + v_{C/B} \times 0.1428 \end{aligned}$$

$$[\because v_B = 11]$$

$$[\because \sin \phi = 0.1428]$$

$$\dots(1)$$

$$\begin{aligned}
 \text{Along } y\text{-axis : } 0 &= v_B \sin 50^\circ - v_{C/B} \cos \phi \\
 &= 11 \sin 50^\circ - v_{C/B} \times 0.9897 \quad [\because v_B = 11 \text{ and } \cos \phi = 0.9897] \\
 &= 8.426 - 0.9897 v_{C/B}
 \end{aligned}$$

$$\text{or } v_{C/B} = \frac{8.426}{0.9897} = 8.514 \text{ m/s} \quad \dots(2)$$

Substituting this value of $v_{C/B}$ in equation (1), we get

$$v_C = 7.07 + 8.514 \times 0.1428 = 8.285 \text{ m/s} \quad \dots(3)$$

$\therefore$ Velocity of piston, $v_C = 8.285 \text{ m/s}$. **Ans.**

(ii) Angular velocity of connecting rod (i.e., ω_{BC})

We know that $v_{C/B} = \omega_{BC} \times BC$

But $v_{C/B} = 8.514 \text{ m/s}$ from equation (2)

$$\therefore 8.514 = \omega_{BC} \times 1.125 \quad (\because BC = 1.125 \text{ m})$$

$$\therefore \omega_{BC} = \frac{8.514}{1.125} = 7.568 \text{ rad/s. Ans.}$$

Problem 10.33. The ends A and B of a slender rod of length 25 cm move in contact with a horizontal floor and an inclined surface as shown in Fig. 10.20. Determine the velocity if the end A has a velocity of 90 cm/s leftward and an acceleration of 18.75 cm/s^2 rightwards. The inclination of the inclined surface with vertical is equal to the inclination of the rod with horizontal. Take $\theta = \tan^{-1} \frac{3}{4}$.

Sol. Given $v_A = 90 \text{ cm/s}$ leftwards ($\leftarrow$)
 $a_A = 18.75 \text{ cm/s}^2$ rightwards ($\rightarrow$)

Inclination of inclined surface with vertical = Inclination of rod with horizontal = θ

$$\therefore \angle OBC = \angle OAB = \theta$$

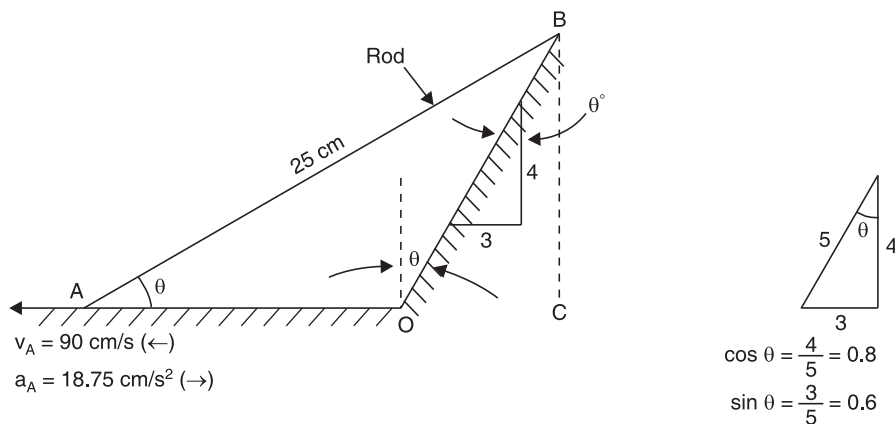


Fig. 10.20

Also $\tan \theta = \frac{3}{4}$ hence $\sin \theta = \frac{3}{5} = 0.6$ and $\cos \theta = \frac{4}{5} = 0.8$ (Refer to Fig. 10.20)

Let v_B = velocity of B and
 a_B = Acceleration of B

Velocity of $B(v_B)$

As the velocity of A is given, hence this point can be considered as pole (or reference point). The rod AB is considered to be rotating about point A . Rod AB is having plane motion.

Now, using the equation (10.20), we get

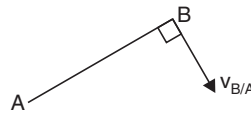
$$v_B = v_A + \rightarrow v_{B/A} \quad \dots(i)$$

where $v_{B/A}$ = relative rotational velocity of B with respect to A

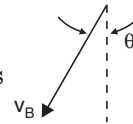
$= \omega \times r$ where r = distance between A and $B = 25$ cm

$= \omega \times 25$ ω = angular velocity of rod AB about pole A

$v_{B/A}$ is acting at B perpendicular to rod AB .



The velocity v_B acting along inclined surface downwards



and $v_A = 90$ cm/s (given) (acting in horizontal direction towards left i.e., v_A)

The velocity of point B can now be determined graphically or analytically.

Analytical Method

$$v_B = v_A + \rightarrow v_{B/A} \text{ (vector sum)}$$

Substituting the values in magnitude and direction, we get

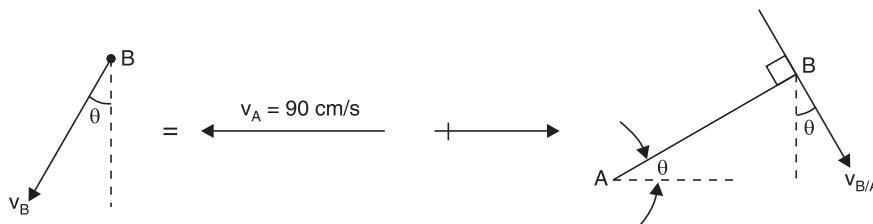


Fig. 10.20 (a)

Resolving the velocities in x and y -directions, we get

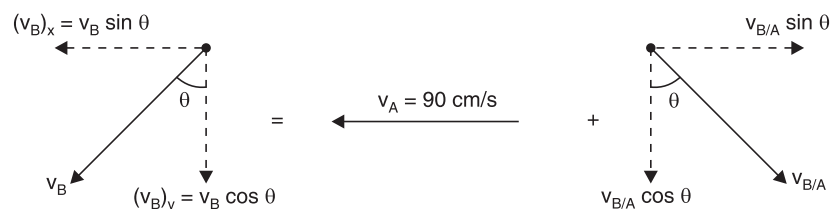


Fig. 10.20 (b)

Equating the horizontal and vertical components of velocity to both sides, we get

For horizontal direction,

$$-v_B \sin \theta = -90 + v_{B/A} \sin \theta$$

$$\text{or } v_B \times 0.6 = 90 - (25\omega) \times 0.6 \quad [\because \sin \theta = 0.6 \text{ and } v_{B/A} = 25\omega]$$

$$\text{or } v_B = \frac{90}{0.6} - 25\omega = 150 - 25\omega \quad \dots(1)$$

For vertical direction

$$v_B \cos \theta = v_{B/A} \cos \theta$$

$$\text{or } v_B = v_{B/A} \quad \text{But } v_{B/A} = 25\omega$$

$$\therefore v_B = 25\omega \quad \dots(2)$$

Solving equations (1) and (2), we get

$$150 - 25\omega = 25\omega \quad [\text{Equating the values of } v_B \text{ in (1) and (2)}]$$

$$\text{or } 150 = 50\omega$$

$$\therefore \omega = \frac{150}{50} = 3 \text{ rad/s}$$

From equation (2), $v_B = 25\omega$

$$= 25 \times 3 = 75 \text{ cm/s. Ans.}$$

Graphical Method (For velocity) [Refer to Fig. 10.20 (c)]

The following are the steps for drawing a velocity diagram.

(i) Take any point o . From o draw a horizontal line oa towards left from o to represent the velocity $v_A = 90 \text{ cm/s}$ to some suitable scale.

(ii) From point o , draw a line ob parallel to the direction of velocity v_B (i.e., parallel to inclined surface downwards) to represent the velocity of point B . The magnitude of v_B is not known.

(iii) From point a , draw a line ab perpendicular to line AB to represent the rotational velocity of B with respect to A . i.e., velocity $v_{B/A}$. This line cuts the line ob at point b . Then ob represents the velocity of b i.e., v_B . Measure line ob .

By measurement,

$$v_B = ob \times \text{scale} = 75 \text{ cm/s. Ans.}$$

$$\begin{aligned} v_{B/A} &= \text{Length } ab \times \text{scale} = 75 \text{ cm/s} = \omega \times r \\ &= \omega \times 25 \end{aligned}$$

$$\therefore \omega = \frac{75}{25} = 3 \text{ rad/s.}$$

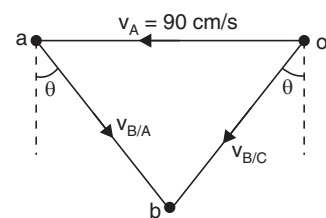


Fig. 10.20 (c)

10.9. INSTANTANEOUS CENTRE OF ZERO VELOCITY

The combined motion of translation and rotation of the link from its initial position AB to the position A_1B_1 as shown in Fig. 10.21 may be assumed to be a motion of entirely rotation about a certain point. This point is known as *instantaneous centre of rotation*. The instantaneous centre of rotation is instantaneously at rest and has zero velocity. The link may seem to be rotating about one point at one instant of time and about another point at the next instant. Hence, instantaneous centre is changing every instant and is not a fixed point. The position of the instantaneous centre of rotation is determined as given below :

Let the link AB in a short interval of time changes its position from AB to A_1B_1 . The point A of the link has moved to point A_1 where the point B of the link has moved to the point B_1 as shown in Fig. 10.21. Draw the right bisectors of chord AA_1 and chord BB_1 . Let CD is the right bisectors of AA_1 whereas EF is the right bisectors of BB_1 . Let these two bisectors meet at the point O . Then the point O is the instantaneous centre of rotation of the link AB . This means the link AB as a whole has rotated about O .

Let V_A = Linear velocity of point A

V_B = Linear velocity of point B

ω = Angular velocity of link AB about O .

This means the angular velocity of point A and point B about O will be ω .

Now, from equation (10.10), we know

Linear velocity = Angular velocity $\times r$

$\therefore$ Linear velocity of point A is given by

$$\begin{aligned} V_A &= \text{Angular velocity of point } A \text{ about } O \\ &\quad \times \text{Distance of } A \text{ from centre of rotation (i.e., from } O) \\ &= \omega \times OA \end{aligned}$$

$$\therefore \omega = \frac{V_A}{OA} \quad \dots(i)$$

Similarly, the linear velocity of point B is given by

$$\begin{aligned} V_B &= \text{Angular velocity of point } B \text{ about } O \\ &\quad \times \text{Distance of } B \text{ from centre of rotation (i.e., from } O) \\ &= \omega \times BO \end{aligned}$$

$$\therefore \omega = \frac{V_B}{BO} \quad \dots(ii)$$

Equating (i) and (ii), we get

$$\frac{V_A}{OA} = \frac{V_B}{BO} \quad \text{or} \quad \frac{V_A}{V_B} = \frac{OA}{BO} \quad \dots(10.22)$$

The direction of the velocity at A will be at right angle to OA whereas the velocity at B will be at right angle to BO .

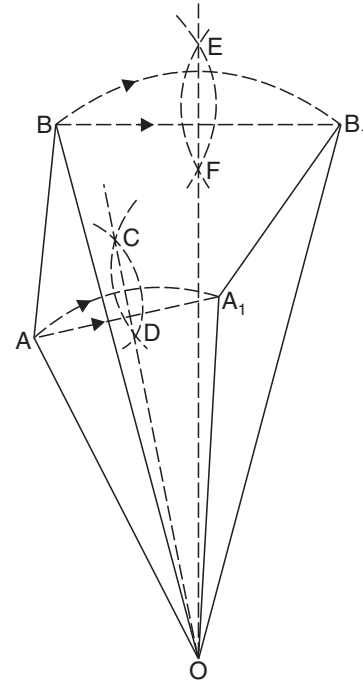


Fig. 10.21

Thus, if the directions of velocities at A and B are known, then the instantaneous centre of AB is obtained by drawing perpendiculars to the directions of the velocities at A and B . The point, where these two perpendiculars, meet is the instantaneous centre.

If the directions of velocities at A and B are parallel but unequal in magnitude, then instantaneous centre of AB is obtained by determining the point of intersection of the line AB with the line joining the extremities of the vectors V_A and V_B as shown in Figs. 10.22 (a) and 10.22 (b). In Fig. 10.22 (a), the instantaneous centre O is outside the link AB whereas in Fig. 10.22 (b) the instantaneous centre O is on the link AB . Hence, instantaneous centre may lie on the link or outside the link.

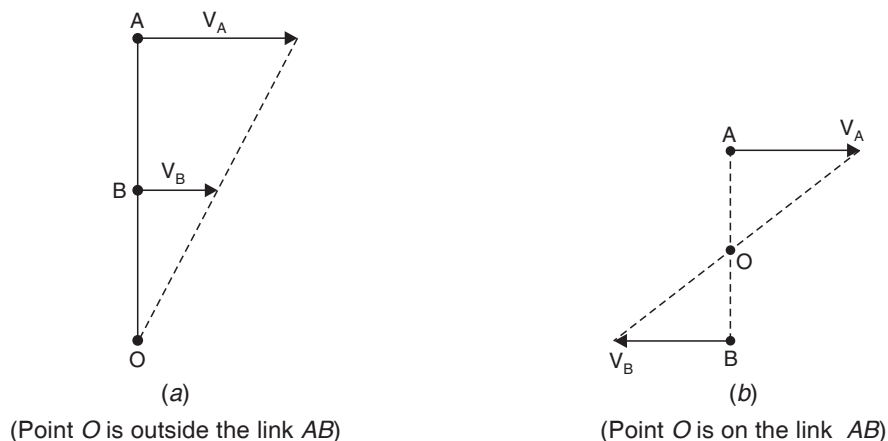


Fig. 10.22

If the directions of velocities at A and B are parallel and also are equal in magnitude, then the instantaneous centre is at infinity and all the points of the link AB have the same velocity as shown in Fig. 10.22 (c). The two parallel lines will meet at infinity.

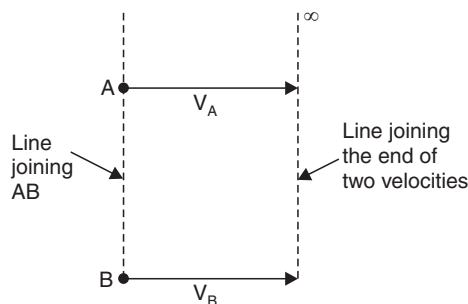


Fig. 10.22 (c)

Notes :

- (i) The instantaneous centre is at rest and has zero velocity.
- (ii) The instantaneous centre may be outside the body or within the body.
- (iii) If any point on the body or outside the body has zero velocity, then that point will be instantaneous centre for the body at that instant.

Problem 10.34. A link AB is moving in a vertical plane. At a certain instant, when the link is inclined at 30° to the horizontal, the point A is moving horizontally at 4 m/s , while B is moving vertically upwards. Find the velocity of B .

Sol. Given :

Inclination of link AB with horizontal = 30°

Horizontal velocity of A , $V_A = 4$ m/s

Let V_B = Velocity of B in the vertically upward direction.

Since the actual directions of motion of A and B are in different directions and are known, the position of the instantaneous centre can be easily determined by drawing perpendiculars to the directions of motions at A and B as shown in Fig. 10.23.

AO is the perpendicular to the direction of motion of A whereas BO is the perpendicular to the direction of motion of B . The point O , at which these two perpendiculars meet, is the instantaneous centre of the link AB .

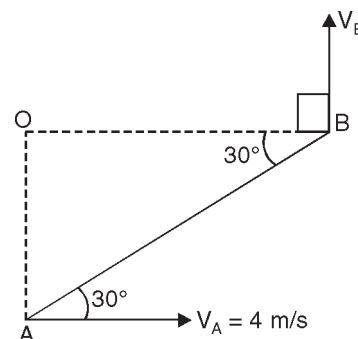


Fig. 10.23

$$\left[\text{or } V_A = \omega \times AO \text{ and } V_B = \omega \times BO \therefore \frac{V_A}{V_B} = \frac{AO}{BO} \right]$$

Now, using equation (10.22), we get

$$\frac{V_A}{V_B} = \frac{OA}{BO} = \tan 30^\circ \quad \left(\because \text{In } \triangle AOB, \tan 30^\circ = \frac{AO}{BO} \right)$$

or

$$\frac{4}{V_B} = \frac{1}{\sqrt{3}}$$

$$\therefore V_B = 4 \times \sqrt{3} = 4 \times 1.732 = 6.928 \text{ m/s. Ans.}$$

Problem 10.35. A cylinder of radius 1 m rolls without slipping along a horizontal plane AB . Its centre has uniform velocity of 20 m/s. Find the velocity of point E and F on the circumference of the cylinder shown in Fig. 10.24 (a). (U.P. Tech. University, 2001–2002)

Sol. Given : $r = 1$ m, $V_C = 20$ m/s

As the cylinder rolls without slipping, hence the point of contact D with the horizontal plane AB at any instant will have zero velocity. The point on the cylinder, which has zero velocity is the instantaneous centre of rotation of the cylinder. Hence, point D is the instantaneous centre at that instant. Hence, the velocity of any other point on the cylinder will be equal to the angular velocity of cylinder multiplied by the distance of that point from instantaneous centre.

$$\therefore \text{Velocity of point } E, \quad V_E = \omega \times DE$$

where ω = Angular velocity of cylinder

$$\text{and Velocity of point } F, \quad V_F = \omega \times DF$$

$$\text{Also velocity of point } C, \quad V_C = \omega \times DC$$

$$\text{But } V_C = 20 \text{ m/s and } DC = r = 1 \text{ m (given)}$$

$$\therefore 20 = \omega \times 1$$

$$\text{or } \omega = 20 \text{ rad/s}$$

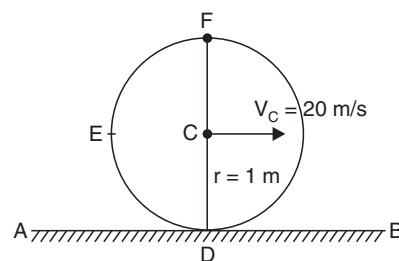


Fig. 10.24 (a)

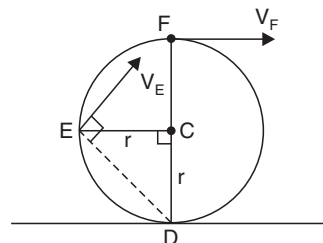


Fig. 10.24 (b)

$$\therefore \text{Velocity of } E, \quad V_E = \omega \times DE = 20 \times DE$$

$$\text{where } DE = \sqrt{r^2 + r^2} = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.414 \text{ m}$$

$$\therefore \quad V_E = 20 \times 1.414 = \mathbf{28.28 \text{ m/s}}$$

The direction of velocity V_E is normal to the line ED as shown in Fig. 10.24 (b).

The velocity of point F ,

$$\begin{aligned} V_F &= \omega \times FD = 20 \times 2 & (\because FD = 2r = 2) \\ &= \mathbf{40 \text{ m/s. Ans.}} \end{aligned}$$

The direction of velocity V_F will be normal to the line FD as shown in Fig. 10.24 (b).

Problem 10.36. A cylindrical roller, 50 cm in diameter is in contact with two conveyor belts at its top and bottom as shown in Fig. 10.25. If the belts run at the uniform speed of 5 m/s and 3 m/s, find linear velocity and angular velocity of the roller. (U.P. Tech. University, 2000–2001)

Sol. Given :

$$\text{Diameter of roller} = 50 \text{ cm}$$

$$\therefore \text{Radius, } r = 25 \text{ cm} = 0.25 \text{ m}$$

$$V_A = 5 \text{ m/s, } V_B = 3 \text{ m/s}$$

Let us first find the instantaneous centre (O). As the directions of velocities at A and B are parallel and unequal in magnitude, hence the instantaneous centre is obtained by determining the point of intersection of the line AB with the line joining the extremities of the vectors V_A and V_B as shown in Fig. 10.25 (a).

$$\text{Here } AL = V_A = 5 \text{ m/s}$$

$$BM = V_B = 3 \text{ m/s}$$

Point O is the intersection of lines AB and LM .

Let ω = Angular velocity of AB or of roller. Then

$$V_A = \omega \times AO$$

$$\text{and } V_B = \omega \times BO$$

$$\text{Now } V_A = 5 \text{ m/s and}$$

$$AO = AB + BO = 0.5 + BO$$

$$= 0.5 + x$$

$$[\text{Let } BO = x]$$

$$V_B = 3 \text{ m/s and } BO = x$$

$$\therefore 5 = \omega \times (0.5 + x) = 0.5 \times \omega + \omega \times x \quad \dots(i)$$

$$3 = \omega \times x \quad \dots(ii)$$

Subtracting equation (ii) from equation (i), we get

$$2 = 0.5 \times \omega \quad \text{or} \quad \omega = \frac{2}{0.5} = \mathbf{4 \text{ rad/s. Ans.}}$$

$$\text{Now } V_C = \omega \times OC = \omega(0.25 + x) \quad \dots(iii)$$

$$\left[\because OC = BC + OB = \frac{0.5}{2} + x = 0.25 + x \right]$$

But x from equation (ii),

$$x = \frac{3}{\omega} = \frac{3}{4} = 0.75$$

$$(\because \omega = 4 \text{ rad/s})$$

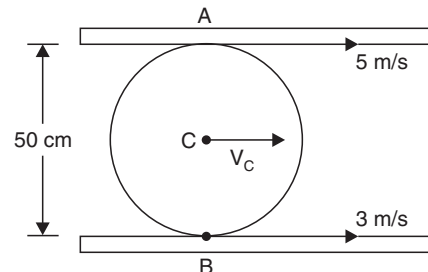


Fig. 10.25

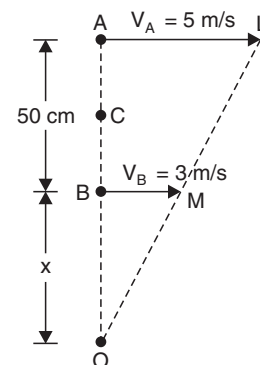


Fig. 10.25 (a)

Substituting the value of x in equation (iii), we get

$$V_C = \omega(0.25 + 0.75) = 4 \times 1 \quad (\because \omega = 4 \text{ rad/s})$$

$$= 4 \text{ m/s. Ans.}$$

$\therefore$ Linear velocity of roller = 4 m/s and angular velocity of roller = 4 rad/s.

Problem 10.37. The crank of a reciprocating engine is rotating at 210 r.p.m. The lengths of the crank and connecting rod are 20 cm and 100 cm respectively. Find the velocity of the point A (i.e., velocity of piston), when crank has turned through an angle of 45° with the horizontal as shown in Fig. 10.26.

Sol. Given :

Rotation of crank, $N = 120 \text{ r.p.m.}$

$\therefore$ Angular velocity of crank, $\omega = \frac{2\pi N}{60} = 2 \times \frac{22}{7} \times \frac{210}{60} = 22 \text{ rad/s.}$

Crank length or radius, $r = 20 \text{ cm} = 0.20 \text{ m}$

Length of connecting rod, $L = 100 \text{ cm} = 1.0 \text{ m}$

Angle turned by crank, $\theta = 45^\circ$.

Fig. 10.26 shows a reciprocating engine mechanism in which AB is the connecting rod and BC is the crank. The crank BC is rotating at a uniform angular velocity in the clockwise direction about point C . The point A connected to the piston rod and connecting rod is having to and fro motion in the horizontal plane. The connecting rod is having a combined motion of translation and rotation.

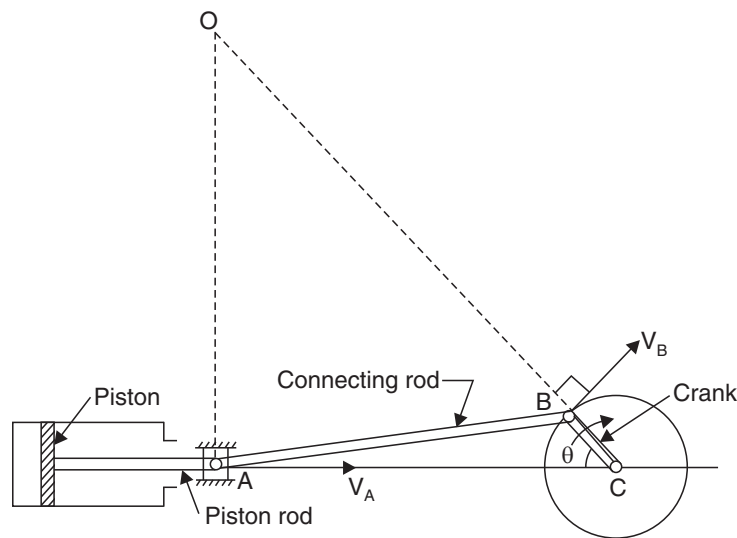


Fig. 10.26

Velocity by Instantaneous Centre Method

Let N = Revolution of crank in r.p.m.

ω = Angular velocity of crank

r = Radius of crank, i.e., length BC

L = Length of connecting rod AB .

$$\text{Then } \omega = \frac{2\pi N}{60}$$

$$\text{Linear velocity of point } B = V_B = \omega \times BC = \omega \times r \quad \dots(i)$$

The direction of the linear velocity at B is along the tangent at B to the crank circle. Hence, V_B will be acting at right angles to BC as shown in Fig. 10.26. The velocity at A (*i.e.*, V_A) is in the horizontal direction acting along AC for the given position of the crank. Since the directions of V_A and V_B are known, the position of the instantaneous centre can be determined by drawing perpendiculars to the directions of the velocities at A and B . The two perpendiculars meet at O , which is the instantaneous centre for the connecting rod AB . Thus for the given position, the connecting rod can be considered as having a motion of entirely rotation about the instantaneous centre *i.e.*, about O . All the points of AB will have same angular velocity.

Let ω_0 = Angular velocity of the connecting rod AB about O .

$$\therefore V_A = \omega_0 \times AO \quad \dots(ii)$$

$$\text{and } V_B = \omega_0 \times BO \quad \dots(iii)$$

Equating equations (i) and (iii), we get

$$\omega \times r = \omega_0 \times BO \quad \text{or } \omega_0 = \frac{\omega \times r}{BO} \quad \dots(A)$$

As the values of ω and r are known, hence the value of ω_0 will be obtained if length BO is measured to the scale.

Substituting the value of ω_0 in equation (ii), we get the velocity of the point A as

$$V_A = \frac{\omega \times r}{BO} \times AO \quad \dots(B)$$

Hence, by measuring to the scale the lengths AO and BO , the velocity at A can be determined from equation (B).

Draw the given mechanism to a suitable scale. Refer to Fig. 10.26 (a).

1. With C as centre, draw a circle of radius 0.2 m.

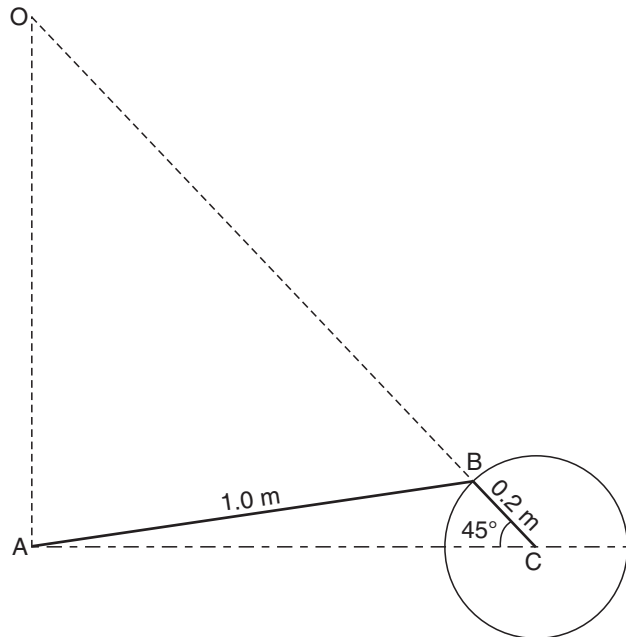


Fig. 10.26 (a)

2. Draw a line BC through C , making an angle of 45° with the horizontal and cutting the circle at B .

3. Draw a horizontal line AC through C .

4. With B as centre, cut off AB equal to 1.0 m.

5. Produce the line CB . And draw a normal to the line AC passing through A . The intersection of the normal through A and produced line BC , gives the instantaneous centre O for the rod AB .

Measuring lengths AO and BO from the diagram, we get

$$AO = 1.15 \text{ m} \quad \text{and} \quad BO = 1.41 \text{ m}.$$

Now, using equation (B), we get

$$V_A = \frac{\omega \times r \times AO}{BO} = \frac{22 \times 0.2 \times 1.15}{1.41} = 3.58 \text{ m/s.} \quad \text{Ans.}$$

Problem 10.38. Fig. 10.27 shows a pin-jointed four bar linkage having the following dimensions :

Fixed link $AD = 4 \text{ m}$,

Driving link $AB = 1.5 \text{ m}$

Driven link $CD = 2.5 \text{ m}$,

Connecting link $BC = 3 \text{ m}$

Angle $BAD = 60^\circ$

Link AB revolves at 25 r.p.m. in the clockwise direction. Determine :

(i) Angular velocity of link CD and

(ii) Angular velocity of link BC .

Sol. Given :

Length $AD = 4 \text{ m}$,

Length AB

$= 1.5 \text{ m}$

Length $CD = 2.5 \text{ m}$,

Length BC

$= 3 \text{ m}$

Angle $BAD = 60^\circ$,

r.p.m. of link AB

$= 25$

$\therefore$ Angular velocity of link AB ,

$$\omega_1 = \frac{2\pi N}{60} = \frac{2\pi \times 25}{60} = 2.619 \text{ rad/s.}$$

Let ω_1 = Angular velocity of link AB , rotating about A in the clockwise direction. The value of ω_1 is given.

ω_2 = Angular velocity of link CD , rotating about D . The value of ω_2 is to be calculated.

ω_0 = Angular velocity of the link BC . The value of ω_0 is also to be calculated.

V_B = Linear velocity of point B in the direction perpendicular to AB .
 $= \omega_1 \times AB \quad \dots(i)$

V_C = Linear velocity of point C in the direction perpendicular to CD .
 $= \omega_2 \times CD \quad \dots(ii)$

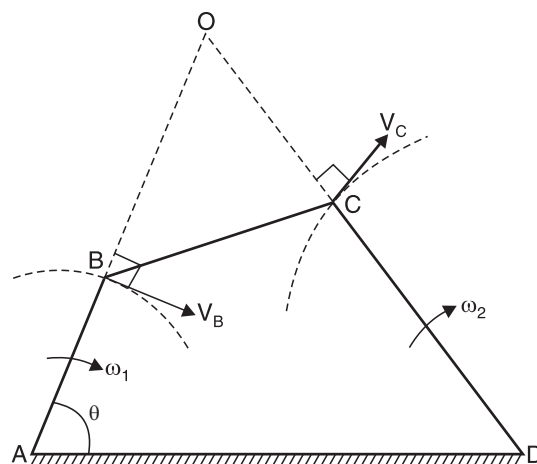


Fig. 10.27

The link AB and link CD are having motion of rotation, whereas the link BC is having motion of translation as well as rotation. The instantaneous centre of link BC for the given position is determined by drawing normals to the directions of velocity V_B and V_C . The normal to the direction V_B is the line AB whereas the normal to the direction V_C is line CD . Hence, produce line AB and CD . The intersection of these lines gives the point O (*i.e.*, instantaneous centre) for the link BC . The linear velocities of any point on link BC is obtained from instantaneous centre. All the points on link BC will have same angular velocity.

Then linear velocity at B from instantaneous centre is given by

$$V_B = \omega_0 \times BO \quad \dots(iii)$$

and linear velocity at C from instantaneous centre is given by

$$V_C = \omega_0 \times CO \quad \dots(iv)$$

Equating the velocity V_B given by equations (i) and (iii), we get

$$\omega_1 \times AB = \omega_0 \times BO$$

$$\therefore \omega_0 = \frac{\omega_1 \times AB}{BO} \quad \dots(A)$$

In equation (A), the values of ω_1 and AB are given. The value of BO is obtained from Fig. 10.27. Hence, ω_0 (angular velocity of link BC) can be determined. Again equating the velocity V_C given by equations (ii) and (iv), we get

$$\omega_2 \times CD = \omega_0 \times CO$$

$$\therefore \omega_2 = \frac{\omega_0 \times CO}{CD} \quad \dots(B)$$

In the above equation value of CD is given, value of CO is obtained from Fig. 10.27 and the value of ω_0 is known from equation (A). Hence, the value of ω_2 (angular velocity of link CD) can be determined.

Instantaneous Centre Method. Draw the link mechanism to a suitable scale as shown in Fig. 10.27 (a). The instantaneous centre O of link BC is obtained by producing line AB and DC . Now, measure the distance CO and BO from Fig. 10.27 (a).

By measurement, $CO = 3$ m and $BO = 4.65$ m.

First calculate the angular velocity of link BC .

Let ω_0 = angular velocity of link BC .

Using equation (A),

$$\begin{aligned} \omega_0 &= \frac{\omega_1 \times AB}{BO} = \frac{2.619 \times 1.5}{4.65} \text{ rad/s.} \\ &(\because AB = 1.5 \text{ m and } BO = 4.65) \\ &= \mathbf{0.845 \text{ rad/s. Ans.}} \end{aligned}$$

The angular velocity of link CD (*i.e.*, ω_2) is obtained by using equation (B).

$$\begin{aligned} \omega_2 &= \frac{\omega_0 \times CO}{CD} = \frac{0.845 \times 3.0}{2.5} \\ &= \mathbf{1.014 \text{ rad/s. Ans.}} \end{aligned}$$

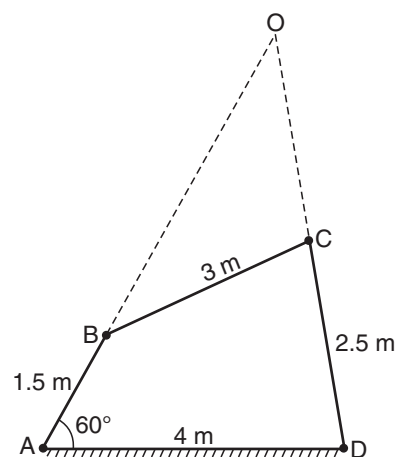


Fig. 10.27 (a)

HIGHLIGHTS

1. The motion of a body in a straight line is called linear motion.
2. The rate of change of displacement of a body is known as velocity. It is given by

$$v = \frac{s}{t} \quad \text{or} \quad \frac{ds}{dt}.$$

3. The rate of change of velocity is known as acceleration. It is represented by 'a'. The acceleration is given by

$$\begin{aligned} a &= \frac{dv}{dt} && \text{..... in terms of velocity} \\ &= \frac{d^2s}{dt^2} && \text{..... in terms of distance} \\ &= \frac{v dv}{ds} && \text{..... in terms of velocity and distance.} \end{aligned}$$

4. When a body is moving with a uniform acceleration, then equations of motions are :

$$(i) \quad v = u + at$$

$$(ii) \quad s = ut + \frac{1}{2} at^2$$

$$(iii) \quad v^2 - u^2 = 2as$$

where u = Initial velocity

v = Final velocity

a = Uniform acceleration

t = Time in seconds

s = Distance covered.

5. The distance travelled by a body moving with uniform acceleration in n th second

$$= u + \frac{a}{2} (2n - 1).$$

6. Equations of motion due to gravity in the downward direction and upward direction are given as :

For downward motion

$$v = u + gt$$

$$h = s = ut + \frac{1}{2} gt^2$$

$$v^2 - u^2 = 2gh$$

For upward motion

$$v = u - gt$$

$$h = ut - \frac{1}{2} gt^2$$

$$v^2 - u^2 = -2gh.$$

7. If a body starts from rest, its initial velocity is zero, i.e., $u = 0$ whereas if a body comes to rest, the final velocity of the body will be zero i.e., $v = 0$.
8. If a body is projected vertically upwards, the final velocity of the body at the highest point is zero.
9. If the equation of motion of a body moving in a straight line is given in terms of displacement (s) and time (t), the velocity of the body is obtained by differentiating the displacement with respect to time. And acceleration is obtained by differentiating the velocity with respect to time.
10. If the equation of motion of a body moving in a straight line is given in terms of acceleration (f) and time (t), then velocity is obtained by integrating the acceleration. The constant of integration for velocity is obtained from the given condition in terms of velocity and time.
The displacement is obtained by integrating the velocity. There will be another constant of integration. This constant of integration is also obtained from given condition in terms of displacement and time.
11. The rate of change of angular displacement is known as angular velocity. Angular displacement is measured in terms of θ . Angular velocity is represented by ω .

$$\therefore \quad \omega = \frac{d\theta}{dt}.$$

12. The relation between linear velocity (V) and angular velocity (ω) is given by $V = \omega \times r$.
13. The rate of change of angular velocity is known as angular acceleration. Angular acceleration is represented by α .

$$\therefore \alpha = \frac{d^2\theta}{dt^2} \quad \dots \text{in terms of } \theta$$

$$= \omega \frac{d\omega}{d\theta} \quad \dots \text{in terms of } \omega$$

14. The relation between linear acceleration (a) and angular acceleration (α) is given by

$$a = r \times \alpha.$$

15. The equations of motion along a circular path are given by

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

where ω_0 = Initial angular velocity in rad/s

ω = Final angular velocity

α = Angular acceleration

θ = Angular displacement.

16. The relation between angular velocity (ω) and r.p.m. (N) is given by

$$\omega = \frac{2\pi N}{60}.$$

17. If a body moves in such a way that all its particle move in parallel planes and travel the same distance, then the body is having motion of translation.
18. If a body rotates about a fixed point in such a way that all its particle move in circular path, the body is having motion of rotation.
19. If a body is having a combined motion of translation and rotation, then the body is assumed to be rotating about a certain point which is known as instantaneous centre of rotation.
20. The instantaneous centre of a link AB , having motion of translation and rotation, is determined if the directions of velocities at A and B are known. The perpendiculars to the directions of velocities at A and B are drawn. The point of intersection of two perpendiculars gives the instantaneous centre for the link AB .
21. If V_A and V_B are the velocities at points A and B of a link AB and AO and BO are the distances of the points A and B from the instantaneous centre, then we have

$$\frac{V_A}{V_B} = \frac{AO}{BO}.$$

EXERCISE 10

(A) Theoretical Questions

- What do you understand by the term kinematics ? Explain different types of plane motion of rigid bodies with suitable example. (U.P. Tech. University, May 2009)
- (a) What do you mean by linear motion ? Give some examples of linear motion.
(b) Distinguish between uniform motion and uniformly accelerated motion.

3. Acceleration can be expressed as

$$a = \frac{d^2s}{dt^2} \quad \text{and} \quad a = \frac{u dv}{ds}$$

Derive the above two expressions.

4. The distance travelled by a body moving with uniform acceleration in the n th second is equal to

(a) $un + \frac{1}{2}an^2$

(b) $2u + a(2n - 1)$

(c) $u + \frac{a}{2}(2n - 1)$

(d) $u + 2a(n - 1)$.

[Ans. (c)]

5. If a body is projected vertically upwards, then at the highest point the final velocity is

(a) maximum

(b) zero

(c) equal to 'g'

(d) none of the above.

[Ans. (b)]

6. The acceleration of a body is given by

(a) $\frac{dv}{ds}$

(b) $\frac{ds}{dt}$

(c) $\frac{s dv}{ds}$

(d) $\frac{v dv}{ds}$.

[Ans. (d)]

7. Define the terms : angular velocity and angular acceleration.

8. What is the relation between angular velocity and linear velocity ?

9. Prove that linear acceleration is equal to r times the angular acceleration, where r is the radius of rotation.

10. What do you mean by circular motion ? Give some examples of circular motion.

11. Derive the following equations of motion of a body moving along a circular path with uniform angular acceleration :

(i) $\omega = \omega_0 + \alpha t$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

(iii) $\omega^2 - \omega_0^2 = 2\alpha\theta$.

12. What is the relations between angular velocity and r.p.m. of a body ?

13. Define the terms : motion of translation, motion of rotation, combined motion of rotation and translation.

14. What do you mean by instantaneous centre of rotation ? How can it be located for a body moving with combined motion of rotation and translation ?
(AMIE Winter, 1982)

15. A link AB is having linear velocities at A and B as V_A and V_B . If the link is having a combined motion of rotation and translation, how will you determine the angular velocity of the link AB ?

16. Distinguish between the linear velocity of a point on a body rotating about a fixed axis and its angular velocity. Derive the relation between them.

(B) Numerical Problems

1. A body is moving with a velocity of 3 m/s. After five seconds the velocity of the body becomes 13 m/s. Find the acceleration of the body.
[Ans. 2 m/s²]

2. A car is moving with a velocity of 20 m/s. The car is brought to rest by applying brakes in 4 seconds. Determine : (i) the retardation and (ii) distance travelled by the car after applying brakes.
[Ans. - 5 m/s², 40 m/s]

3. A bullet, moving at the rate of 200 m/s, is fired into a log of wood. The bullet penetrates to a depth of 50 cm. If the bullet moving with the same velocity is fired into a similar piece of wood 25 cm thick, with what velocity would it emerge. Take the resistance to be uniform in both the case.
[Ans. 141.4 m/s]

4. A burglar's car had a start with an acceleration of 2 m/s^2 . A police vigilant party came after 5 seconds and continued to chase the burglar's car with a uniform velocity of 27 m/s . Find the time taken, in which the police van will overtake the car. [Ans. 5 s]
5. A body is moving with uniform acceleration and covers 20 m in 4th second and 30 m in 8th second. Determine : (i) the initial velocity of the body and (ii) acceleration of the body. [Ans. (i) 11.25 m/s , (ii) 2.5 m/s^2]
6. A tower is 100 m in height. A particle is dropped from the top of the tower and at the same time another particle is projected upward from the foot of the tower. Both the particles meet at a height of 40 m. Find the velocity with which the second particle is projected upward. [Ans. 28.59 m/s]
7. A stone dropped into a well is heard to strike the water after 6 seconds. Find the depth of the well, if the velocity of sound is 350 m/s . [Ans. 151.63 m]
8. A stone is thrown vertically upwards with a velocity of 29.40 m/s from the top of a tower 49 m high. Calculate : (i) time required for the stone to reach the ground, (ii) velocity of the stone in its downward travel at the point in the same level as the point of projection, and (iii) the maximum height to which the stone will rise in its flight. [Ans. (i) 7.358 s , (ii) 29.4 m/s , (iii) 44.1 m]
9. A stone is dropped from a height. After falling four seconds from rest, the stone breaks the glass pane and in breaking, the stone loses 25% of its velocity. Find the distance travelled by the stone in the next second. Take $g = 9.80 \text{ m/s}^2$. [Ans. 34.3 m/s]
10. A particle moves along a straight line so that its displacement is metre from a fixed point is given by, $s = 2t^3 + 4t^2 - 6t + 8$. Find : (i) velocity at start, (ii) velocity after 5 seconds, (iii) acceleration at start and (iv) acceleration after 5 seconds. [Ans. (i) 6 m/s , (ii) 184 m/s , (iii) 8 m/s^2 , (iv) 68 m/s^2]
11. The equation of motion of a particle moving in a straight line is given by

$$s = 15t + 3t^2 - t^3$$

where s is the distance covered from the starting point in metre at the end of t seconds. Find :

- (i) the velocity and acceleration at start,
- (ii) the time, when the particle reaches its maximum velocity, and
- (iii) the maximum velocity of the particle. [Ans. (i) 15 m/s , 6 m/s^2 , (ii) $t = 1 \text{ s}$, (iii) 18 m/s]
12. A body moves along a straight line and its acceleration (a) which varies with time (t) is given by : $a = 4 - 5t$. After 4 seconds from start of observations its velocity is observed to be 16 m/s . After 8 seconds from start of observations, the body was 70 m from the origin. (i) Determine its acceleration, velocity and distance from the origin at the start of observations, (ii) determine the time after start of observation in which the velocity becomes zero and its distance from the origin. [Ans. (i) 4 m/s^2 , 40 m/s , 48.67 m , (ii) 8.4 s , 31.87 m]
13. The velocity of a particle moving along x -axis is defined by $v = kx^3 - 4x^2 + 6x$ where v is in m/s and x in m and k is a constant. If $k = 2$, compute the acceleration when $x = 2 \text{ m}$. Also find the smallest value of k that will make acceleration = 16 m/s^2 at $x = 3 \text{ m}$. (Osmania, 1993)

[Hint. $a = \frac{v dv}{ds}$ or $= \frac{v dv}{dx}$. But $v = kx^3 - 4x^2 + 6x$ and $\frac{dv}{dx} = 3kx^2 - 8x + 6$

$$\therefore a = (kx^3 - 4x^2 + 6x)(3kx^2 - 8x + 6)$$

When $k = 2$ and $x = 2$ then $a = (2 \cdot 2^3 - 4 \cdot 2^2 + 6 \cdot 2)(3 \cdot 2 \cdot 2^2 - 8 \cdot 2 + 6)$ or $a = (16 - 16 + 12)(24 - 16 + 6) = 12 \times 14 = 168 \text{ m/s}^2$. **Ans.]**

14. A body is rotating with an angular velocity of 8 radian/s. After 5 seconds, the angular velocity of the body becomes 28 rad/s. Determine the angular acceleration of the body. [Ans. 4 rad/s^2]
15. A wheel, rotating about a fixed axis at 30 r.p.m. is uniformly accelerated for 50 seconds, during which time it makes 40 revolution. Find : (i) angular velocity at the end of this interval, and (ii) time required for the speed to reach 80 revolution per minute. [Ans. (i) 6.91 rad/s , (ii) 69.44 s]

16. A wheel is rotating about its axis with a constant acceleration of 2 rad/s^2 . If the initial and final angular velocities are 5 rad/s and 15 rad/s , determine the total angle turned through during the time interval this change of angular velocity took place. [Ans. 50 rad.]
17. (a) A flywheel starts rotating from rest and is given an acceleration of 1.5 rad/s^2 . Find the angular velocity and speed in r.p.m. after 60 seconds. [Ans. 90 rad/s, 859.4 r.p.m.]
 (b) If the flywheel is brought to rest with a uniform angular retardation of 0.5 rad/s^2 , determine the time taken by the flywheel in seconds to come to rest. [Ans. 3 Minute]
18. A flywheel is rotating at 150 r.p.m. and after 8 seconds it is rotating at 120 r.p.m. If the retardation is uniform, determine number of revolution made by the flywheel and the time taken by the flywheel before it comes to rest from the speed of 150 r.p.m. [Ans. 50 rev., 40 s]
19. The angle of rotation of a body is given by the equation $\theta = t^3 - 4t^2 + 10t + 5$ where θ is expressed in radians and t in seconds. Determine :
 (i) Angular velocity and (ii) angular acceleration of the body when $t = 0$ and $t = 5$ seconds. [Ans. (i) 10 rad/s, 45 rad/s, (ii) 6 rad/s², 22 rad/s²]
20. A link AB is moving in a vertical plane. At a certain instant, when the link is inclined at 60° to the horizontal, the point A is moving horizontally at 5 m/s , while B is moving vertically upwards. Find the velocity of B . [Ans. 2.88 m/s]
21. The crank of a reciprocating engine is rotating at 180 r.p.m. The lengths of the crank and connecting rod are 25 cm and 1.25 m respectively. Find the velocity of the piston, when crank has turned through an angle of 30° with the horizontal. [Ans. 2.766 m/s]
22. Find the acceleration of the piston of problem 21 at the instant when the angle $\theta = 30^\circ$. [Ans. 86.28 m/s²]
23. A four bar mechanism $ABCD$ has the following dimensions :
 Find link $AD = 5 \text{ m}$, Driving link $AB = 20 \text{ m}$
 Driven link $CD = 3.0 \text{ m}$, Connecting link $BC = 4.0 \text{ m}$
 Angle $BAD = 60^\circ$. Link AB rotates at 30 r.p.m. Determine :
 (i) Angular velocity of link CD , and
 (ii) Angular velocity of link BC . [Ans. (i) 0.91 rad/s, (ii) 1.44 rad/s]

11

Kinetics of Rigid Bodies

11.1. INTRODUCTION

Kinetics is that branch of Engineering Mechanics which deals with the force system which produces acceleration and resulting motion of bodies. In the previous chapter *i.e.*, chapter of Kinematics of Rigid Bodies, we have dealt with the motion of the bodies (*i.e.*, displacement, velocity and acceleration of Rigid bodies) without consideration of the forces which produce these motion.

11.2. FORCE, MASS AND ACCELERATION

11.2.1. Force. Force is defined as an agent which tends to change the state of rest of a body to which it is applied. For the complete definition of a force, we must know : (i) the magnitude of force, (ii) the point of application of force, and (iii) its direction. The magnitude of a force is obtained in S.I. units by a unit force, which is usually taken as the Newton. Newton is defined as a force which is acting on a mass of one kilogram and produces an acceleration of one metre per second square (*i.e.*, $1 \text{ N} = 1 \text{ kg} \times 1 \text{ m/s}^2$).

The point of application of a force acting on a body is that point in the body at which the force can be assumed to be concentrated. The direction of the force is the direction, along a straight line through its point of application, in which the force tends to move a body to which it is applied. This line is called the line of action of the force.

11.2.2. Mass. The quantity of matter combined in a body is known as the mass of the body. Mass is a scalar quantity. In C.G.S. units, mass is expressed in gramme (gm) whereas in S.I. units mass is expressed in kilogramme (kg).

11.2.3. Acceleration. It is defined as the rate of change of velocity of a body. It is measured in metre per second square and is written as m/s^2 in S.I. system.

$$\therefore \text{Acceleration} = \frac{\text{Change of velocity}}{\text{Time}} = \frac{dv}{dt}$$

$$\text{or} \quad a = \frac{d}{dt} \left(\frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \quad \left(\because v = \frac{ds}{dt} \right)$$

$$\text{Also} \quad \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v \quad \left(\because v = \frac{ds}{dt} = \text{rate of change of displacement} \right)$$

But $\frac{dv}{dt} = a = \text{acceleration.}$

$$\therefore a = \frac{dv}{ds} \cdot v = v \frac{dv}{ds}$$

11.2.4. Weight. Weight of a body is defined as the force, by which the body is attracted towards the centre of the Earth. Mathematically weight of a body is given by

$$\text{Weight} = \text{mass} \times \text{acceleration due to gravity} = \text{mass} \times g \quad \dots(11.1)$$

If mass is taken in kilogram (kg) and acceleration due to gravity in metre per second square (m/s^2), then weight is expressed in newton (N). But if mass is taken in gramme (gm) and acceleration due to gravity in centimetre per second square (cm/s^2), then weight is expressed in dyne. The relation between newton (N) and dyne is given as

$$\text{One Newton} = 10^5 \text{ dyne.}$$

11.2.5. Momentum. The product of the mass of a body and its velocity is known as momentum of the body. Momentum is a vector quantity. Mathematically, momentum is given by

$$\text{Momentum} = \text{Mass} \times \text{Velocity.}$$

11.3. NEWTON'S LAWS OF MOTION

When a body is at rest or moving in a straight line or rotating about an axis, the body obeys certain laws of motion. These laws are called Newton's laws of motion. There are three laws of motion. These laws for linear motions and angular motions are given in the next articles.

Newton's Law for Linear Motion

First Law. It states that a body continues in its state of rest or of uniform motion in a straight line unless it is compelled by an external force to change that state.

Second Law. It states that the rate of change of momentum of a body is proportional to the external force applied on the body and takes place in the direction of the force.

Third Law. It states that to every action, there is always an equal and opposite reaction.

11.3.1. Newton's First Law of Motion. It consists of two parts. First part states that a body continues in its state of rest unless it is compelled by an external force to change that state. A book lying on a table remains at rest, unless it is lifted by some external force.

Second part states that a body continues in its state of uniform motion in a straight line unless it is compelled by an external force to change that state. In actual practice, we see that when a body is moving with a uniform velocity in a straight line, the body does not continue in its state of uniform motion but comes to rest after some time. This is due to frictional force acting on the body. For an ideal case (*i.e.*, when there is no frictional force acting on the body), the body will continue to move with uniform velocity in a straight line, unless compelled by an external force to change that state.

11.3.2. Newton's Second Law of Motion. This law enables us to measure a force. Let a body of mass ' m ' is moving with a velocity ' u ' along a straight line. It is acted upon by a force F and the velocity of the body becomes v in time t . Then we have

u = Initial velocity of the body,

v = Final velocity of the body,

m = Mass of the body,

a = Uniform linear acceleration,

F = Force acting on the body, which changes the velocity u to v in time t ,

t = Time in second to change the velocity from u to v . Initial momentum of the body

$$= \text{mass} \times \text{initial velocity} = m \times u$$

$$\text{Final momentum of the body} = m \times v.$$

$\therefore$ Change of momentum

$$= \text{Final momentum} - \text{Initial momentum} = mv - mu = m(v - u)$$

$$\text{Rate of change of momentum} = \frac{\text{Change of momentum}}{\text{Time}} = \frac{m(v - u)}{t} \quad \dots(i)$$

But we know that,

$$\frac{v - u}{t} = a \quad (\text{i.e., linear acceleration})$$

Substituting the value of $\left(\frac{v - u}{t}\right)$ in equation (i), we get

$$\text{Rate of change of momentum} = m \times a.$$

But according to Newton's second law of motion, the rate of change of momentum is directly proportional to the external force acting on the body.

$$\therefore F \propto m \times a \quad \text{or} \quad F = k \times m \times a \quad \dots(ii)$$

where k is a constant of proportionality.

In equation (ii), k and m (mass of a body) are constants for a given body and hence force acting on a body is proportional to the acceleration produced by the force. This means that for a given body, greater force produces greater acceleration while a smaller force produces smaller acceleration. The acceleration produced will be zero if no force is applied on the body.

Two important conclusions are drawn from the first two Newton's laws of motion :

(i) There will be no acceleration, if no external force is applied on the body. This means the body will continue in its state of existing uniform motion in a straight line.

(ii) Force applied on the body is proportional to the product of mass of the body and the acceleration produced by the force.

11.3.3. Unit of Force. Let us first define a 'unit force'. A unit force can be suitably defined so as to make the value of k in equation (ii) equal to one. A unit force (i.e., Force = 1.0) is that which produces unit acceleration on an unit mass. Then by substituting $F = 1.0$, $m = 1.0$ and $a = 1.0$ in equation (ii) (i.e., $F = k \times m \times a$), we get

$$1 = k \times 1 \times 1 \quad \text{or} \quad k = 1.$$

Substituting the value of $k = 1$, in equation (ii), we get

$$F = m \times a \quad \dots(11.2)$$

(i) If mass (m) = 1 kg and acceleration produced (a) = 1 m/s² the unit of force is known as newton (which is written as N). Thus newton is defined as that force which acts on a body of mass one kg and produces an acceleration of 1 m/s² in the direction of force. Newton is the unit of force in S.I. system.

$$\therefore 1 \text{ N} = 1 \text{ kg} \times 1 \text{ m/s}^2 = 1 \text{ kg-m/s}^2.$$

(ii) If mass (m) = 1 gm and acceleration (a) = 1 cm/s², then unit of force is known as 'dyne'. Thus a dyne may be defined as that force which acts on a body of mass one gm and produces an acceleration of 1 cm/s². Dyne is the unit of force in C.G.S. system.

$$\therefore 1 \text{ dyne} = 1 \text{ gm} \times 1 \text{ cm/s}^2 = \text{gm cm/s}^2$$

By definition,

$$\begin{aligned} 1 \text{ N} &= 1 (\text{kg}) \times 1 (\text{m/s}^2) = 1 \times 1000 (\text{gm}) \times 1 \times 100 (\text{cm/s}^2) = 10^5 (\text{gm cm/s}^2) \\ &= 10^5 \text{ dyne} \quad (\because 1 \text{ dyne} = 1 \text{ gm} \times 1 \text{ cm/s}^2 = \text{gm cm/s}^2) \end{aligned}$$

Note (i) The body will have acceleration if the external force is acting on the body in the direction of motion of the body.

Note (ii) The body will have retardation if the external force is acting opposite to the direction of motion of the body.

Problem 11.1. A particle of mass 1 kg moves in a straight line under the influence of a force which increases linearly with time at the rate of 60 N per sec. At time $t = 0$, the initial force may be taken as 50 N. Determine the acceleration and velocity of the particle 4 sec after it started from rest at the origin. (U.P. Tech. University, 2000–2001)

Sol. Given :

Mass, $m = 1 \text{ kg}$; rate increase of force $\frac{dF}{dt} = 60 \text{ N/s}$

At $t = 0$, $F = 50 \text{ N}$, time = 4 sec.

Find : (i) velocity and (ii) acceleration.

Force is increasing linearly with time. Hence applied force on the particle is a function of time.

Let $F = At + B$... (i)

where A and B are constant.

When $t = 0$, $F = 50 \text{ N}$. Now equation (i) becomes,

$$50 = A \times 0 + B = B \quad \therefore B = 50 \text{ N}$$

To find the value of A , differentiate equation (i).

$$\therefore \frac{dF}{dt} = A + 0 \quad \left[\because B \text{ is constant, hence } \frac{dB}{dt} = 0 \right]$$

But $\frac{dF}{dt} = 60 \text{ N/s}$ (given) $\therefore A = 60 \text{ N/s}$

Substituting the values of A and B in equation (i),

$$F = 60t + 50 \quad \dots (ii)$$

(i) Velocity of particle after 4 sec

We know that $F = m \times a = m \times \frac{dV}{dt} \quad \left[\because a = \frac{dV}{dt} \right]$

Substituting this value of F in equation (ii),

$$m \times \frac{dV}{dt} = 60t + 50 \quad \dots (iii)$$

But mass, $m = 1 \text{ kg}$ (given). Hence the above equation becomes as

$$1 \times \frac{dV}{dt} = 60t + 50 \quad \text{or} \quad \frac{dV}{dt} = 60t + 50$$

Integrating w.r.t. time, we get

$$\int dV = \int (60t + 50) dt \quad \text{or} \quad V = \int_0^4 (60t + 50) dt$$

or

$$V = \left[\frac{60t^2}{2} + 50t \right]_0^4 = 30 \times 4^2 + 50 \times 4 = 480 + 200 = 680 \text{ m/s. Ans.}$$

(ii) Acceleration of particle after 4 sec

Equation (iii) is $m \times \frac{dV}{dt} = 60t + 50$ when $m = 1$ kg, this equation becomes as

$$\frac{dV}{dt} = 60t + 50 \quad \text{But} \quad \frac{dV}{dt} = a = \text{acceleration}$$

$$\therefore a = 60t + 50 \quad \therefore \text{Acceleration (a) after 4 sec,}$$

$$a = 60 \times 4 + 50 = \mathbf{290 \text{ m/s}^2}. \quad \text{Ans.}$$

Problem 11.2. A bullet of mass 81 gm and moving with a velocity of 300 m/s is fired into a log of wood and it penetrates to a depth of 10 cm. If the bullet moving with the same velocity, were fired into a similar piece of wood 5 cm thick, with what velocity would it emerge? Find also the force of resistance, assuming it to be uniform.

Sol. Given :

$$\text{Mass of bullet,} \quad m = 81 \text{ gm} = \frac{81}{1000} \text{ kg} = 0.081 \text{ kg}$$

$$\text{Initial velocity of bullet,} \quad u = 300 \text{ m/s}$$

$$\text{Distance travelled,} \quad s = 10 \text{ cm} = 0.10 \text{ m}$$

$$\text{Final velocity,} \quad v = 0.$$

As the force of resistance is acting in the opposite direction of motion of bullet, hence force of resistance will produce retardation on the bullet.

$$\text{Let} \quad a = \text{retardation}$$

Now, using the relation,

$$v^2 - u^2 = -2as \quad (-\text{ve sign is taken due to retardation})$$

$$\text{or} \quad 0^2 - 300^2 = -2 \times a \times 0.1$$

$$\therefore a = \frac{300 \times 300}{2 \times 0.1} = 450000 \text{ m/s}^2.$$

Let F = Force of resistance offered by wood to the bullet.

Using equation (11.2), we get

$$F^* = m \times a = 0.081 \times 450000 = \mathbf{36450 \text{ N.}} \quad \text{Ans.}$$

Velocity of the bullet with which the bullet will come out from a piece of wood of 5 cm thick.

Let v = velocity with which the bullet emerges from the piece of wood of 5 cm thick.

$$\text{Initial velocity,} \quad u = 300 \text{ m/s.}$$

As the resistance offered by wood is uniform, hence retardation will be same as before.

$$\therefore a = 450000 \text{ m/s}^2.$$

$$\text{Distance travelled,} \quad r = 5 \text{ cm} = .05 \text{ m}$$

Using the relation,

$$v^2 - u^2 = -2as \quad (-\text{ve sign is taken due to retardation})$$

$$\therefore v^2 - 300^2 = -2 \times 450000 \times .05 \quad \text{or} \quad v^2 = 300 \times 300 - 45000$$

$$= 90000 - 45000 = 45000$$

$$\therefore v = \sqrt{45000} = \mathbf{212.132 \text{ m/s.}} \quad \text{Ans.}$$

Problem 11.3. A car, moving on a straight level road, skidded for a total distance of 60 metres after the brakes were applied. Determine the speed of the car, just before the brakes were applied, if the co-efficient of friction between the car tyres and the road is 0.4. Take $g = 9.80 \text{ m/s}^2$.

Sol. Given :

Let u = velocity of car just before applying the brakes.

Final velocity of car, $v = 0$

Distance travelled, $s = 60 \text{ m}$

Co-efficient of friction between car tyres and road,

$$\mu = 0.4$$

Let W = Weight of car in Newton

R = Normal reaction

$$\therefore \text{Mass of car } m = \frac{W}{g}$$

Now, frictional resistance

$$= \mu R = \mu W \quad (\because R = W)$$

$$= 0.4 W \text{ Newton}$$

$$\therefore F = 0.4 W.$$

As frictional force is acting in the opposite direction of motion, hence the frictional force will produce retardation.

Using equation (11.2),

$$\text{Force} = \text{mass} \times a \quad \dots(i)$$

$$\therefore 0.4 W = \frac{W}{g} \times a \quad \text{or} \quad a = \frac{0.4 W \times g}{W} = 0.4 \times 9.80 = 3.92 \text{ m/s}^2$$

$$\therefore \text{Retardation} = 3.92 \text{ m/s}^2.$$

Now, using the relation,

$$v^2 - u^2 = -2as \quad (-\text{ve sign is due to retardation})$$

$$\therefore 0^2 - u^2 = -2 \times 3.92 \times 60$$

$$\therefore u = \sqrt{2 \times 3.92 \times 60} = \sqrt{470.4} = 21.688 \text{ m/s}$$

$$= \frac{21.688}{1000} \times 60 \times 60 \text{ km/hour} = \mathbf{78.077 \text{ km/hour. Ans.}}$$

Problem 11.4. The tractive force, exerted by a railway car weighing 50 kN, is 2000 N. If the frictional resistance is 5 N per kN of the railway car's weight, determine the acceleration when the railway car is moving on a level track.

Sol. Given :

Tractive force exerted by railway car,

$$F_1 = 2000 \text{ N}$$

Weight of car, $W = 50 \text{ kN} = 50 \times 1000 \text{ N}$

$$\therefore \text{Mass of car, } m = \frac{W}{g} = \frac{50 \times 1000}{9.81} \text{ kg}$$

Frictional resistance, $F_2 = 5 \text{ N per kN of car's weight}$

$$= 5 \text{ N} \times \text{weight of car in kN} = 5 \times 50 \text{ N} = 250 \text{ N.}$$

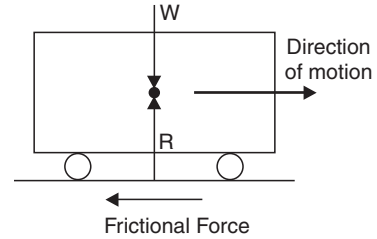


Fig. 11.1

The tractive force is acting in the direction of motion, while frictional resistance is acting in opposite direction of motion.

∴ Net force in the direction of motion,

$$F = F_1 - F_2 = 2000 - 250 = 1750 \text{ N.}$$

As the net force is acting in the direction of motion, it will produce acceleration.

Let a = acceleration produced

Using equation (11.2), we have

$$F = m \times a$$

$$\text{or} \quad 1750 = \frac{50 \times 1000}{9.81} \times a \quad \left(\because m = \frac{50 \times 1000}{9.81} \right)$$

$$\therefore a = \frac{1750 \times 9.81}{50 \times 1000} = 0.343 \text{ m/s}^2. \quad \text{Ans.}$$

11.4. D' ALEMBERT'S PRINCIPLE AND DYNAMIC EQUILIBRIUM

11.4.1. D' Alembert's Principle Applicable to Plane Motion. Let a body of mass ' m ' be moving with a uniform acceleration ' a ' under the action of external force F (also known as effective force). Then according to Newton's second law of motion, we have

$$F = m \times a \quad \dots(i)$$

The equation (i) can also be written as

$$F - ma = 0 \quad \dots(ii)$$

From equation (ii), it is clear that by applying a force $-ma$ on the body, the body will be in equilibrium as the sum of all forces acting on the body is zero. Such equilibrium is called *dynamic equilibrium* and the force $-ma$ is called **D' Alembert force** or *inertia force*.

The body will be in dynamic equilibrium under the action of external force F and the inertia force. This is known as D' Alembert's Principle.

Hence D' Alembert's Principle states that the net external forces acting on the system and the resultant inertia force are in equilibrium. This is D' Alembert's principle applicable to plane motion.

The inertia force on a body = mass of the body $\times$ acceleration.

Inertia force acts in the opposite direction of motion of the body and passes through the C.G. of the body. The above method is considerably quicker for problems where a number of bodies are connected as this method does not require the consideration of tension in the string.

To illustrate this principle, which is applicable to plane motion consider the two following cases :

1. *Analysis of the motion of two bodies connected by a string.* Two bodies of weights W_1 and W_2 are connected to the two ends of a light inextensible string passing over a smooth pulley as shown in Fig. 11.1(a). The weight W_1 is more than weight W_2 .

Let a = Acceleration of the system.

Now, the net external force acting on the system in the direction of motion

$$= (W_1 - W_2). \quad \dots(i)$$

$$\begin{aligned}
 &\text{Inertia force on weight } W_1 \\
 &= - \text{Mass of } W_1 \times \text{Acceleration} \\
 &= - \frac{W_1}{g} \times a \quad \left(\because \text{Mass of } W_1 = \frac{W_1}{g} \right)
 \end{aligned}$$

$$\begin{aligned}
 &\text{Inertia force on weight } W_2 \\
 &= - \text{Mass of } W_2 \times \text{Acceleration} \\
 &= - \frac{W_2}{g} \times a
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Resultant inertia force} \\
 &= - \frac{W_1}{g} \times a + \left(- \frac{W_2}{g} \times a \right) = - \frac{a}{g} [W_1 + W_2].
 \end{aligned}$$

But according to D'Alembert's Principle the net external force acting on the system and the resultant inertia force should be in equilibrium.

$$\therefore \text{Net external force} + \text{resultant inertia force} = 0$$

$$\text{or} \quad (W_1 - W_2) - \frac{a}{g} (W_1 + W_2) = 0$$

$$\text{or} \quad (W_1 - W_2) = \frac{a}{g} (W_1 + W_2)$$

$$\therefore \quad a = \frac{g(W_1 - W_2)}{(W_1 + W_2)}.$$

2. Analysis of motion of two bodies connected by a string when one body is lying on a smooth horizontal surface and other is hanging free.

Let a = Acceleration of the system.

Net external force acting on the system shown in Fig. 11.1(b) in the direction of motion = W_1 .

(Kindly note here, the weight W_2 is not acting in the direction of motion).

$$\begin{aligned}
 &\text{Inertia force on weight } W_1 \\
 &= - (\text{Mass of } W_1) \times (\text{Acceleration}) \\
 &= - \frac{W_1}{g} \times a \quad \left(\because \text{Mass} = \frac{W_1}{g} \right)
 \end{aligned}$$

$$\begin{aligned}
 &\text{Inertia force on weight } W_2 \\
 &= - (\text{Mass of } W_2) \times \text{Acceleration} \\
 &= - \frac{W_2}{g} \times a.
 \end{aligned}$$

(This force will be acting on W_2 in the opposite direction of motion of W_2 i.e., this force acts in the horizontal direction towards left of W_2 and will pass through the C.G. of the body of weight W_2).

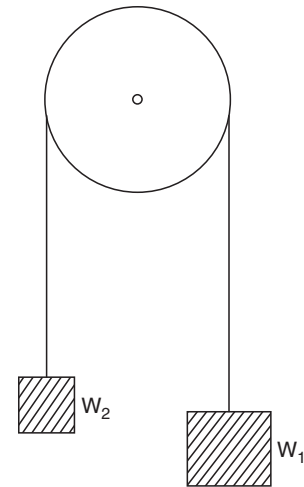


Fig. 11.1(a)

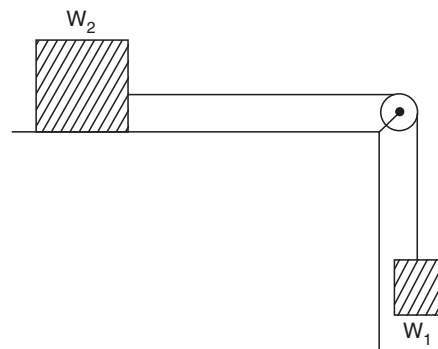


Fig. 11.1(b)

∴ Resultant inertia force

$$= -\frac{W_1}{g} a + \left(-\frac{W_2}{g} a \right) = -\frac{a}{g} [W_1 + W_2].$$

But according to D'Alembert's Principle, the net external force acting on the system in the direction of motion and the resultant inertia forces should be in equilibrium.

∴ Net external force + resultant inertia force = 0.

or
$$W_1 + \left[-\frac{a}{g} (W_1 + W_2) \right] = 0$$

or
$$W_1 = \frac{a}{g} [W_1 + W_2] \quad \therefore \quad a = \frac{gW_1}{(W_1 + W_2)}.$$

Problem 11.5. Two weights 800 N and 200 N are connected by a thread and move along a rough horizontal plane under the action of a force 400 N applied to the first weight of 800 N as shown in Fig. 11.1(c). The co-efficient of friction between the sliding surfaces of the weights and the plane is 0.3. Determine the acceleration of the weights and the tension in the thread using D'Alembert's principle. (U.P.Tech. University, May 2009)

Sol. Given :

Weights, $W_1 = 200 \text{ N}$

$W_2 = 800 \text{ N}$

Force applied, $F = 400 \text{ N}$

Co-efficient of friction, $\mu = 0.3$

Find : (i) Acceleration of weights

(ii) Tension in the threads

Let a = Acceleration of weights and

T = Tension in the threads.

Total weight of the system = $W_1 + W_2 = 200 + 800 = 1000 \text{ N}$

Total mass of the system = $\frac{\text{Total weight}}{g} = \frac{1000}{g} \text{ kg}$

∴ Inertia force on the system = $-(\text{Mass of system}) \times \text{acceleration of the system}$

$$= -\left(\frac{1000}{g} \right) \times a \text{ Newton.}$$

Force of friction on the system = $\mu \times \text{Total weight}$

$$= 0.3 \times (1000) = 300 \text{ N (} \leftarrow \text{)}$$

This is also acting in the backward direction.

The free body diagram of the system is shown in Fig. 11.1(d).

By D'Alembert principle, the sum of the inertia force and the net external forces on the system is zero.

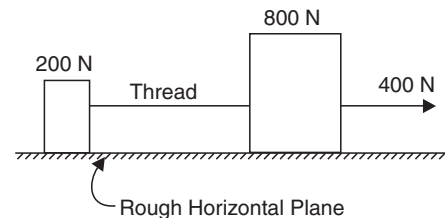


Fig. 11.1(c)

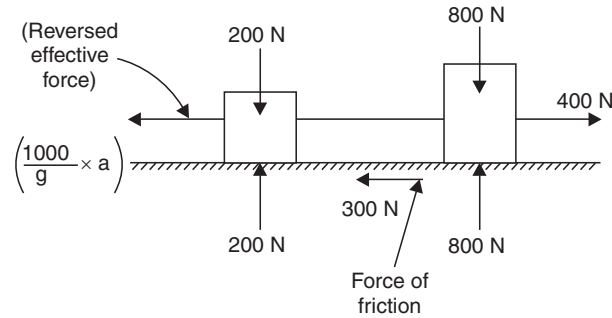


Fig. 11.1(d)

Force applied – Force of friction + Inertia force = 0

$$\therefore 400 - 300 - \frac{1000}{g} \times a = 0 \quad \text{or} \quad 100 = \frac{1000 \times a}{g}$$

or
$$a = \frac{100 \times g}{1000} = \frac{g}{10} = \frac{9.80}{10} = 0.98 \text{ m/s}^2. \quad \text{Ans.}$$

Tension in the thread

Let T = Tension in the thread

Consider the motion of 200 N weight.

The free body diagram of 200 N weight is shown in Fig. 11.1(e).

$$\text{Force of friction} = \mu \times 200 = 0.3 \times 200 = 60 \text{ N}$$

$$\text{Inertia force on it} = \text{Mass} \times \text{Acceleration}$$

$$= \left(\frac{200}{g} \right) \times a = \frac{200}{g} \times \frac{g}{10} = 20 \text{ N} \quad \left(\frac{200}{g} \times a \right)$$

$$\text{By D' Alembert's principle, } T - 60 - 20 = 0$$

$$\therefore T = 60 + 20 = 80 \text{ N.} \quad \text{Ans.}$$

11.4.2. D' Alembert's Principle Applicable to Rotary Motion. It states that when external torques (also called active torques) acts on a system having rotating motion, then the algebraic sum of all the torques acting on the system due to external forces and reversed active forces including the inertia torques (taken in the opposite direction of the angular acceleration) is zero.

To illustrate the above principle, consider the two following cases :

1. Where Rotation due to a weight W attached to one end of a string, passing over a pulley of weight W_0 .

A weight W is attached to one end of an inextensible string, which passes over a pulley of weight W_0 , the other end of the string is attached to the periphery of the pulley as shown in Fig. 11.1(f). When the weight W moves downwards, the rotation to the pulley is caused in clockwise direction.

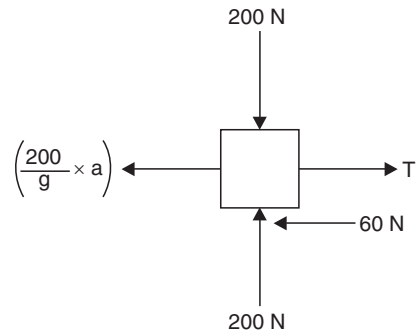


Fig. 11.1(e)

Let a = Linear acceleration of the weight W .

α = Angular acceleration of the pulley

R = Radius of the pulley

I = Moment of inertia of the pulley about the axis of rotation.

But we know

Linear acceleration = Angular acceleration $\times$ radius.

$$\therefore a = \alpha \times R$$

Inertia force on weight W

$$= - (\text{Mass of } W) \times \text{Acceleration}$$

$$= - \left(\frac{W}{g} \right) \times a = - \left(\frac{W}{g} \right) \times \alpha \times R$$

$$(\because a = \alpha \times R)$$

Now torque due to external force = $W \times R$.

Torque due to inertia force = inertia $\times R$

$$= \left(- \frac{W}{g} \times \alpha \times R \right) \times R = - \frac{W}{g} \times \alpha \times R^2.$$

Torque on the pulley due to angular acceleration

$$= I \times \alpha.$$

$\therefore$ Inertia torque, taken in opposite direction of the angular acceleration

$$= - I \times \alpha.$$

Then according to D' Alembert's Principle, the algebraic sum of all the torques acting on the system due to external forces and reversed active forces including inertia torques (taken in the opposite direction of the angular acceleration) should be zero.

$$\therefore W \times R - \frac{W}{g} \times \alpha \times R^2 - I \times \alpha = 0$$

$$\text{or } W \times R = \frac{W}{g} \times \alpha \times R^2 + I \times \alpha = \alpha \left[\frac{W}{g} R^2 + I \right]$$

$$\therefore \alpha = \left(\frac{W \times R}{\frac{W}{g} R^2 + I} \right) \quad \dots(i)$$

If linear acceleration ' a ' is to be determined, then

$$\alpha = \frac{a}{R}.$$

$\therefore$ Equation (i) becomes as

$$\frac{a}{R} = \frac{W \times R}{\frac{W}{g} R^2 + I}$$

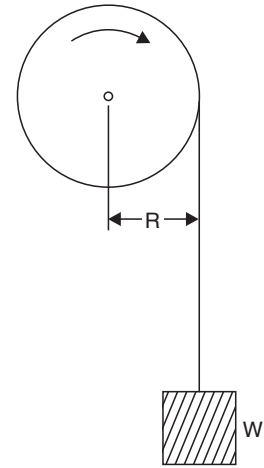


Fig. 11.1(f)

$$\therefore a = \frac{WR^2}{\frac{W}{g}R^2 + I} \quad \dots(ii)$$

If the pulley is considered to be a solid disc, then moment of inertia of the pulley is given by equation (11.10) as

$$I = \frac{MR^2}{2}, \text{ where } M = \text{Mass of pulley} = \left(\frac{W_0}{g}\right) = \left(\frac{W_0}{g}\right) \frac{R^2}{2}.$$

Substituting this value in equation (ii), we get

$$a = \frac{WR^2}{\frac{W}{g}R^2 + \frac{W_0}{g} \frac{R^2}{2}} = \frac{gW}{\left(W + \frac{W_0}{2}\right)}$$

2. *Rotation due to weights attached to the two ends of a string, which passes over a rough pulley of weight W_0 .*

Let R = Radius of pulley

a = Linear acceleration of the system

α = Angular acceleration of the pulley

W_1, W_2 = Two weights attached

W_0 = Weight of pulley

I = Moment of inertia of the pulley

$$= \frac{MR^2}{2} \text{ if pulley is considered a solid disc}$$

$$= \frac{W_0}{g} \frac{R^2}{2} \quad \left(\because \text{Mass of pulley} = \frac{\text{Weight of pulley}}{g} \right)$$

Consider $W_1 > W_2$, so that the rotation of the pulley is caused in clockwise direction.

Resultant torque caused by the weights

$$= W_1 \times R - W_2 R = (W_1 - W_2) \times R.$$

Inertia force on weight W_1

$$= -(\text{Mass of } W_1) \times \text{Acceleration} = -\frac{W_1}{g} \times a.$$

$\therefore$ Torque due to inertia force on weight W_1

$$= -\frac{W_1}{g} \times a \times R$$

Inertia force on weight W_2

$$= -\frac{W_2}{g} \times a$$

$\therefore$ Torque due to inertia force on weight W_2

$$= -\frac{W_2}{g} \times a \times R.$$

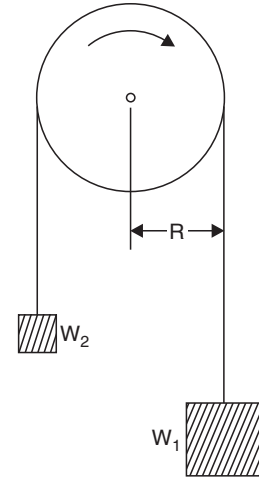


Fig. 11.1(g)

Torque due to angular acceleration on pulley = $I \times \alpha$

$\therefore$ Inertia torque taken in opposite direction = $-I \times \alpha$.

Now, according to D' Alembert's Principle, the algebraic sum of all the torques acting on the system due to external forces and inertia forces including inertia torques (taken in opposite direction) should be zero.

$$\therefore (W_1 - W_2) \times R - \frac{W_1}{g} \times a \times R - \frac{W_2}{g} \times a \times R - I \times \alpha = 0$$

$$\text{or } (W_1 - W_2) \times R - \frac{W_1}{g} \times a \times R - \frac{W_2}{g} \times a \times R - I \times \frac{a}{R} = 0 \quad \left(\because \alpha = \frac{a}{R} \right)$$

$$\text{and } I = \frac{W_0}{g} \frac{R^2}{2}.$$

11.5. LAWS OF MOTION APPLIED TO PLANER TRANSLATION

We shall now consider the kinetics of the rigid bodies for the following cases :

- (i) Motion on an inclined surface
- (ii) Motion on an inclined rough surface,
- (iii) Analysis of lift motion,
- (iv) Analysis of the motion of two bodies connected by a string,
- (v) Analysis of the motion of two bodies connected by a string when one body is lying on a horizontal surface and other is hanging free.
- (vi) Analysis of the motion of two bodies connected by a string when one body is lying on inclined plane and other is hanging free in air.

11.5.1. Motion on an Inclined Smooth Surface. Fig. 11.2 shows a body of weight W , sliding down on a smooth inclined plane.

Let θ = Angle made by inclined plane with horizontal

W = Weight of the body

a = Acceleration of the body

$$m = \text{Mass of the body} = \frac{\text{Weight}}{g}$$

$$= \frac{W}{g}.$$

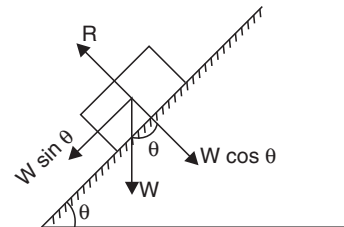


Fig. 11.2

As the surface of the plane is smooth, hence the frictional force will be zero. Hence the forces acting on the body are its own weight W and reaction R of the plane. The resolved part of W perpendicular to the plane is $W \cos \theta$, which is balanced by R , while the resolved part parallel to the plane is $W \sin \theta$, which produces acceleration down the plane. This force is responsible for the movement of the body down the plane.

$\therefore$ Net force acting on the body down the pane is

$$F = W \sin \theta$$

Now using the equation (11.2), we have

$$F = m \times a.$$

Substituting the values of F and m in the above equation, we get

$$W \sin \theta = \frac{W}{g} \times a$$

$$\therefore a = g \sin \theta \quad \dots(11.3)$$

If the body is moving up the plane, the corresponding acceleration will be $-g \sin \theta$.

11.5.2. Motion on an Inclined Rough Surface. Fig. 11.3 shows a body of weight W , sliding down the rough inclined surface.

Let a = Acceleration of the body

$$m = \text{mass of the body} = \frac{W}{g}$$

θ = Inclination of the plane with horizontal

μ = Co-efficient of friction

F_1 = Force of friction.

As the body is moving down the plane, the force of friction will be acting up the plane as shown in Fig. 11.3.

$$\begin{aligned} \text{Now force of friction, } F_1 &= \mu R \\ &= \mu \times W \cos \theta \end{aligned}$$

$$(\because R = W \cos \theta)$$

Force acting down the plane, $F_2 = W \sin \theta$

$\therefore$ Net force acting on the body down the plane,

$$F = F_2 - F_1 = W \sin \theta - \mu W \cos \theta.$$

Now using the equation (11.2), $F = m \times a$

$$\text{or } (W \sin \theta - \mu W \cos \theta) = \frac{W}{g} \times a \quad \left(\because m = \frac{W}{g} \right)$$

$$\text{or } W (\sin \theta - \mu \cos \theta) = \frac{W}{g} \times a \quad \text{or } a = W (\sin \theta - \mu \cos \theta) \times \frac{g}{W}$$

$$\therefore a = g(\sin \theta - \mu \cos \theta). \quad \dots(11.4)$$

Problem 11.6. A body of weight 200 N is initially stationary on a 45° inclined plane. What distance along the inclined plane must the body slide, before it reaches a speed of 2 m/s? The co-efficient of friction between the body and the plane = 0.1.

Sol. Given : (Refer to Fig. 11.4).

Weight of body, $W = 200 \text{ N}$

$$\therefore \text{Mass of body, } m = \frac{W}{g} = \frac{200}{9.81} \text{ kg}$$

Angle of plane, $\theta = 45^\circ$

Initial velocity, $u = 0$

Final velocity, $v = 2 \text{ m/s}$

Co-efficient of friction, $\mu = 0.1$.

$$R = 200 \cos 45^\circ$$

$$\begin{aligned} F_1 &= \mu R \\ &= 0.1 \times 200 \cos 45^\circ \end{aligned}$$

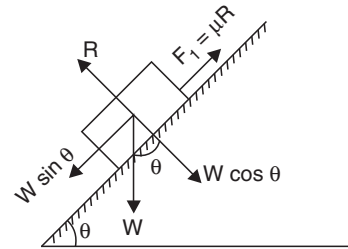


Fig. 11.3

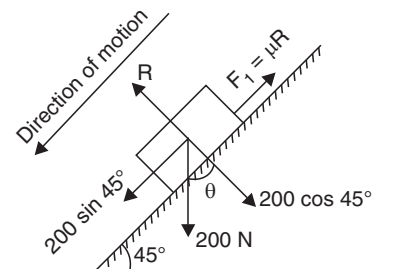


Fig. 11.4

$$\begin{aligned}\text{Net force down the plane} &= (200 \sin 45^\circ - 0.1 \times 200 \cos 45^\circ) \\ &= 200 (\sin 45^\circ - 0.1 \times \cos 45^\circ)\end{aligned}$$

$$\text{Net force} = \text{Mass} \times \text{Acceleration}$$

$$200 (\sin 45^\circ - 0.1 \times \cos 45^\circ) = \frac{200}{9.81} \times a$$

$$\begin{aligned}\therefore a &= g[\sin \theta - \mu \cos \theta] = 9.81[\sin 45^\circ - 0.1 \cos 45^\circ] \\ &= 9.81[0.707 - 0.1 \times 0.707] = 6.242 \text{ m/s}^2.\end{aligned}$$

Now using the relation,

$$v^2 - u^2 = 2as \quad \text{or} \quad 2^2 - 0^2 = 2 \times 6.242 \times s$$

$$\therefore s = \frac{2 \times 2}{2 \times 6.242} = 0.32 \text{ m} = \mathbf{32 \text{ cm.} \quad \text{Ans.}}$$

Problem 11.7. Two bodies directly in line and 10 m apart are held stationary on an inclined plane having inclination of 20° . The co-efficient of friction between the plane and lower body is 0.08 and that between the plane and the upper body is 0.05. If both the bodies are set in motion at the same instant, calculate the distance through which each body travels before they meet together.

Sol. Given :

Initial velocity, $u = 0$

Angle of inclined plane, $\theta = 20^\circ$

Co-efficient of friction between the plane and lower body, $\mu_1 = 0.08$.

Co-efficient of friction between the plane and upper body, $\mu_2 = 0.05$.

Distance between two bodies, $d = 10 \text{ m}$.

Let the two bodies meet after ' t ' seconds.

If s_1 = Distance travelled by lower body in time ' t ', and

s_2 = Distance travelled by upper body in time ' t '.

Then $s_2 = s_1 + 10$

...(i)

Let a_1 = Acceleration of the lower body down the plane, and

a_2 = Acceleration of the upper body down the plane.

(i) Consider the motion of lower body

Net force down the plane = Mass $\times$ Acceleration

$$W_1 \sin 20^\circ - \mu_1 \times W_1 \cos 20^\circ = \frac{W_1}{g} \times a_1$$

$\therefore$ Acceleration of lower body,

$$\begin{aligned}a_1 &= g(\sin 20^\circ - \mu_1 \cos 20^\circ) \\ &= 9.81(\sin 20^\circ - 0.08 \times \cos 20^\circ) \quad (\because \mu_1 = 0.08) \\ &= 9.81(0.342 - 0.08 \times 0.9397) \\ &= 9.81(0.342 - 0.0752) = 2.617 \text{ m/s}^2.\end{aligned}$$

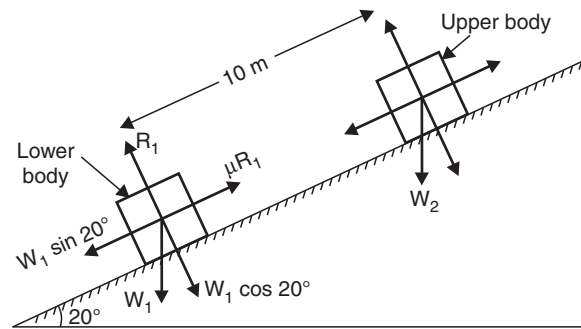


Fig. 11.5

(ii) Consider the motion of upper body

Net force down the plane = Mass $\times$ a_2

$$\text{or} \quad (W_2 \sin 20^\circ - \mu_2 \times W_2 \cos 20^\circ) = \frac{W_2}{g} \times a_2$$

$$\text{or} \quad W_2 (\sin 20^\circ - \mu_2 \times \cos 20^\circ) = \frac{W_2}{g} \times a_2$$

Acceleration of upper body,

$$\begin{aligned} a_2 &= g(\sin 20^\circ - \mu_2 \cos 20^\circ) \\ &= 9.81 (\sin 20^\circ - 0.05 \cos 20^\circ) \quad (\because \mu_2 = 0.05) \\ &= 9.81 (0.342 - 0.05 \times .9397) \\ &= 9.81 (0.342 - 0.04698) = 2.894 \text{ m/s}^2. \end{aligned}$$

The distance travelled by the body is given by

$$s = ut + \frac{1}{2}at^2$$

$\therefore$ Distance travelled by the lower body is given by

$$\begin{aligned} s_1 &= ut + \frac{1}{2}a_1t^2 \\ &= 0 \times t + \frac{1}{2} \times 2.617 \times t^2 \quad (\because u = 0 \text{ and } a_1 = 2.617) \\ &= \frac{1}{2} \times 2.617 \times t^2 = 1.3085 t^2 \quad \dots(ii) \end{aligned}$$

Distance travelled by the upper body is given by

$$\begin{aligned} s_2 &= ut + \frac{1}{2}a_2t^2 = 0 \times t + \frac{1}{2} \times 2.894 \times t^2 \\ &= \frac{1}{2} \times 2.894 \times t^2 = 1.447 t^2 \quad \dots(iii) \end{aligned}$$

Substituting the values of s_1 and s_2 in equation (i), we get

$$1.447 t^2 = 1.3085 t^2 + 10$$

$$\text{or} \quad 1.447 t^2 - 1.3085 t^2 = 10 \quad \text{or} \quad 0.1385 t^2 = 10$$

$$\therefore \quad t = \sqrt{\frac{10}{.1385}} = \sqrt{72.202} = 8.497 \text{ s.}$$

Substituting the value of t in equations (ii) and (iii), we get

$$s_1 = 1.3085 \times 8.497^2 = \mathbf{94.476 \text{ m.} \quad \text{Ans.}}$$

and

$$s_2 = 1.447 \times 8.497^2 = \mathbf{104.476 \text{ m.} \quad \text{Ans.}}$$

Problem 11.7A. A truck weighing 6 kN just moves freely (engine is not running) at 36 kilometre per hour down a slope of 1 in 40, the road resistance at this speed just being sufficient to prevent any acceleration. Find the road resistance per kN weight of truck.

What power will the engine have to exert to run up the same slope at double the speed when the road resistance remains the same ?

Sol. Given :

$$\begin{aligned} \text{Weight of truck,} \quad W &= 6 \text{ kN} = 6 \times 1000 \text{ N} \\ &= 6000 \text{ N} \end{aligned}$$

$$\begin{aligned} \text{Speed of truck,} \quad u &= 36 \text{ km/hr} \\ &= \frac{36 \times 1000}{60 \times 60} = 10 \text{ m/s} \end{aligned}$$

$$\text{Slope of the road} = 1 \text{ in } 40$$

$$\therefore \quad \sin \theta = \frac{1}{40}$$

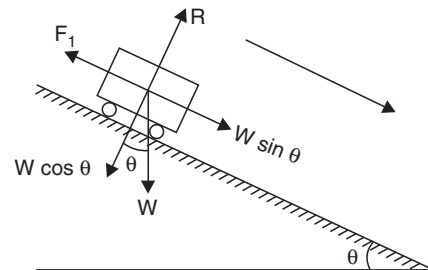


Fig. 11.6

1st Case. Fig. 11.6 shows the position of the truck, when it is moving down the plane. The road resistance (F_1) is acting in the opposite direction of the motion of the truck. The truck is not having any acceleration and hence it is moving with a constant velocity of 10 m/s. Hence the net force on the truck in the direction of motion should be zero. But net force on the truck in the direction of motion (See Fig. 11.6).

$$\begin{aligned} &= W \sin \theta - F_1 \\ &= 6000 \times \frac{1}{40} - F_1 \quad \left\{ \because W = 6000 \text{ N}, \sin \theta = \frac{1}{40} \right\} \end{aligned}$$

But the net force should be zero as truck is moving with uniform velocity.

$$\therefore 6000 \times \frac{1}{40} - F_1 = 0$$

$$\therefore F_1 = 6000 \times \frac{1}{40} = \frac{6000}{40} = 150 \text{ N}$$

$\therefore$ Frictional force per kN weight of truck

$$= \frac{F_1}{\text{Weight of truck in kN}} = \frac{150}{6} = 25 \text{ N. Ans.}$$

2nd Case. Truck is moving up an inclined plane of slope 1 in 40 with double speed. Road resistance is same.

$$\therefore \text{Speed of truck} = 2u = 2 \times 10 = 20 \text{ m/s}$$

$$\sin \theta = \frac{1}{40}.$$

$$\text{Road resistance} = F_1 = 150 \text{ N.}$$

The truck is moving at constant speed of 20 m/s up the plane. The truck can move upwards only when engine is running and exerts force on the truck upwards.

Let P = force exerted by engine up the plane.

As the truck is moving with uniform speed, the net force on the truck along the plane should be zero (See Fig. 11.7). But net force on the truck in the upward direction

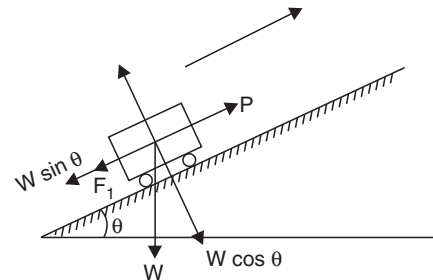


Fig. 11.7

$$= P - W \sin \theta - F_1$$

$$\therefore P - W \sin \theta - F_1 = 0$$

$$\therefore P = W \sin \theta + F_1 = 6000 \times \frac{1}{40} + 150 = 150 + 150 = 300 \text{ N}$$

*Power exerted by engine

$$\begin{aligned} &= \frac{(\text{Force exerted by engine in N}) \times \text{Speed of engine in m/s}}{1000} \\ &= \frac{(300 \text{ N}) \times 20 \text{ m/s}}{1000} = \frac{300 \times 20}{1000} = 6 \text{ kW. Ans.} \end{aligned}$$

$$\text{*Power in kW} = \frac{\text{Force in N} \times \text{Speed in m/s}}{1000}.$$

11.5.3. Analysis of Lift Motion. Fig. 11.8 shows a lift (elevator or cage) carrying some weight and moving with a uniform acceleration.

Let W = Weight carried by the lift

$$m = \text{Mass carried by lift} = \frac{W}{g}$$

a = Uniform acceleration of the lift

T = Tension in cable supporting the lift. This is also called the *reaction of the lift.

The lift may be moving upwards or moving downwards.

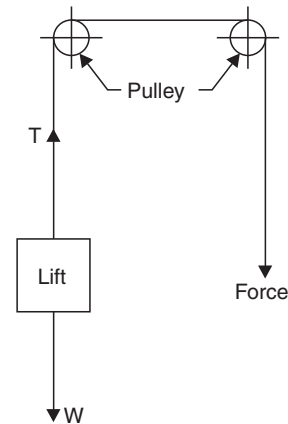


Fig. 11.8

1st Case. Let the lift is moving upwards as shown in Fig. 11.9. The weight carried by lift is acting downwards while the tension in the cable is acting upwards. As the lift is moving up, the net force which is equal to $(T - W)$ is acting upwards.

$\therefore$ Net force in upward direction = $T - W$.

This net force produces an acceleration ' a '

Hence using,

Net force = mass $\times$ acceleration

$$\text{or} \quad (T - W) = \left(\frac{W}{g} \right) \times a \quad \left(\because \text{Mass} = \frac{W}{g} \right)$$

$$\text{or} \quad T = W + \frac{W}{g} \times a = W \left(1 + \frac{a}{g} \right) \quad \dots(11.5)$$

2nd Case. As the lift is moving downwards as shown in Fig. 11.10, the net force is acting downwards. Hence in this case W is more than T (tension in string).

$\therefore$ Net force in downward direction = $(W - T)$.

This net force produces an acceleration ' a '.

Hence using, Net force = mass $\times$ acceleration

$$\text{or} \quad (W - T) = \frac{W}{g} \times a$$

$$\text{or} \quad T = W - \frac{W}{g} \times a$$

$$= W \left(1 - \frac{a}{g} \right) \quad \dots(11.6)$$

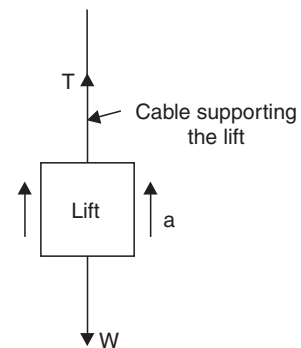


Fig. 11.9. Lift is moving upwards

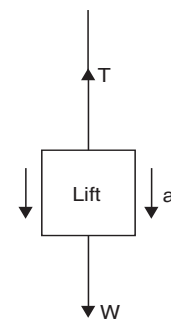


Fig. 11.10. Lift is moving downwards

*Reaction of lift is equal to the tension (T) in cable supporting the lift.

Problem 11.8. A lift carries a weight of 100 N and is moving with a uniform acceleration of 2.45 m/s^2 . Determine the tension in the cables supporting the lift, when :

(i) Lift is moving upwards, and

(ii) Lift is moving downwards. Take $g = 9.80 \text{ m/s}^2$.

Sol. Given :

Weight carried by lift, $W = 100 \text{ N}$.

Uniform acceleration, $a = 2.45 \text{ m/s}^2$.

(i) Lift is moving upwards [Refer to Fig. 11.11 (a)]

Let $T =$ Tension in the cables supporting the lift.

Net force = Mass $\times$ Acceleration

$$T - W = \frac{W}{g} \times a$$

or
$$T = \frac{W}{g} \times a + W$$

or
$$T = W \left(1 + \frac{a}{g} \right)$$

$$= 100 \left(1 + \frac{2.45}{9.80} \right) = 100 (1.25)$$

$$= 125 \text{ N. Ans.}$$

(ii) Lift is moving downwards [Refer to Fig. 11.11 (b)]

$$(W - T) = \frac{W}{g} \times a$$

or
$$T = W \left(1 - \frac{a}{g} \right)$$

$$= 100 \left(1 - \frac{2.45}{9.80} \right) = 100(1 - 0.25) = 100 \times .75$$

$$= 75 \text{ N. Ans.}$$

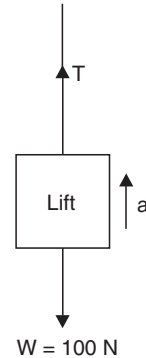


Fig. 11.11 (a)

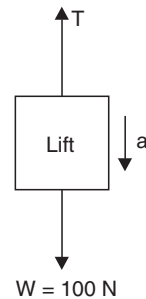


Fig. 11.11 (b)

Problem 11.9. A cage, carrying 10 men each weighing 500 N, starts moving downwards from rest in a mine vertical shaft. The cage attains a speed of 12 metres/s in 20 metres. Find the pressure exerted by each man on the floor of the cage. Take $g = 9.80 \text{ m/s}^2$.

Sol. Given :

Weight of one man, $= 500 \text{ N}$

Total weight of 10 men on the cage, $W = 500 \times 10 = 5000 \text{ N}$

Initial velocity of cage, $u = 0$

Final velocity of cage, $v = 12 \text{ m/s}$

Distance travelled, $s = 20 \text{ m}$

Acceleration due to gravity, $g = 9.80 \text{ m/s}^2$.

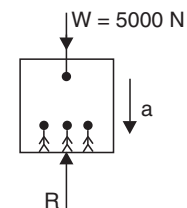


Fig. 11.11 (c)

The pressure exerted by the men on the floor of the cage will be same as the pressure exerted by floor on the men.

Let R = Pressure exerted by 10 men on the floor = Pressure exerted by floor on 10 men

a = Acceleration of the cage.

The acceleration will be obtained by using the relation,

$$v^2 - u^2 = 2as \quad \text{or} \quad 12^2 - 0 = 2a \times 20$$

$$\therefore a = \frac{12 \times 12}{2 \times 20} = 3.6 \text{ m/s}^2.$$

The cage is moving downwards, hence

Net force = Mass $\times$ Acceleration

$$\text{or} \quad (W - R) = \frac{W}{g} \times a \quad \text{or} \quad R = W - \frac{W}{g} \times a$$

$$R = W \left(1 - \frac{a}{g} \right) \quad \text{or} \quad R = 5000 \left(1 - \frac{3.6}{9.8} \right)$$

$$= 5000 (1 - 0.3673) = 3163.26 \text{ N.}$$

$\therefore$ Pressure exerted by floor on one man,

$$= \frac{R}{10} = \frac{3163.26}{10} = 316.326 \text{ N.}$$

But pressure exerted by floor on each man is the same as the pressure exerted by each man on the floor of the cage.

$\therefore$ Pressure exerted by each man on the floor = **316.326 N. Ans.**

11.5.4. Analysis of the Motion of Two Bodies Connected by a String. Fig. 11.12 shows a light and inextensible string passing over a smooth and weightless pulley. Two bodies of weights W_1 and W_2 are attached to the two ends of the string. Let W_1 be greater than W_2 . As $W_1 > W_2$, the weight W_1 will move downwards, whereas the smaller weight (W_2) will move upwards. For an inextensible string, the upward acceleration of the weight W_2 will be equal to the downward acceleration of the weight W_1 .

As the string is light and inextensible and passing over a smooth pulley, the tension* of the string will be the same on both sides of the pulley.

Let T = Tension in both strings,

a = Acceleration of the bodies.

Consider the motion of weight W_1 . Refer to Fig. 11.12 (a). The weight W_1 is moving downwards with an acceleration a . The forces acting on W_1 are (i) its weight W_1 acting downwards and (ii) tension T acting upwards. As the weight W_1 is moving downwards, hence net force on the weight W_1 , is acting downwards.

*Tension of the string on both sides of the pulley will be different if the string is heavy (*i.e.*, weight of string is considered), extensible and does not pass over a smooth pulley.

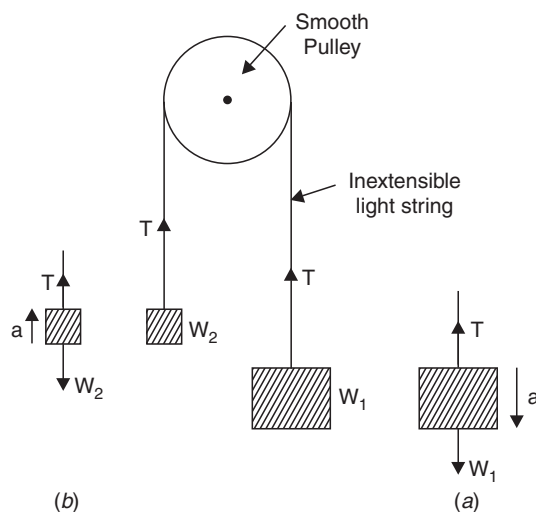


Fig. 11.12

$\therefore$ Net downward force $= (W_1 - T)$

But net force = mass $\times$ acceleration

$$\therefore (W_1 - T) = \frac{W_1}{g} \times a \quad \left(\because \text{mass} = \frac{\text{Weight}}{g} = \frac{W_1}{g} \right) \quad \dots(i)$$

Now consider the motion of weight, W_2 . Refer to Fig. 11.12 (b). The forces acting on W_2 are : (i) its weight W_2 acting downwards and (ii) tension T acting upwards. But the weight W_2 is moving upwards, hence net force on weight W_2 , is acting upwards.

$\therefore$ Net upward force $= (T - W_2)$

But net upward force = mass $\times$ acceleration

$$\text{or} \quad (T - W_2) = \frac{W_2}{g} \times a \quad \left(\because \text{mass} = \frac{W_2}{g} \right) \quad \dots(ii)$$

Adding equation (i) and (ii), we get the value of acceleration a .

$$\therefore (W_1 - W_2) = \frac{a}{g} (W_1 + W_2)$$

$$\text{or} \quad a = \frac{g(W_1 - W_2)}{(W_1 + W_2)} \quad \dots(11.7)$$

Equation (11.7) is used for finding the acceleration. If the value of this acceleration is substituted either in equation (i) or in equation (ii) the value of tension (T) is obtained.

Problem 11.10. Two bodies of weight 50 N and 30 N are connected to the two ends of a light inextensible string. The string is passing over a smooth pulley. Determine :

- (i) The acceleration of the system, and
- (ii) Tension in the string. Take $g = 9.80 \text{ m/s}^2$.

Sol. Given :

Heavier weight, $W_1 = 50 \text{ N}$

Lighter weight, $W_2 = 30 \text{ N}$

Let a = Acceleration of the system, and

T = Tension in the string.

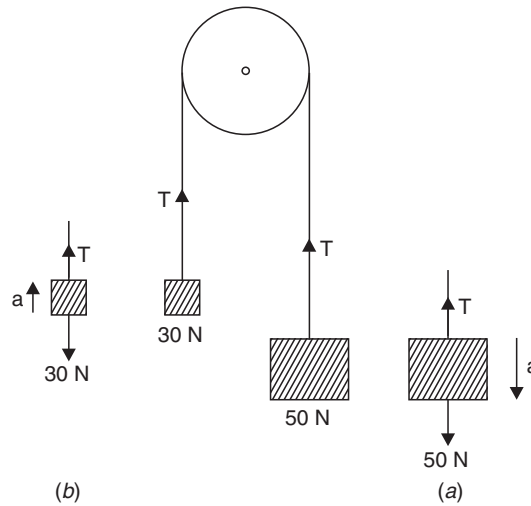


Fig. 11.13

Refer to Fig. 11.13 (a) i.e., *FBD* of 50 N.

$$(50 - T) = \left(\frac{50}{g}\right) \times a \quad \dots(i)$$

Refer to Fig. 11.13 (b).

$$(T - 30) = \left(\frac{30}{g}\right) \times a \quad \dots(ii)$$

Adding equations (i) and (ii), we get the value of a

$$20 = \frac{80}{g} \times a \quad \therefore a = \frac{20 \times g}{80} = \frac{g}{4} = \frac{9.8}{4} = 2.45 \text{ m/s}^2. \quad \text{Ans.}$$

Substitute this value in equation (ii), we get tension, T .

$$\therefore T - 30 = \frac{30}{g} \times 2.45$$

$$\begin{aligned} \therefore T &= 30 + \frac{30}{9.8} \times 2.45 \\ &= 30 + 7.5 = 37.5 \text{ N.} \quad \text{Ans.} \end{aligned}$$

Problem 11.11. Find the tension in the string and accelerations of blocks A and B weighing 200 N and 50 N respectively, connected by a string and frictionless and weightless pulleys as shown in Fig. 11.14. (U.P. Tech. University, 2002–2003)

Sol. $W_A = 200 \text{ N}, \quad W_B = 50 \text{ N}$

$$\therefore M_A = \left(\frac{200}{9.81}\right) \text{ kg}, \quad M_B = \left(\frac{50}{9.81}\right)$$

Let a = Acceleration of the block B and

T = Tension in the string.

As the pulley is smooth, the tension in the string will be same throughout.

Let x_B = Displacement of block B

x_A = Displacement of block A

From given figure, it is clear that $x_B = 2x_A$ or $x_A = \frac{x_B}{2}$

Differentiating the above equation w.r.t. time, we get

$$\frac{d}{dt}(x_A) = \frac{d}{dt}\left[\frac{x_B}{2}\right] \quad \text{or} \quad V_A = \frac{V_B}{2}$$

Similarly, $\frac{d}{dt}(V_A) = \frac{d}{dt}\left[\frac{V_B}{2}\right] \quad \text{or} \quad a_A = \frac{1}{2}a_B$

The acceleration of block A will be equal to half the acceleration of block B.

$$\therefore \text{Acceleration of block A} = \frac{a}{2}$$

To find the directions in which block A and B are moving, assume that block B is stationary. Then $T = 50$ N. Now assume that block A is stationary. Then $2T = 200$ N or $T = 100$ N.

Take the mean of these two values as T the tension in the string. The mean value is $\frac{50 + 100}{2} = 75$ N.

Hence, assume tension in the string as 75 N, throughout. Due to this tension of 75 N, the block B will move upwards and block A will move downwards.

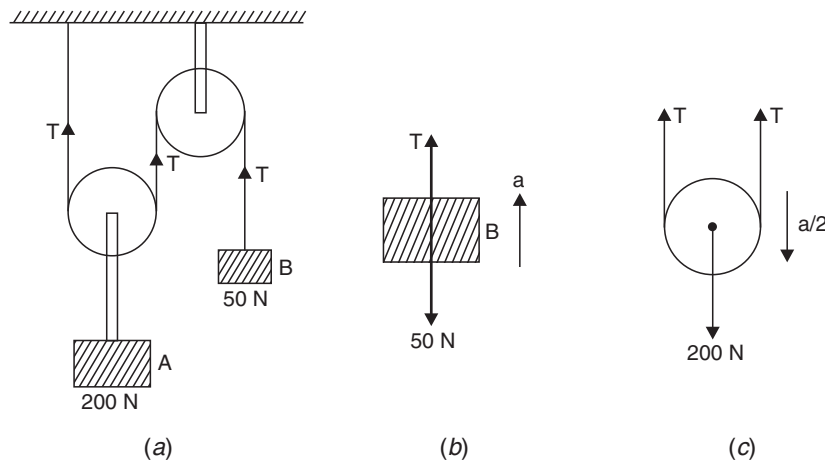


Fig. 11.14

(i) Consider the motion of Block B

Block B is moving upward with acceleration 'a'. Net force in upward direction
= Mass $\times$ acceleration

$$(T - 50) = M_B \times a$$

$$(T - 50) = \frac{50}{9.81} \times a \quad \dots(i)$$

(ii) Consider the motion of Block A

Block A is moving downward with acceleration = $\frac{a}{2}$

$\therefore$ Net force in downward direction = Mass $\times$ acceleration

$$(200 - 2T) = M_A \times \frac{a}{2}$$

$$(200 - 2T) = \left(\frac{200}{9.81} \right) \times \frac{a}{2}$$

or

$$(200 - 2T) = \frac{100}{9.81} \times a \quad \dots(ii)$$

or

$$(100 - T) = \frac{50a}{9.81} \quad [\text{Dividing (ii) by 2}] \quad \dots(iii)$$

Adding equations (i) and (iii),

$$50 = \frac{100a}{9.81} \quad \therefore \quad a = \frac{50 \times 9.81}{100} = 4.905 \text{ m/s}^2. \quad \text{Ans.}$$

Substituting the value of a in equation (ii)

$$T - 50 = \frac{50}{9.81} \times 4.905 = 25 \quad \therefore \quad T = 75 \text{ N.} \quad \text{Ans.}$$

$\therefore$ Acceleration of block $B = a = 4.905 \text{ m/s}^2. \quad \text{Ans.}$

Acceleration of block $A = \frac{a}{2} = \frac{4.905}{2} = 2.45 \text{ m/s}^2. \quad \text{Ans.}$

Problem 11.12. A cord runs over two pulleys A and B with fixed axles, and carries a movable pulley C . If $P = 40 \text{ N}$, $P_1 = 20 \text{ N}$ and $P_2 = 30 \text{ N}$ and the cord lies in the vertical plane determine, the acceleration of pulley C . Neglect friction and weights of the pulleys.

(U.P. Tech. University, 2000–2001)

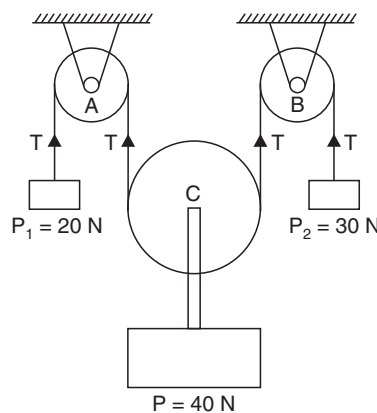


Fig. 11.15

Sol. As the weights of pulleys and friction are neglected, the tension in the cord will be same throughout.

Let T = Tension in the cord

a = Acceleration of pulley C ,

a_1 = Acceleration of weight $P_1 = 20 \text{ N}$, and

a_2 = Acceleration of weight $P_2 = 30 \text{ N}$.

To find the directions in which the weights P_1 , P and P_2 are moving, assume first the weight P_1 as stationary. Then $T = P_1 = 20$ N. Now assume weight P_2 to be stationary, then $T = P_2 = 30$ N. Take the mean of these two values as the tension in the cord. The mean of these two values are $(20 + 30) \div 2 = 25$ N.

Hence, assume tension in the cord as 25 N throughout. This tension is only assumed value. Due to tension 25 N throughout cord, the load P_1 will be moving upwards whereas the load P_2 will be moving downwards.

Let x_1 = Displacement of P_1 upwards

x_2 = Displacement of P_2 downwards

x = Net displacement of P upwards (assumed)

Due to the displacement of P_1 upwards, the load P will move downwards by a distance of $\frac{x_1}{2}$. Also due to the displacement of P_2 downwards, the load P will move upwards by a

distance of $\frac{x_2}{2}$. Let the load P is moving upwards. Then displacement of P upwards = $\frac{x_2}{2} - \frac{x_1}{2}$

$$\therefore x = \frac{x_2}{2} - \frac{x_1}{2} \quad \text{or} \quad 2x = x_2 - x_1$$

Differentiating the above equation with respect to time,

$$2 \frac{dx}{dt} = \frac{dx_2}{dt} - \frac{dx_1}{dt} \quad \text{or} \quad 2v = v_2 - v_1$$

$[v_2 = \text{Velocity of } P_2 \text{ and } v_1 = \text{Velocity of } P_1 \text{ and } v = \text{Velocity of } P]$

Differentiating again with respect to time, we get

$$2 \frac{dv}{dt} = \frac{dv_2}{dt} - \frac{dv_1}{dt} \quad \text{or} \quad 2a = a_2 - a_1$$

$[a_2 = \text{Acc. of } P_2, a_1 = \text{Acc. of } P_1 \text{ and } a = \text{Acc. of } P]$

$$\text{or} \quad a = \left(\frac{a_2 - a_1}{2} \right) \quad \dots(i)$$

(i) Consider the Free Body Diagram of load P_1

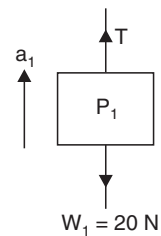
The load P_1 is moving upwards with an acceleration a_1 .

$\therefore$ Net force = Mass $\times$ Acceleration

$$T - P_1 = \frac{P_1}{g} \times a_1$$

$$(T - 20) = \left(\frac{20}{9.81} \right) \times a_1$$

$\dots(ii)$



$$\left[\because \text{Mass of } P_1 = \frac{20}{9.81} = \frac{W_1}{g} \text{ and Acc.} = a_1 \right]$$

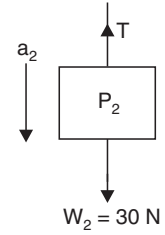
(ii) Consider the Free Body Diagram of load P_2

The load P_2 is moving downwards with an acceleration of a_2

∴ Net force = Mass × Acceleration

$$(P_2 - T_1) = \frac{P_2}{g} \times a_2$$

$$(30 - T) = \frac{30}{9.81} \times a_2 \quad \dots(iii)$$



(iii) Consider the Free Body Diagram of load P

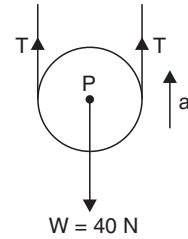
The load P is moving upwards with an acceleration of a

∴ Net force = Mass × Acceleration

$$2T - P = \frac{P}{g} \times a$$

$$(2T - 40) = \frac{40}{9.81} \times a = \frac{40}{9.81} \times \left(\frac{a_2 - a_1}{2} \right) \quad \dots(iv)$$

$$\left[\because a = \frac{a_2 - a_1}{2} \text{ from equation (i)} \right]$$



or

$$2T - 40 = \frac{20}{9.81} (a_2 - a_1) \quad \dots(v)$$

Adding equations (ii) and (v), we get

$$3T - 60 = \frac{20}{9.81} \times a_2 \quad \dots(vi)$$

Multiplying the above equation by $\left(\frac{3}{2}\right)$, we get

$$\frac{3}{2} (3T - 60) = \frac{30}{9.81} \times a_2$$

or

$$4.5T - 90 = \frac{30}{9.81} \times a_2 \quad \dots(vii)$$

Substituting equation (iii) from (vii),

$$(4.5T - 90) - (30 - T) = \frac{30}{9.81} \times a_2 - \frac{30 \times a_2}{9.81}$$

or

$$5.5T - 120 = 0$$

or

$$T = \frac{120}{5.5} = 21.818 = 21.8 \text{ N}$$

Substituting this value of T in equation (iv),

$$(2 \times 21.8 - 40) = \frac{40}{9.81} \times a$$

or

$$3.6 = \frac{40}{9.81} \times a$$

or

$$a = \frac{3.6 \times 9.81}{40} = 0.883 \text{ m/s}^2. \quad \text{Ans.}$$

As 'a' is +ve, hence our assumption that pulley C is moving upward is correct.

11.5.5. Analysis of the Motion of Two Bodies Connected by a String when One Body is Lying on a Horizontal Surface and other is Hanging Free

(a) **1st Case. The horizontal surface is smooth and the string is passing over a smooth pulley.** Fig. 11.16 shows the two weights W_1 and W_2 connected by a light inextensible string, passing over a smooth pulley. The weight W_2 is placed on a smooth horizontal surface, whereas the weight W_1 is hanging free.

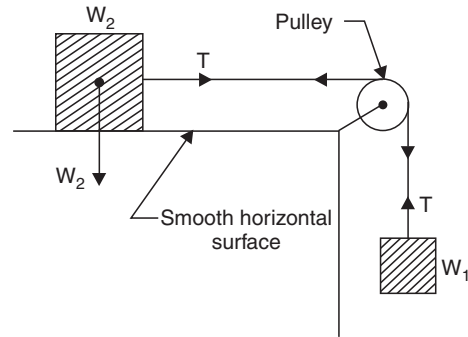


Fig. 11.16

The weight W_1 is moving downwards, whereas the weight W_2 is moving on smooth horizontal surface. The velocity and acceleration of W_1 will be same as that of W_2 .

As the string is light and inextensible and passing over a smooth pulley, the tensions of the string will be same on both sides of the pulley.

Let T = Tension in the string

a = Acceleration of the weight W_1 and also of W_2

(i) Consider the motion of the hanging weight W_1

The weight W_1 is moving downwards with an acceleration a . The forces acting on W_1 are : (i) its weight W_1 acting downwards, and (ii) tension T acting upwards.

$\therefore$ Net downward force = $(W_1 - T)$.

Using, net force = mass $\times$ acceleration

$$\therefore (W_1 - T) = \frac{W_1}{g} \times a \quad \left(\because \text{Mass} = \frac{\text{Weight}}{g} \right) \quad \dots(i)$$

(ii) Consider the motion of weight W_2

The weight W_2 is moving on the horizontal surface with an acceleration of ' a '. As the weight W_2 is moving in the horizontal direction, the only force causing the motion is T . The weight W_2 is acting downwards and hence the component of this weight in horizontal direction is $W_2 \cos 90^\circ$, which is zero.

Using, force = mass $\times$ acceleration

$$\text{or} \quad T = \frac{W_2}{g} \times a \quad \left(\because \text{Mass} = \frac{\text{Weight}}{g} \right) \quad \dots(ii)$$

Adding equations (i) and (ii), we get the value of acceleration, a .

$$\therefore W_1 = \frac{W_1}{g} \times a + \frac{W_2}{g} \times a = \frac{a}{g} [W_1 + W_2]$$

$$\therefore a = \frac{g \times W_1}{(W_1 + W_2)} \quad \dots(iii)$$

Equation (iii) gives the acceleration of the system.

To find the tension (T) in the string, substitute the value of ' a ' in equation (ii).

$$\therefore T = \frac{W_2}{g} \times \frac{g \times W_1}{(W_1 + W_2)} = \frac{W_1 W_2}{(W_1 + W_2)} \quad \dots(iv)$$

Equation (iv) gives the tension in the string.

(b) 2nd Case. The horizontal surface is rough and string is passing over a smooth pulley. Fig. 11.17 shows the two weights W_1 and W_2 connected by a light inextensible string, passing over a smooth pulley. The weight W_1 is hanging free, whereas the weight W_2 is placed on a rough horizontal surface. Hence in this case force of friction will be acting on the weight W_2 in the opposite direction of the motion of weight W_2 as shown in Fig. 11.17.

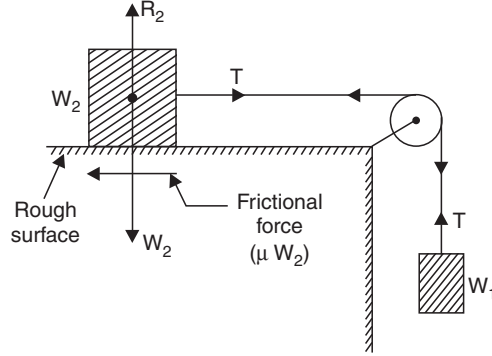


Fig. 11.17

Let μ = Co-efficient of friction between weight W_2 and horizontal surface

a = Acceleration of the system

T = Tension in the string

R_2 = Normal reaction at the horizontal rough surface = W_2

$$\therefore \text{Force of friction} = \mu R_2 = \mu W_2 \quad (\because R_2 = W_2) \quad \dots(i)$$

(i) Consider the motion of the hanging weight W_1

The weight W_1 is moving downwards with an acceleration ' a '. The net downward force acting on weight $W_1 = (W_1 - T)$.

Using, net force = mass $\times$ acceleration

$$W_1 - T = \frac{W_1}{g} \times a \quad \left(\because \text{Mass} = \frac{\text{Weight}}{g} \right) \quad \dots(ii)$$

(ii) Consider the motion of the weight W_2 .

The weight W_2 is moving on the rough horizontal surface towards right with an acceleration ' a '. The forces acting in the horizontal directions are : (i) tension (T) towards right, and (ii) force of friction = $\mu R_2 = \mu W_2$ towards left.

$\therefore$ Net horizontal force towards right = $T - \mu W_2$.

Using, net force = mass $\times$ acceleration

$$\text{or} \quad T - \mu W_2 = \frac{W_2}{g} \times a \quad \dots(iii)$$

Adding equations (ii) and (iii), we get the value of acceleration a .

$$\therefore W_1 - \mu W_2 = \frac{a}{g} (W_1 + W_2)$$

$$\therefore a = \frac{g(W_1 - \mu W_2)}{(W_1 + W_2)} \text{ m/s}^2$$

To find the tension T , substitute the value of ' a ' in equation (ii).

Problem 11.13. Find the acceleration of bodies and tension in the string joining A and B shown in Fig. 11.18.
(U.P. Tech. University, 2002–2003)

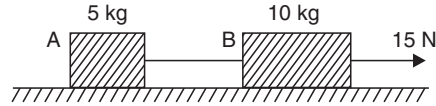


Fig. 11.18

Sol. Given :

Mass of A, $m_A = 5 \text{ kg}$; mass of B, $m_B = 10 \text{ kg}$
 Force applied $= 15 \text{ N}$
 Let $a =$ Acceleration of the system and
 $T =$ Tension in the string

(i) Consider the equilibrium of body B

Weight of B $= m_B \times g$
 $= 10 \times 9.81 \text{ N} = 98.1 \text{ N}$

The body B is moving with acceleration 'a'.

Net force in horizontal direction

$$= (15 - T) \text{ N}$$

But Net Force = Mass $\times$ Acceleration

$$\therefore (15 - T) = 10 \times a \quad \dots(i)$$

(ii) Consider the motion of body A

Only tension is acting in horizontal direction.

$\therefore$ Net force = Mass $\times$ Acceleration

$$T = 5 \times a \quad \dots(ii)$$

Adding equations (i) and (ii), we get the value of acceleration a .

$$\therefore 15 = 15a$$

$$\therefore a = 1 \text{ m/s}^2. \text{ Ans.}$$

Substituting this value of a in (ii), to get the value of tension T .

$$\therefore T = 5 \times 1 = 5 \text{ N. Ans.}$$

Problem 11.14. Two blocks shown in Fig. 11.19 have weights $A = 20 \text{ N}$ and $B = 10 \text{ N}$ and co-efficient of friction between the block A and the horizontal plane is $\mu = 0.25$. If the system is released from the rest and the block B falls through a vertical distance of 2 m, what is the velocity attained by block B? Neglect the friction in the pulley and the extension of the string.

Sol. Given :

Weight of block A, $W_2 = 20 \text{ N}$

Weight of block B, $W_1 = 10 \text{ N}$

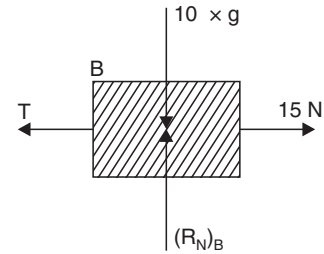
Co-efficient of friction, $\mu = 0.25$

Distance moved by block B, $s = 2 \text{ m}$

Initial velocity of block B, $u = 0$.

Let $v =$ Final velocity of block B

$a =$ Acceleration of the system or acceleration of block B.



(i) Consider the motion of block B. [Refer to Fig. 11.19 (a)]

Net force = Mass $\times$ Acceleration

$$(10 - T) = \frac{10}{9.8} \times a \quad \dots(i)$$

(ii) Consider the motion of block A. [Refer to Fig. 11.19 (b)].

Net force = Mass $\times$ Acceleration

$$(T - \mu R) = \frac{20}{9.8} \times a$$

or
$$T - 0.25 \times 20 = \frac{20}{9.8} \times a$$

or
$$T - 5 = \frac{20}{9.8} \times a \quad \dots(ii)$$

Adding equations (i) and (ii), we get acceleration a .

$$\therefore 5 = \frac{30}{9.8} \times a$$

$$\therefore a = \frac{5 \times 9.8}{30} = 1.633 \text{ m/s}^2. \quad \text{Ans.}$$

Now using the relation,

$$v^2 = u^2 + 2as = 0 + 2 \times 1.633 \times 2 = 6.54$$

$$\therefore v = \sqrt{6.54} = 2.557 \text{ m/s.} \quad \text{Ans.}$$

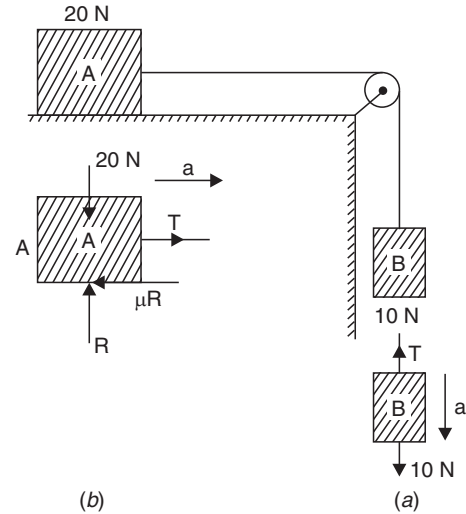


Fig. 11.19

11.5.6. Analysis of the Motion of two Bodies Connected by a String when One Body is Lying on Inclined Plane and the other is Hanging Free in Air

(a) First case when the inclined surface is smooth

Fig. 11.20 shows two bodies of weights W_1 and W_2 connected by a light inextensible string, which passes over a smooth and weightless pulley. The weight W_2 is placed on an inclined smooth plane of inclination θ with the horizontal, whereas the weight W_1 is hanging free in air.

As the inclined plane is smooth and hence the friction between the weight W_2 and the inclined plane will be neglected. When the weight W_1 is moving downwards, the weight W_2 will be moving upwards along the inclined plane. The velocity and acceleration of the weight W_1 will be same as that of weight W_2 . Since the pulley is smooth and string is light and inextensible, the tension* in the string on both sides of the pulley will be same.

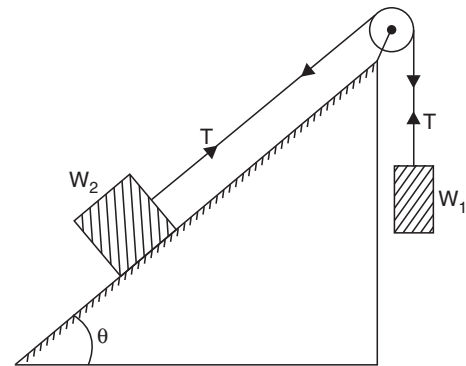


Fig. 11.20

*If the string is inextensible and passes over a smooth pulley, then tension on both sides of the pulley will be same. The tension is independent of the inclination.

Let a = Acceleration of the system *i.e.*, acceleration of weight W_1 as well as acceleration of weight W_2 ,

T = Tension in the string,

θ = Inclination of the inclined plane.

Consider the motion of weight W_1 . The weight W_1 is moving downwards with an acceleration ' a '. The forces acting on W_1 are : (i) its weight W_1 acting downwards, and (ii) tension T acting upwards.

$$\therefore \text{Net downwards force} = W_1 - T$$

But net downwards force = mass $\times$ acceleration.

$$\therefore (W_1 - T) = \frac{W_1}{g} \times a \quad \left(\because \text{mass} = \frac{W_1}{g} \right) \quad \dots(i)$$

Now consider the motion of Weight W_2 . The weight W_2 is moving upwards along the inclined plane with an acceleration a . The forces acting on W_2 along the plane are shown in Fig. 11.21. They are :

(i) $W_2 \sin \theta$ downwards, and

(ii) Tension T upwards.

$\therefore$ Net force acting on W_2 along the plane in the upwards direction = $T - W_2 \sin \theta$.

But net force = mass $\times$ acceleration

$$\therefore T - W_2 \sin \theta = \frac{W_2}{g} \times a$$

$$\left(\because \text{Mass} = \frac{\text{Weight}}{g} \right) \quad \dots(ii)$$

Adding equations (i) and (ii), we get acceleration a .

$$W_1 - W_2 \sin \theta = \frac{W_1}{g} \times a + \frac{W_2}{g} \times a = \frac{a}{g} (W_1 + W_2)$$

$$\therefore a = \frac{g(W_1 - W_2 \sin \theta)}{(W_1 + W_2)} \text{ m/s}^2$$

To find the value of tension T , in the string, the value of a is substituted in equation (i).

(b) **Second Case when the inclined surface is rough**

As the surface of the inclined plane is not smooth, hence a force of friction equal to μR will be acting on the weight W_2 in the opposite direction of motion of weight W_2 as shown in Fig. 11.22.

Let μ = Co-efficient of friction between the weight W_2 and inclined surface,

θ = Angle of the inclination of the plane,

a = Acceleration of the system,

T = Tension in the string,

R = Normal reaction acting on W_2 .

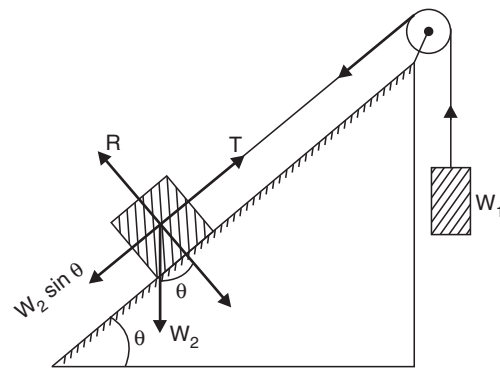


Fig. 11.21

The forces acting on the weight W_2 are shown in Fig. 11.22. Equating the forces normal to the plane, we get

$$R = W_2 \cos \theta.$$

$$\therefore \text{Frictional force, } F = \mu R \\ = \mu W_2 \cos \theta \quad \dots(i)$$

Consider the motion of weight W_1 . The weight W_1 is moving downwards with an acceleration a .

The net downward force acting on

$$W_1 = (W_1 - T).$$

But net force = mass $\times$ acceleration

$$\therefore (W_1 - T) = \frac{W_1}{g} \times a \\ \left(\because \text{Mass} = \frac{W_1}{g} \right) \dots(ii)$$

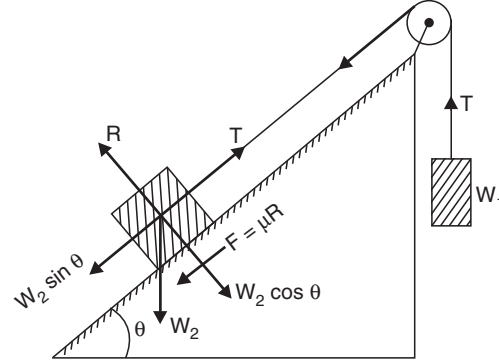


Fig. 11.22

Now consider the motion of weight W_2 . The weight W_2 is moving upwards along the inclined plane with an acceleration a . The net upward force along the inclined plane acting on weight W_2

$$= T - W_2 \sin \theta - \mu R \\ = T - W_2 \sin \theta - \mu W_2 \cos \theta \quad (\because \text{From (i), } \mu R = \mu W_2)$$

Using, Net force = mass $\times$ acceleration

$$\therefore T - W_2 \sin \theta - \mu W_2 \cos \theta = \frac{W_2}{g} \times a \\ \left(\because \text{Mass} = \frac{W_2}{g} \right) \dots(iii)$$

Adding equations (ii) and (iii), we get the value of a

$$W_1 - W_2 \sin \theta - \mu W_2 \cos \theta = \frac{W_1}{g} \times a + \frac{W_2}{g} \times a = \frac{a}{g} [W_1 + W_2] \\ \therefore a = \frac{g(W_1 - W_2 \sin \theta - \mu W_2 \cos \theta)}{(W_1 + W_2)} \quad \dots(iv)$$

To find tension T in the string, the value of ' a ' from equation (iv) is substituted in equation (ii).

Problem 11.15. Two bodies of weights 40 N and 25 N are connected to the two ends of a light inextensible string, which passes over a smooth pulley. The weight 40 N is placed on a rough inclined plane, while the weight 25 N is hanging free in air. If the angle of the plane is 15° , and co-efficient of friction between the weight 40 N and inclined surface is 0.2 determine :

- (i) Acceleration of the system, and
- (ii) Tension in the string. Take $g = 9.80 \text{ m/s}^2$,
- (iii) Distance moved by weight 25 N in 3 seconds starting from rest.

Sol. Given :

Weight placed on inclined plane, $W_2 = 40 \text{ N}$

Weight hanging free in air, $W_1 = 25 \text{ N}$

Angle of inclination, $\theta = 15^\circ$

Acceleration due to gravity, $g = 9.80 \text{ m/s}^2$

Co-efficient of friction, $\mu = 0.2$.

Let a = Acceleration of the system,

T = Tension in the string, and

s = Distance travelled by weight 15 N in 3 seconds starting from rest.

Consider the motion of weight 25 N. [Refer to Fig. 11.23 (a)].

Weight 25 N is moving downwards.

Net force = Mass $\times$ Acceleration

$$\therefore 25 - T = \frac{25}{9.8} \times a \quad \dots(i)$$

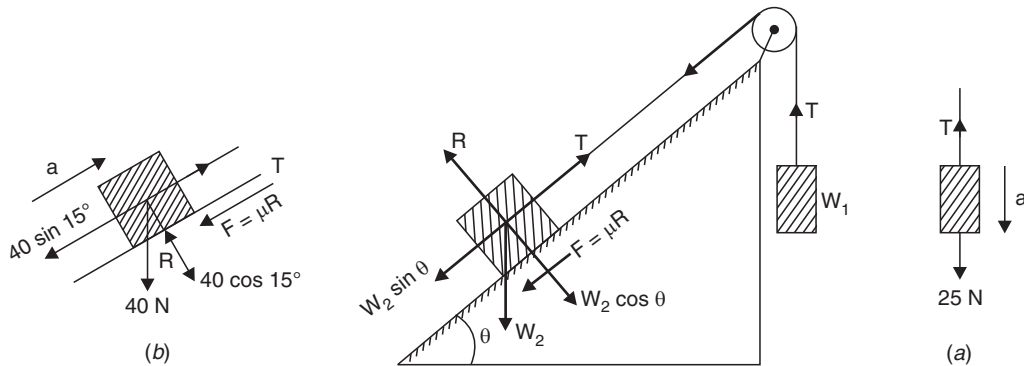


Fig. 11.23

Consider the motion of weight 40 N. [Refer to Fig. 11.23 (b)]

Weight 40 N is moving upwards.

Net force = Mass $\times$ Acceleration

$$T - 40 \sin 15^\circ - \mu R = \frac{40}{9.8} \times a \quad \dots(ii)$$

(i) Acceleration of the system

To find acceleration, add equations (i) and (ii)

$$\therefore 25 - 40 \sin 15^\circ - \mu R = \frac{25}{9.8} \times a + \frac{40}{9.8} \times a$$

$$\text{or } 25 - 10.35 - 0.2 \times 40 \cos 15^\circ = \frac{65}{9.8} \times a \quad (\because \mu = 0.2 \text{ and } R = 40 \cos 15^\circ)$$

$$\text{or } 25 - 10.35 - 7.73 = \frac{65}{9.8} \times a$$

$$\text{or } 6.92 = \frac{65}{9.8} \times a$$

$$\text{or } a = \frac{6.92 \times 9.8}{65} = 1.043 \text{ m/s}^2. \quad \text{Ans.}$$

(ii) Tension in the string

To find tension, substitute the value of acceleration 'a' in equation (i)

$$\therefore 25 - T = \frac{25}{9.8} \times 1.043$$

$$\therefore T = 25 - \frac{25}{9.8} \times 1.043 = 22.34 \text{ N.} \quad \text{Ans.}$$

(iii) Distance moved by the weight 25 N in 3 seconds, starting from rest.

Initial velocity, $u = 0$

Time, $t = 3$ seconds

Acceleration, $a = 1.043 \text{ m/s}^2$

Then distance s is given by

$$\begin{aligned} s &= ut + \frac{1}{2} at^2 = 0 \times 3 + \frac{1}{2} \times 1.043 \times 3^2 \\ &= 0 + 4.6935 = \mathbf{4.6935 \text{ m.}} \quad \text{Ans.} \end{aligned}$$

11.6. MOMENTUM AND ANGULAR MOMENTUM (OR MOMENT OF MOMENTUM)

11.6.1. Momentum. The product of mass and velocity of a body is known as momentum of the body. If ' m ' is the mass of a body and v is the velocity of the body, the momentum of the body is equal to $m \times v$.

11.6.2. Moment of Momentum or Angular Momentum. The product of mass moment of inertia and angular velocity of a rotating body is known as moment of momentum or angular momentum.

If ω = Angular velocity of a body rotating about an axis, and

I = Moment of inertia of the body about the axis.

Then angular momentum = $\omega \times I$.

...(11.8)

The equation (11.8) is derived as given below :

Consider a body of mass ' m ' rotating in a circle about its centre O .

Let dm = Mass of the elementary strip

r = Radius of the mass ' dm '

ω = Angular velocity of the body or angular velocity of the mass ' dm '. (As angular velocity is constant, hence the angular velocity of the body will be same as angular velocity of mass ' dm ')
 v = Linear velocity of mass ' dm ' = $\omega \times r$.

Now momentum of elementary mass

= Elementary mass $\times$ velocity

= $dm \times v$

= $dm \times \omega r$

($\because v = \omega r$)

Moment of momentum of elementary mass ' dm ' about O

= Momentum $\times$ radius

= $(dm \times \omega r) \times r$

= $dm \times \omega r^2$

...(i)

The moment of momentum of the entire mass about O is obtained by integrating the above equation.

$\therefore$ Moment of momentum of the entire mass

= $\int dm \times \omega r^2$

= $\omega \int r^2 dm$ (ω is constant and can be taken outside the integral sign)

But $\int r^2 dm$ = moment of inertia* of the whole body about $O = I$.

*Please refer to chapter of moment of inertia.

Substituting this value, we get

$$\therefore \text{Moment of momentum of the entire mass} \\ = \omega \times I \quad \text{or} \quad I \times \omega \quad \dots(11.9)$$

But moment of momentum is also known as **angular momentum**.

11.6.3. Moment of Inertia of a Circular Section. Consider a circular section as shown in Fig. 11.24.

Let R = Radius of the section

m = Mass per unit area of the section

Then total mass,

$$M = \text{Area of the section} \times m = \pi R^2 \times m.$$

Consider an elementary circular ring of thickness ' dr ' at a radius r .

Mass of the circular ring

$$= m \times \text{Area of circular ring}$$

$$= m \times 2\pi r \times dr.$$

Moment of inertia of the circular ring about an axis passing through O and perpendicular to the plane of paper

$$= (\text{Mass of circular ring}) \times \text{radius}^2$$

$$= m \times 2\pi r \times dr \times r^2 = m \times 2\pi r^3 \times dr \quad \dots(i)$$

Moment of inertia of the whole circular section is obtained by integrating equation (i) between the limits 0 to R .

Hence moment of inertia of the whole section,

$$\begin{aligned} I_{zz} &= \int_0^R m \times 2\pi r^3 \times dr \\ &= m \cdot 2\pi \int_0^R r^3 dr = m \times 2\pi \times \left[\frac{r^4}{4} \right]_0^R \\ &= 2m \times \frac{\pi R^4}{4} = m \times \frac{\pi R^4}{2} = m \times \pi R^2 \times \frac{R^2}{2} \\ &= M \frac{R^2}{2} \quad (\because m \times \pi R^2 = M) \quad \dots(11.10) \end{aligned}$$

But from the theorem of perpendicular axis, the moment of inertia about X-X or Y-Y axis,

$$I_{XX} = I_{YY} = \frac{I_{ZZ}}{2} = \frac{MR^2}{2 \times 2} = \frac{MR^2}{4} \quad \dots(11.11)$$

11.6.4. Moment of Inertia of Solid Cylinder. The moment of inertia of a solid cylinder is also given by equation (11.10). But in case of solid cylinder, M is equal to the mass of the solid cylinder.

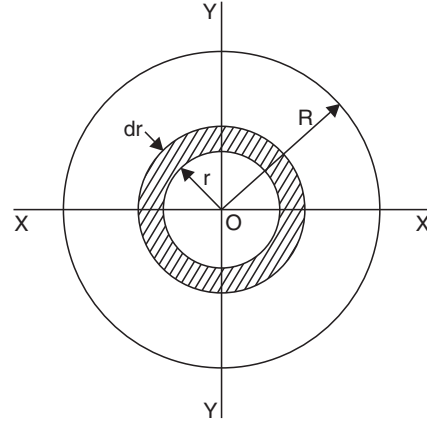


Fig. 11.24

11.6.5. Moment of Inertia of a Solid Sphere.

Consider a solid sphere of radius R with O as centre, as shown in Fig. 11.25.

Let m = Mass of sphere per unit volume

M = Total mass of the sphere

= $m \times$ volume of sphere

$$= m \times \frac{4\pi R^3}{3}.$$

Consider a strip of thickness dx at a distance of x from O as shown in Fig. 11.25. The strip under consideration for sphere is of circular section of radius y , where

$$y = \sqrt{R^2 - x^2}.$$

Mass of the strip

= $m \times$ Volume of strip = $m \times$ Area of strip $\times$ Thickness of strip

$$= m \times \pi y^2 \times dx = m \times \pi \times (R^2 - x^2) \times dx \quad (\because y = \sqrt{R^2 - x^2})$$

Moment of inertia of the strip about $X-X$ axis

$$= \text{Mass of the strip} \times \frac{\text{Radius}^2}{2}$$

$$= m\pi(R^2 - x^2) dx \times \frac{y^2}{2} = m\pi(R^2 - x^2) \times dx \times \frac{(R^2 - x^2)}{2}$$

$$= \frac{m\pi}{2} \times (R^2 - x^2)^2 dx = \frac{m\pi}{2} \times (R^4 + x^4 - 2R^2x^2)dx$$

The mass moment of inertia of the whole sphere is obtained by integrating the above equation between the limits $-R$ to $+R$. Therefore,

$$\begin{aligned} I &= \int_{-R}^{+R} \frac{m\pi}{2} (R^4 + x^4 - 2R^2x^2) dx = \frac{m\pi}{2} \int_{-R}^{+R} (R^4 + x^4 - 2R^2x^2) dx \\ &= \frac{m\pi}{2} \left[R^4x + \frac{x^5}{5} - \frac{2R^2x^3}{3} \right]_{-R}^{+R} \\ &= \frac{m\pi}{2} \left[R^4 \times R + \frac{R^5}{5} - \frac{2R^2 \times R^3}{3} - \left\{ R^4(-R) + \frac{(-R)^5}{5} - \frac{2R^2(-R)^3}{3} \right\} \right] \\ &= \frac{m\pi}{2} \left[R^5 + \frac{R^5}{5} - \frac{2R^5}{3} - \left\{ -R^5 - \frac{R^5}{5} + \frac{2R^5}{3} \right\} \right] \\ &= \frac{m\pi}{2} \left[2R^5 + \frac{2R^5}{5} - \frac{4R^5}{3} \right] = \frac{m\pi}{2} \left[\frac{30 + 6 - 20}{15} \right] R \\ &= \frac{m\pi}{2} \times \frac{16}{15} R^5 = \frac{8}{15} m\pi R^5 = \frac{2}{5} \times \frac{4}{3} m\pi R^3 \times R^2 \\ &= \frac{2}{5} \times M \times R^2 \quad \left(\because \frac{4}{3} m\pi R^3 = M \right) \\ &= \frac{2MR^2}{5} \quad \dots(11.12) \end{aligned}$$

The moment of inertia of the sphere about any axis is same as sphere is symmetrical.

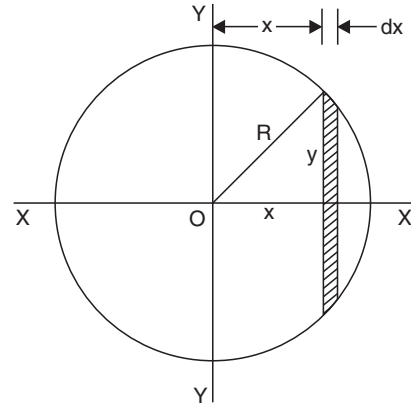


Fig. 11.25

11.6.6. Radius of Gyration (K). It is defined as the distance of a body from the axis of reference where whole mass of the body is assumed to be concentrated so as not to alter the moment of inertia about the given axis.

The moment of inertia of a body in terms of radius of gyration is given by

$$I = Mk^2$$

$$\therefore \text{Radius of gyration, } k = \sqrt{\frac{I}{M}}. \quad \dots(11.13)$$

Problem 11.16. Calculate the moment of inertia with respect to the axis of rotation of : (i) a circular lamina of diameter 60 cm and of mass 0.001 kg/cm², (ii) a circular cylinder of diameter 80 cm of height 15 cm and of mass 0.002 kg/cm³, and (iii) a solid sphere of diameter 40 cm and of mass 0.0015 kg/cm³.

Sol. Given :

(i) **For a circular lamina :**

Diameter, $D = 60 \text{ cm}$

$$\therefore \text{Radius, } R = \frac{D}{2} = \frac{60}{2} = 30 \text{ cm}$$

Mass per unit area, $m = 0.001 \text{ kg/cm}^2$

$$\therefore \text{Total mass, } M = m \times \text{Area of circular lamina} \\ = 0.001 \times \pi R^2 = 0.001 \times \pi \times 30^2 = 2.827 \text{ kg.}$$

Moment of inertia of the circular section about the axis of rotation is given by equation (11.10) as,

$$I_{ZZ} = \frac{MR^2}{2} = \frac{2.827}{2} \times (30)^2 = 1272.15 \text{ kg/cm}^2. \quad \text{Ans.}$$

(ii) **For a circular cylinder :**

Diameter, $D = 80 \text{ cm}$

$$\therefore \text{Radius, } R = \frac{D}{2} = \frac{80}{2} = 40 \text{ cm}$$

Height, $h = 15 \text{ cm.}$

Mass per unit volume, $m = 0.002 \text{ kg/cm}^3$

$$\therefore \text{Total mass, } M = m \times \text{Volume of the cylinder} \\ = m \times \pi R^2 \times h \\ = 0.002 \times \pi \times 40^2 \times 15 = 150.796 \text{ kg.}$$

Moment of inertia of a circular cylinder about the axis of rotation is given by equation (11.10) as

$$I_{ZZ} = \frac{MR^2}{2} = \frac{150.796 \times 40^2}{2} = 120636.8 \text{ kg/cm}^2. \quad \text{Ans.}$$

(iii) **For a solid sphere :**

Diameter, $D = 40 \text{ cm}$

$$\therefore \text{Radius, } R = \frac{D}{2} = \frac{40}{2} = 20 \text{ cm}$$

Mass per unit volume, $m = 0.0015 \text{ kg/cm}^3$

$$\begin{aligned}\therefore \text{ Total mass, } M &= m \times \text{Volume of the sphere} \\ &= m \times \frac{4\pi R^3}{3} = \frac{0.0015 \times 4 \times \pi \times 20^3}{3} \text{ kg} = 50.265 \text{ kg.}\end{aligned}$$

Moment of inertia of the solid sphere is given by equation (11.12) as

$$I = \frac{2}{5} MR^2 = \frac{2}{5} \times 50.265 \times 20^2 \text{ kg/cm}^2 = \mathbf{8042.4 \text{ kg/cm}^2}. \quad \text{Ans.}$$

Problem 11.17. Calculate the moment of inertia of a grindstone 90 cm in diameter and 10 cm thick with respect to its axis of rotation. Stone mass is 0.0026 kg/cm³.

Sol. Given :

Diameter of grindstone, $D = 90 \text{ cm}$

$$\therefore \text{ Radius, } R = \frac{90}{2} = 45 \text{ cm}$$

Thickness of grindstone, $t = 10 \text{ cm}$

Mass per unit volume, $m = 0.0026 \text{ kg/cm}^3$

$$\begin{aligned}\therefore \text{ Total mass, } M &= m \times \text{Volume of grindstone} \\ &= m \times \pi R^2 \times t \\ &= 0.0026 \times \pi \times 45^2 \times 10 = 165.4 \text{ kg.}\end{aligned}$$

Moment of inertia (I_{ZZ}) of grindstone (which is similar to cylinder) about the axis of rotation is given by equation (11.10) as

$$I_{ZZ} = \frac{MR^2}{2} = \frac{165.4 \times 45^2}{2} = \mathbf{167467.5 \text{ kg/cm}^2}. \quad \text{Ans.}$$

11.7. LAWS OF MOTION APPLIED TO ROTARY MOTION

11.7.1. First Law. It states that a body continues in its state of rest or of rotation about an axis with constant angular velocity unless it is compelled by an external torque to change the state.

In actual practice, we see that when a body is rotating about an axis with constant angular velocity, the body does not continue in its state of rotation, but comes to rest after some time. This is due to air resistance and friction between the body and its bearing or axis. If these forces had been absent, the body would have gone on rotating indefinitely.

11.7.2. Second Law. It states that the rate of change of angular momentum of a rotating body is proportional to the external torque applied on the body and takes place in the direction of the torque.

Consider a body of moment of inertia I rotating with an angular velocity ω_0 . It is acted upon a torque T and the angular velocity of the body becomes ω . Then we have

ω_0 = Initial angular velocity,

ω = Final angular velocity,

I = Moment of inertia of the body = Mk^2 ,

M = Mass of the body and k = radius of gyration. At this radius* the whole mass of the body is assumed to concentrate in rotation.

*Please refer to chapter of moment of inertia for radius of gyration.

T = Torque acting on the body,

t = Time in seconds.

Initial angular momentum of the body

$$= \text{Moment of Inertia} \times \text{Initial angular velocity} = I \times \omega_0 = I\omega_0$$

Final angular momentum

$$= I \times \omega = I\omega$$

$\therefore$ Change of angular momentum

$$= \text{Final angular momentum} - \text{Initial angular momentum}$$

$$= I\omega - I\omega_0 = I(\omega - \omega_0)$$

$\therefore$ Rate of change of angular momentum

$$= \frac{\text{Change of momentum}}{\text{Time}} = \frac{I(\omega - \omega_0)}{t}.$$

But we know that,

$$\frac{(\omega - \omega_0)}{t} = \alpha = \text{Angular acceleration.}$$

$\therefore$ Rate of change of angular momentum $= I \times \alpha = I\alpha$.

But according to second law of motion for rotation, the rate of change of angular momentum is directly proportional to the external torque applied on the body.

$\therefore$ Torque $\propto$ Rate of change of angular momentum

$$\text{or } T \propto I\alpha \quad \therefore T = kI\alpha \quad \dots(i)$$

where k is a constant of proportionality.

If we define unit torque as that which produces unit angular acceleration on unit moment of inertia, then we have $T = 1$, $I = 1$ and $\alpha = 1$. Substituting these values in equation (i), we get

$$1 = k \times 1 \times 1 = k.$$

Substituting this value of $k = 1$ in equation (i), we get

$$T = I\alpha \quad \dots(11.14)$$

Units of Torque :

The units of torque depends upon the units of I , where I = moment of inertia $= Mk^2$. k is the radius of gyration from the axis of the rotation. It is at this radius that the whole mass of the rotating body is assumed to be concentrated. If I is in kg m^2 , then T is in Newton-metre written as Nm.

Note. If the value of M is taken in kg, then the unit of torque is in N m. But if the value of M is taken as $\frac{W}{g}$, where W is in kgf, then unit of torque will be in kgf-m.

11.7.3. Third Law. It states that to every torque there is always an equal and opposite torque.

11.7.4. Kinetic Energy due to Rotation. Consider a rigid body rotating about O as shown in Fig. 11.26.

Let ω = Angular velocity of the body

dm = Elementary mass of the body

r = Radius of elementary from O

V = Tangential velocity of elementary mass

$$= \omega \times r.$$

K.E. of the elementary mass

$$= \frac{1}{2} \times \text{mass} \times \text{velocity}^2 = \frac{1}{2} \times dm \times V^2.$$

K.E. of the whole body is obtained by integrating the above equation. Hence K.E. of the body

$$= \int \frac{1}{2} dm \times V^2 = \frac{1}{2} \int dm \times (\omega r)^2 \quad (\because V = \omega r)$$

$$= \frac{1}{2} \int \omega^2 r^2 dm = \frac{1}{2} \omega^2 \int r^2 dm. \quad (\because \omega \text{ is a constant})$$

But $\int r^2 dm = I = \text{Moment of Inertia of the body about } O.$

$$\therefore \text{K.E. of the body} = \frac{1}{2} \omega^2 I. \quad \dots(11.15)$$

If mass is taken in kg, the unit of K.E. will be Nm.

11.7.5. Total Kinetic Energy of a Body. Equation (11.15) gives only the K.E. of the body due to rotation. But if the body possesses both the motion of translation as well as the motion of rotation, then the total kinetic energy of body is equal to the K.E. due to rotation plus K.E. due to translation. If the linear velocity of the body is V , then K.E. due to linear velocity $= \frac{1}{2} m V^2$.

$$\therefore \text{Total Kinetic Energy} = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2. \quad \dots(11.16)$$

Problem 11.18. A uniform homogeneous cylinder rolls without slip along a horizontal level surface with a translational velocity of 20 cm/s. If its weight is 0.1 N and its radius is 10 cm, what is its total kinetic energy ?

Sol. Given :

Linear velocity, $V = 20 \text{ cm/s} = 0.20 \text{ m/s}$

Weight of cylinder, $W = 0.1 \text{ N}$

$$\therefore \text{Mass,} \quad M = \frac{W}{g} = \frac{0.1}{9.81} \text{ kg}$$

Radius, $R = 10 \text{ cm} = 0.1 \text{ m}$

Total K.E. is given by equation (11.16)

$$= \frac{1}{2} I \omega^2 + \frac{1}{2} m V^2 \quad \dots(i)$$

where $I = \text{Moment of inertia of solid cylinder}$

$$= \frac{MR^2}{2}$$

[see equation (11.10)]

$$= \frac{0.1}{9.80} \times \frac{0.1^2}{2} = 0.000051$$

$\omega = \text{Angular velocity}$

$$= \frac{V}{R} = \frac{0.20}{0.10} = 2 \text{ rad/s} \quad (\because V = \omega R)$$

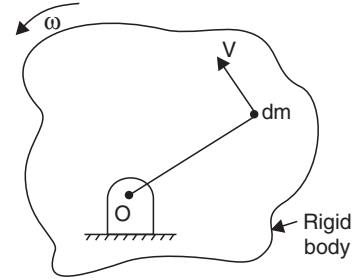


Fig. 11.26. Rigid body rotating about O .

Substituting these values in equation (i), we get

$$\begin{aligned}\text{Total K.E.} &= \frac{1}{2} \times 0.000051 \times 2^2 + \frac{1}{2} \times \frac{0.1}{9.81} \times .2^2 \\ &= 0.000102 + 0.000204 = \mathbf{0.000306 \text{ Nm.}} \quad \text{Ans.}\end{aligned}$$

11.7.6. Rotation due to a Weight W Attached to one end of a String Passing over a Pulley of Weight W_0 . Fig. 11.27 shows a weight W , attached to one end of an inextensible string passing over a pulley. The other end of the string is attached to the periphery of the pulley. Let the weight W is moving downwards. It is required to find the acceleration with which the weight W will move down and the tension P in the string.

Let W = Weight attached to the string,

W_0 = Weight of pulley,

R = Radius of the pulley,

P = Tension in string,

a = Acceleration with which the weight W moves down,

I = Moment of Inertia of the pulley about the axis of rotation, and

α = Angular acceleration of the pulley.

(i) Consider the motion of weight W .

The weight W is moving downwards with an acceleration a , hence net force on this weight is action downwards. [Refer to Fig. 11.27 (a)]

$\therefore$ Net downward force on the weight $W = (W - P)$.

Using, net force = mass $\times$ acceleration

$$\therefore W - P = \frac{W}{g} \times a \quad \left(\because \text{Mass} = \frac{\text{Weight}}{g} \right) \quad \dots(i)$$

(ii) Now consider the rotation of pulley

The torque on the pulley is given by equation (11.14), as

$$T = I \times \alpha \quad \dots(ii)$$

where I = Moment of inertia of the pulley about the axis of rotation.

If pulley is considered as solid disc, then moment of inertia of pulley is given by equation (11.10).

$$\therefore I = \frac{MR^2}{2} \quad (\because \text{Solid disc is like cylinder})$$

In the above equation M = Mass of the pulley

$$= \frac{\text{Weight of the pulley}}{g} = \frac{W_0}{g}$$

$$\therefore I = \frac{W_0}{g} \times \frac{R^2}{2}$$

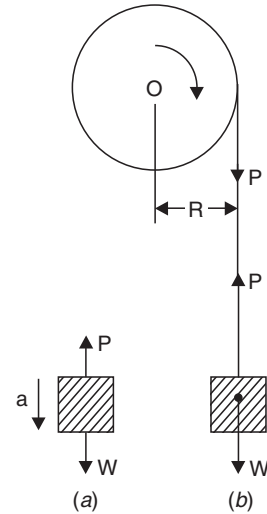


Fig. 11.27

Substituting this value of I in equation (ii), we get

$$T = \left(\frac{W_0}{g} \times \frac{R^2}{2} \right) \times \alpha \quad \dots(iii)$$

But the relation between angular acceleration (α) and linear acceleration is given by as

$$a = \alpha \times R$$

$$\therefore \alpha = \frac{a}{R}.$$

Substituting the value of α in equation (iii), we get

$$T = \left(\frac{W_0}{g} \times \frac{R^2}{2} \right) \times \frac{a}{R} \quad \dots(iv)$$

But the torque on the pulley is due to the tension P^* and its magnitude is given by

$$T = P \times R.$$

Substituting this value of T in equation (iii), we get

$$P \times R = \left(\frac{W_0}{g} \times \frac{R^2}{2} \right) \times \frac{a}{R}$$

$$\text{or} \quad P = \frac{W_0}{2g} \times a. \quad \dots(v)$$

Adding equations (i) and (v), we get the value of acceleration a .

$$\therefore W = \frac{W}{g} \times a + \frac{W_0}{2g} \times a = \frac{a}{g} \left[W + \frac{W_0}{2} \right]$$

$$\therefore a = \frac{gW}{\left(W + \frac{W_0}{2} \right)} \text{ m/s}^2$$

To find the tension (P) in the string, the value of a is substituted in equation (i).

Problem 11.19. A weight of 5 N is suspended by a light rope wound round a pulley of weight 50 N and radius 30 cm, the other end of the rope being fixed to the periphery of the pulley. If the weight is moving downwards, determine :

(i) Acceleration of the weight 5 N, and

(ii) Tension in the string. Take $g = 9.80 \text{ m/s}^2$.

Sol. Given :

Weight suspended by a rope, $W = 5 \text{ N}$

Weight of pulley, $W_0 = 50 \text{ N}$

Radius of pulley, $R = 30 \text{ cm} = 0.30 \text{ m}^{**}$

Acceleration due to gravity, $g = 9.80 \text{ m/s}^2$

Let a = Acceleration of weight 5 N, and

P = Tension in the string.

*If the weight W is not moving, even then the Torque $= P \times R$. But in that case $W = P$ and hence $T = W \times R$. When weight W is moving, $W \neq P$.

**Please, note that the value of R here is extra data. It has not been used.

(i) Consider the motion of weight 5 N. [Refer to Fig. 11.28(a)]. Weight 5 N is moving downwards,

$$\therefore \text{Net force} = \text{Mass} \times \text{acceleration}$$

$$\text{or} \quad (5 - P) = \left(\frac{5}{9.8} \right) \times a \quad \dots(i)$$

(ii) Consider the rotation of the pulley.

$$\text{Net torque} = I \times \alpha$$

$$\text{or} \quad P \times R = I \times \alpha$$

$$\text{where } I = \frac{M_0 R^2}{2} \text{ and } \alpha = \frac{a}{R}$$

$$\therefore P \times R = \frac{M_0 \times R^2}{2} \times \frac{a}{R}$$

$$\text{or} \quad P \times R = \frac{M_0}{2} \times R \times a$$

$$\text{or} \quad P = \frac{M_0}{2} \times a \quad \text{But } M_0 = \frac{W_0}{g} = \frac{50}{9.8}$$

$$\therefore P = \frac{50}{9.8} \times \frac{1}{2} \times a$$

$$\text{or} \quad P = \frac{25}{9.8} \times a \quad \dots(ii)$$

Adding equations (i) and (ii), we get the value of acceleration a .

$$\therefore 5 = \frac{(5 + 25)}{9.8} \times a$$

$$\therefore a = \frac{5 \times 9.8}{30} = \frac{9.8}{6} = 1.633 \text{ m/s}^2. \quad \text{Ans.}$$

To get the value of tension P , substitute this value of a in equation (ii).

$$\therefore P = \frac{25}{9.8} \times 1.633 = 4.166 \text{ N.} \quad \text{Ans.}$$

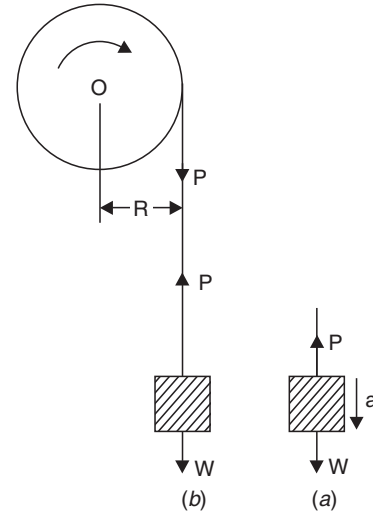


Fig. 11.28

11.7.7. Rotation due to Weights Attached to the two ends of a String, which passes over a Rough Pulley of Weight W_0 . Fig. 11.29 shows the two weights W_1 and W_2 attached to the two ends of a string, which passes over a rough pulley of radius R . As pulley is rough and having certain weight, the tensions on both sides of the string will not be same. If $W_1 > W_2$, the weight W_1 will move downwards whereas the weight W_2 will move upwards with the same acceleration.

Let a = Acceleration of the system ;

T_1 = Tension in the string to which weight W_1 is attached ;

T_2 = Tension in the string to which weight W_2 is attached ;

R = Radius of the pulley ;

I = Moment of inertia of the pulley about the axis of rotation ;

α = Angular acceleration ; and

W_0 = Weight of the pulley.

(i) Consider the motion of weight W_1 . The weight W_1 is moving downwards with an acceleration a . [Refer to Fig. 11.29 (a)]

∴ The net downwards force on weight

$$W_1 = (W_1 - T_1).$$

Mass of weight, $W_1 = \frac{W_1}{g}$.

Now using,

net force = mass $\times$ acceleration

or $(W_1 - T_1) = \frac{W_1}{g} \times a. \quad \dots(i)$

(ii) Now consider the motion of weight W_2 . The weight W_2 is moving upwards with an acceleration a . [Refer to Fig. 11.29 (b)]

∴ Net upwards force = $(T_2 - W_2)$.

Using, net force = Mass $\times$ Acceleration

or $(T_2 - W_2) = \frac{W_2}{g} \times a \quad \dots(ii)$

(iii) Now consider the rotation of the pulley. The pulley is rotating with an angular acceleration α .

But angular acceleration, $\alpha = \frac{a}{R}$.

If the pulley is considered as a solid disc, then moment of inertia of the pulley is given by equation (11.10) as

$$I = \frac{MR^2}{2} \quad (\because \text{Solid disc is like a cylinder})$$

where $M = \text{Mass of pulley} = \frac{\text{Weight of pulley}}{g} = \frac{W_0}{g}$.

$$\therefore I = \frac{W_0}{g} \frac{R^2}{2}$$

The torque on the pulley is given by equation (11.14) as

$$T = I \times \alpha = \frac{W_0}{g} \frac{R^2}{2} \times \frac{a}{R} \quad \left(\because \alpha = \frac{a}{R} \right) \dots(iii)$$

But torque on the pulley = Torque due to T_1 - Torque due to T_2
 $= T_1 \times R - T_2 \times R = R[T_1 - T_2].$

Substituting the value of torque in equation (iii), we get

$$R[T_1 - T_2] = \frac{W_0}{g} \frac{R^2}{2} \times \frac{a}{R}$$

or $T_1 - T_2 = \frac{W_0}{2g} \times a. \quad \dots(iv)$

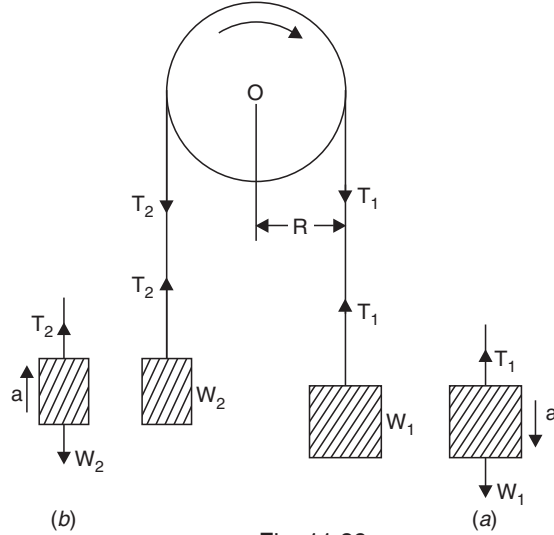


Fig. 11.29

Adding equations (i), (ii) and (iv), we get the value of acceleration a .

$$\therefore W_1 - W_2 = \frac{W_1}{g} \times a + \frac{W_2}{g} \times a + \frac{W_0}{2g} \times a = \frac{a}{g} \left[W_1 + W_2 + \frac{W_0}{2} \right]$$

$$\therefore a = \frac{g[W_1 - W_2]}{\left(W_1 + W_2 + \frac{W_0}{2} \right)}$$

To find the tensions T_1 and T_2 , substitute the value of a in equations (i) and (ii).

Problem 11.20. Two blocks weighing 100 N and 40 N are supported at the ends of a rope of negligible weight which is passing over the rough surface of a pulley mounted on a horizontal axle. The pulley may be assumed as a solid disc with a weight of 50 N. Friction in the bearings of the pulley may be neglected. Find the tension on the two parts of the two rope and the linear acceleration of the blocks.

Sol. Given :

Bigger Weight, $W_1 = 100$ N

Smaller weight, $W_2 = 40$ N

Weight of pulley, $W_0 = 50$ N

Pulley is given as a solid disc.

Let a = Linear acceleration of the system

T_1 = Tension in the rope to which bigger weight is attached

T_2 = Tension in the rope which smaller weight is attached.

(i) Consider the motion of weight 100 N [Refer to Fig. 11.30 (a)]

$$(100 - T_1) = \frac{100}{9.81} \times a \quad \dots(i)$$

(ii) Consider the motion of weight 40 N [Refer to Fig. 11.30 (b)]

$$(T_2 - 40) = \frac{40}{9.81} \times a \quad \dots(ii)$$

(iii) Consider the rotation of the pulley

$$\text{Net torque} = I \times \alpha$$

$$\text{or} \quad (T_1 \times R - T_2 \times R) = \left(\frac{M_0 \times R^2}{2} \right) \times \frac{a}{R} \quad \left[\because I = \frac{M_0 \times R^2}{2} \text{ and } \alpha = \frac{a}{R} \right]$$

$$\text{or} \quad (T_1 - T_2) \times R = \frac{M_0}{2} \times R \times a$$

$$\text{or} \quad (T_1 - T_2) = \frac{M_0}{2} \times a \quad \text{But } M_0 = \frac{W_0}{g} = \frac{50}{9.81}$$

$$\therefore (T_1 - T_2) = \frac{50}{2 \times 9.81} \times a$$

$$\text{or} \quad (T_1 - T_2) = \frac{25}{9.81} \times a \quad \dots(iii)$$

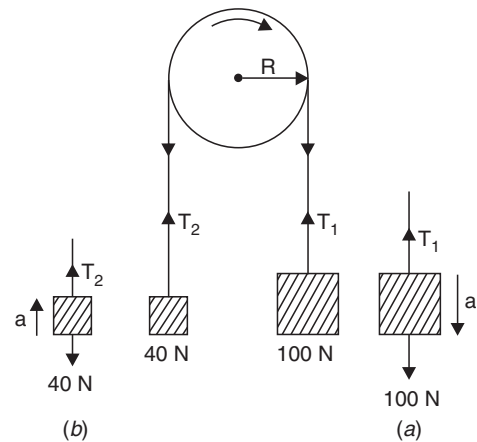


Fig. 11.30

Add equations (i), (ii) and (iii) to get the value of a .

$$\therefore 100 - 40 = \left(\frac{100 + 40 + 25}{9.81} \right) \times a$$

or $60 = \frac{165}{9.81} \times a$

$$\therefore a = \frac{60 \times 9.81}{165} = 3.567 \text{ m/s}^2. \quad \text{Ans.}$$

Substituting the value of a in equations (i) and (ii), we get the values of T_1 and T_2 .
From equation (i), we get

$$(100 - T_1) = \frac{100}{9.81} \times 3.567$$

or $100 - \frac{100}{9.81} \times 3.567 = T_1$

$$\therefore T_1 = 100 - 36.36 = 33.64 \text{ N.} \quad \text{Ans.}$$

From equation (ii), we get

$$T_2 - 40 = \frac{40}{9.81} \times 3.567 = 14.54$$

$$\therefore T_2 = 40 + 14.54 = 54.54 \text{ N.} \quad \text{Ans.}$$

Problem 11.21. A homogeneous solid cylinder of weight 100 N whose axis is horizontal rotates about its axis, in frictionless bearings under the action of the weight of a 10 N block which is carried by a rope wrapped round the cylinder. What will be the angular velocity of the cylinder two seconds after the motion starts? Assume diameter of the cylinder as 100 cm.

Sol. Given :

Weight of cylinder, $W_0 = 100 \text{ N}$

Weight of block, $W = 10 \text{ N}$

Diameter of cylinder, $D = 100 \text{ cm} = 1 \text{ m}$

$\therefore$ Radius of cylinder $R = \frac{1}{2} = 0.5 \text{ m}$

The cylinder is rotating due to the action of weight of 10 N attached to the rope. This means that when weight 10 N is moving downwards, the cylinder is rotating.

Let a = Linear acceleration of the system

P = Tension in the rope

α = Angular acceleration

$$= \frac{a}{R} = \frac{a}{0.5}$$

$$\therefore a = 0.5 \times \alpha.$$

Consider the motion of block of weight 10 N.
The block is moving downwards with an acceleration a .

$\therefore$ Net force = Mass $\times$ Acceleration

or $(10 - P) = \frac{10}{9.80} \times a$

...(i)

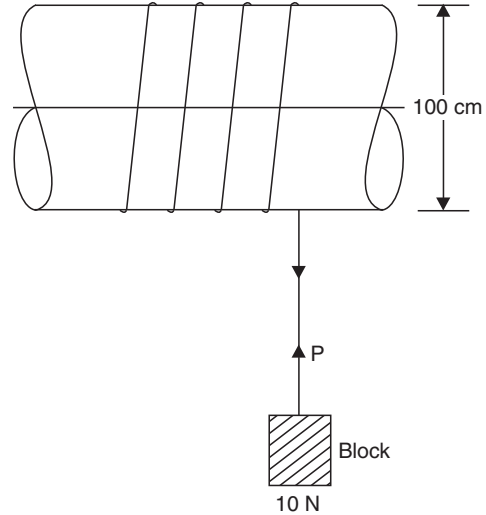


Fig. 11.31

Now consider the rotation of cylinder

Moment of inertia of cylinder about the axis of rotation,

$$I = \frac{MR^2}{2}, \text{ where } M = \text{Mass of cylinder} = \frac{100}{g}$$

$$= \frac{100}{g} \times \frac{(0.5)^2}{2} = \frac{100 \times 0.25}{9.8 \times 2}$$

Net torque,

$$T = I \times \alpha$$

$$\therefore 0.5 P = I \times \alpha \quad (\because T = 0.5 P)$$

$$= \frac{100 \times 0.25}{9.8 \times 2} \times \alpha \quad \left(\because I = \frac{100 \times 0.25}{9.8 \times 2} \right)$$

Multiplying the above equation by 2, we get

$$0.5 \times 2 \times P = \frac{100 \times 0.25}{9.8 \times 2} \times 2 \times \alpha \quad \left(\because \alpha = \frac{a}{R} = \frac{a}{0.5} \right)$$

or

$$P = \frac{25}{9.8} \alpha = \frac{25}{9.8} \times \frac{a}{0.5} = \frac{50a}{9.8} \quad \dots(ii)$$

Adding equations (i) and (ii), we get the value of a .

$$\therefore 10 = \frac{10}{9.8} \times a + \frac{50}{9.8} a = \frac{60}{9.8} \times a$$

$$\therefore a = \frac{10 \times 9.8}{60} = 1.633 \text{ m/s}^2$$

and

$$\alpha = \frac{a}{R} = \frac{1.633}{0.5} = 3.266 \text{ rad/s}^2$$

Now, initial angular velocity of cylinder,

$$\omega_0 = 0$$

(Cylinder starts from rest)

Angular velocity after 2 seconds = ω

Angular acceleration, $\alpha = 3.266 \text{ rad/s}^2$.

Using,

$$\omega = \omega_0 + \alpha t$$

$$= 0 + 3.266 \times 2 = \mathbf{6.532 \text{ rad/s. Ans.}}$$

11.8. LAWS OF MOTION APPLIED TO PLANE MOTION

Plane motion has been defined as the motion of translation and motion of rotation. Hence, plane motion is a combination of translation motion and rotary motion. The laws of plane motion are also the combine laws of translation and rotation. Briefly they are given as :

1. Net force in any direction = Mass $\times$ Acceleration in that direction

$$\text{or } \Sigma F_x = \text{mass} \times a_x = m \times a_x$$

$$\text{and } \Sigma F_y = \text{mass} \times a_y = m \times a_y$$

2. Net torque = $I \times \alpha$

where I = Moment of inertia

α = Angular acceleration.

11.9. WORK AND ENERGY

Work is defined as the product of force and displacement of the body on which force is acting. The force and displacement should be in the same direction. *Energy* is defined as the capacity to do work. The energy exists in many forms like, mechanical, electrical, heat, chemical and light etc. But in engineering mechanics, we only consider mechanical energy. This chapter deals with work and energy.

11.9.1. Work. As defined above, work is the product of force and distance. The distance should be in the direction of the force. If a force P is acting on a body and the body moves a distance of S in the direction of the force, as shown in Fig. 11.32 (a), then the work done on the body is given by :

$$\begin{aligned}\text{Work done} &= \text{Force} \times \text{Distance} \\ &= P \times S\end{aligned}\quad \dots(11.17)$$

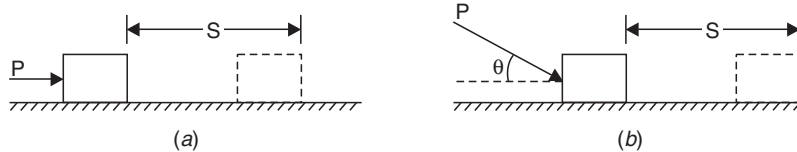


Fig. 11.32

But if the force acting on the body and the distance moved are not in the same direction as shown in Fig. 11.32 (b), then the work done on the body is given by :

$$\begin{aligned}\text{Work done} &= \text{Component of force in the direction of motion} \times \text{Distance} \\ &= P \cos \theta \times S\end{aligned}\quad \dots(11.18)$$

where θ is the angle made by the force with the direction of motion.

If $\theta = 90^\circ$, $\cos \theta = 0$ and hence work done will be zero. Hence if the force and displacement are at right angles, then work done will be zero. If a man of weight W is moving on a horizontal road and the distance moved by man is S , then the work done by the weight of man will be zero. This is due to the fact that W and S are at right angle.

11.9.2. Units of Work. Work is the product of force and distance. In S.I. system, force is expressed in Newton (N) whereas the distance is expressed in metre (m). Hence the unit of work is Nm.

The work done by a force of 1 N on a body which moves a distance of 1 m, is called 1 Nm. This is also known as one Joule (J). Hence one Joule is the work done by a force of 1 N on a body whose displacement is 1 m.

Problem 11.22. A block of wood of weight 800 N is placed on a smooth inclined plane which makes an angle of 30° with the horizontal. Find the work done in pulling the block up for a length of 5 m.

Sol. Given :

Weight, $W = 800 \text{ N}$

Angle, $\theta = 30^\circ$

Distance along the plane moved by block = 5 m.

The inclined plane is given smooth. Hence co-efficient of friction will be zero. This means the force of friction will be also zero.

Let P = Force applied on the wooden block when the block is on the verge of moving upwards.

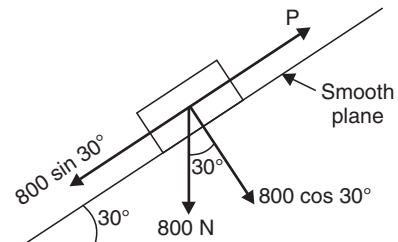


Fig. 11.33

From Fig. 11.33, it is clear that

$$P = W \sin 30^\circ$$

$$\therefore P = 800 \times \sin 30^\circ = 400 \text{ N}$$

$\therefore$ Work done in pulling the block up

$$= P \times \text{Distance moved} = 400 \times 5 = \mathbf{2000 \text{ Nm. Ans.}}$$

Problem 11.23. *If in problem 11.22, the inclined plane is rough and the co-efficient of friction between the block and plane is 0.3, then find the work done in pulling the block up for the length of 5 m.*

Sol. Given :

From Problem 11.22, $W = 800 \text{ N}$, $\theta = 30^\circ$, $S = 5 \text{ m}$

Co-efficient of friction, $\mu = 0.3$

Let P = Force applied so that the block starts moving up the plane.

The forces acting on the block are shown in Fig. 11.34.

For equilibrium of the block

$$R = 800 \cos 30^\circ$$

(Force normal to inclined plane)

$\therefore$ Force of friction,

$$F = \mu R = 0.3 \times 800 \cos 30^\circ = 207.846 \text{ N}$$

Consider forces along the plane,

$$P = 800 \sin 30^\circ + F$$

$$= 400 + 207.846 = 607.846 \text{ N}$$

$\therefore$ Work done $= P \times S$

$$= 607.846 \times 5 = \mathbf{3039.23 \text{ Nm. Ans.}}$$

Problem 11.24. *Determine the work done by electric motor in winding up a uniform cable which hangs from a hoisting drum if its free length is 10 m and weighs 500 N. The drum is rotated by the motor.*

Sol. Given :

Free length of cable, $L = 10 \text{ m}$

Weight of 10 m length cable $= 500 \text{ N}$

Weight of cable per m length $= \frac{500}{10} = 50 \text{ N}$

Refer to Fig. 11.35.

Consider an element of length dx at a distance of x m from free end. This element is to be raised to a height of $(10 - x)$ m. The work done on this element is equal to the product of its weight and the height through which it is raised.

Weight of element = Weight of cable per m length $\times$ length of element

$$= 50 \times dx = 50dx$$

Work done on the element,

$$dW = \text{Weight of element} \times \text{Height through which it is raised}$$

$$= (50dx) \times (10 - x) = 50 \times (10 - x)dx$$

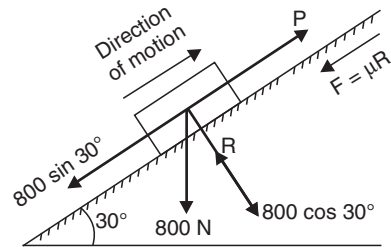


Fig. 11.34

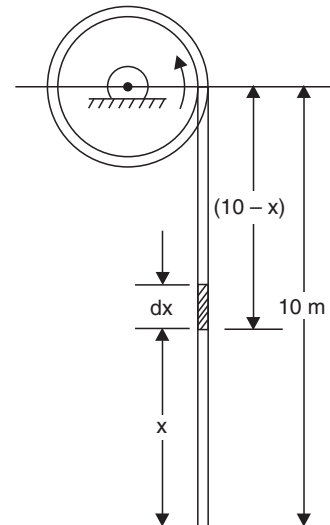


Fig. 11.35

Total work done will be obtained by integrating the above equation from 0 to 10 m.

$$\begin{aligned}\therefore W &= \int_0^{10} 50(10 - x) dx = 50 \left[10x - \frac{x^2}{2} \right]_0^{10} \\ &= 50 \left[10 \times 10 - \frac{10^2}{2} \right] = 50 [100 - 50] = 50 \times 50 = \mathbf{2500 \text{ Nm.}} \quad \text{Ans.}\end{aligned}$$

11.9.3. Power. The rate of doing work is known as power. Hence, power can be obtained by dividing the total work done by time. Or in other words, the power is the work done per second. The unit of power in S.I. system is Nm/s or Watt. It is denoted by W . The power in kilowatts is obtained by dividing watt by 1000.

$$\begin{aligned}\therefore \text{Power} &= \text{Work done per second} \\ &= \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \text{Force} \times \frac{\text{Distance}}{\text{Time}} = \text{Force} \times \text{Velocity} \quad \dots(11.19)\end{aligned}$$

The force and velocity should be in the same direction.

Problem 11.25. A train of weight 2000 kN is pulled by an engine on a level track at a constant speed of 36 kilometre per hour. The resistance due to friction is 10 N per kN of the train's weight. Find the power of the engine.

Sol. Given :

$$\begin{aligned}\text{Weight of train,} & \quad W = 2000 \text{ kN} \\ \text{Speed of train,} & \quad v = 36 \text{ km/hr} \\ & \quad = \frac{36 \times 1000}{60 \times 60} = 10 \text{ m/s}\end{aligned}$$

$$\begin{aligned}\text{Resistance due to friction, } F &= 10 \text{ N per kN weight of train} \\ &= 10 \times \text{weight of train in kN} = 10 \times 2000 = 20,000 \text{ N}\end{aligned}$$

Let P = Force exerted by the engine to pull the train at constant speed.

$$\therefore \text{Net force in the direction of motion} = P - F = (P - 20,000) \text{ N}$$

As engine is moving with uniform speed, the acceleration of engine will be zero. The net force, which is equal to mass $\times$ Acceleration, will also be zero.

$$\begin{aligned}\therefore \text{Net force} &= 0 \\ \text{or } P - 20,000 &= 0 \\ \text{or } P &= 20,000 \text{ N}\end{aligned}$$

Now, the power is given by equation (11.19), as

$$\begin{aligned}\text{Power} &= \text{Force exerted by engine} \times \text{Velocity} \\ &= 20,000 \times 10 \text{ Nm/s} \\ &= 200,000 \text{ W} \quad (\because \text{Watt, } W = \text{Nm/s}) \\ &= \mathbf{200 \text{ kW.}} \quad \text{Ans.}\end{aligned}$$

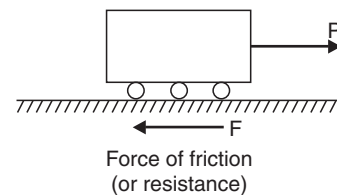


Fig. 11.36

Problem 11.26. If in problem 11.26, the train is to move with an acceleration of 0.5 m/s^2 on the level track after attaining a speed of 36 km/hr, then find the power of the engine.

Sol. Given :

$$\begin{aligned}\text{From problem 11.26,} & \quad W = 2000 \text{ kN} \\ \text{Resistance force,} & \quad F = 20,000 \text{ N} \\ \text{Velocity,} & \quad v = 10 \text{ m/s} \\ \text{Acceleration,} & \quad a = 0.5 \text{ m/s}^2\end{aligned}$$

$$\begin{aligned}\text{Now, mass of the train, } m &= \frac{W}{g} = \frac{2000 \text{ kN}}{9.81 \text{ m/s}^2} \\ &= \frac{2000 \times 1000 \text{ N}}{9.81 \text{ m/s}^2} = \frac{2000 \times 1000}{9.81} \text{ kg}\end{aligned}$$

Let P = Force exerted by engine, when train is moving with an acceleration of 0.5 m/s^2
Now, using the relation,

Net force = Mass $\times$ Acceleration

$$\text{or } (P - F) = m \times a \quad \text{or } F - 20,000 = \left(\frac{2000 \times 1000}{9.81} \right) \times 0.5$$

$$\therefore P = 20,000 + \frac{2000 \times 1000}{9.81} \times 0.5 = 20,000 + 101936.1 = 121936.8 \text{ N}$$

$$\begin{aligned}\therefore \text{Power} &= \text{Force exerted by engine} \times \text{Velocity} = 121936.8 \times 10 \text{ N m/s} \\ &= 1219368 \text{ W} = \mathbf{1219.368 \text{ kW. Ans.}}\end{aligned}$$

Problem 11.27. A train of weight 1500 kN , is ascending (i.e., going upwards) a slope of 1 in 100 with a uniform speed of 36 km/hr . Find the power exerted by the engine, if the road resistance is $5 \text{ N per kN weight of the train}$.

Sol. Given :

$$\begin{aligned}\text{Weight of train, } W &= 1500 \text{ kN} \\ &= 1500 \times 1000 \text{ N} \\ \text{Slope*} &= 1 \text{ in } 100 \\ \therefore \sin \theta &= \frac{1}{100} \\ \text{Speed, } v &= 36 \text{ km/hr} \\ &= \frac{36 \times 1000}{60 \times 60} \text{ m/s} = 10 \text{ m/s}\end{aligned}$$

Resistance force (or Force of friction)

$$\begin{aligned}F &= 5 \text{ N per kN weight of train} \\ &= 5 \text{ N} \times \text{Weight of train in kN} = 5 \times 1500 = 7500 \text{ N}\end{aligned}$$

Let F = Force exerted by the engine.

Fig. 11.37 shows the various forces acting on the train, when it is moving up. The net force acting along the plane in the upward direction.

$$\begin{aligned}\therefore \text{Net force} &= P - W \sin \theta - F \\ &= P - (1500 \times 1000) \times \frac{1}{100} - 7500 = P - 15000 - 7500\end{aligned}$$

But the train is moving with uniform speed, and hence acceleration of the train is zero. If acceleration is zero, then net force acting on the engine should also be zero.

$$\therefore P - 15000 - 7500 = 0$$

$$\text{or } P = 15000 + 7500 = 22500 \text{ N}$$

$$\begin{aligned}\therefore \text{Power exerted by engine} &= \text{Force exerted by engine} \times \text{Velocity} \\ &= 22500 \times 10 \text{ Nm/s} = 225000 \text{ W} = \mathbf{225 \text{ kW. Ans.}}\end{aligned}$$

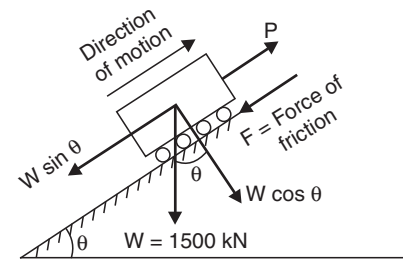


Fig. 11.37

* When slope is very small, then $\theta = \tan \theta = \sin \theta$.

Problem 11.28. A train of weight 2000 kN moves down a slope of 1 in 150 at 18 km/hr and engine develops a power of 35 kW. If the train is pulled up at the same speed, what power will be required to pull the train.

Sol. Given :

Weight, $W = 2000 \text{ kN} = 2000 \times 1000 \text{ N}$

Slope $= 1 \text{ in } 150$

$$\therefore \sin \theta = \frac{1}{150}$$

$$\text{Speed of train, } v = 18 \text{ km/hr} = \frac{18 \times 1000}{60 \times 60} = 5 \text{ m/s}$$

$$\text{Power developed by engine} = 35 \text{ kW} = 35 \times 1000 \text{ W} = 35000 \text{ W.}$$

1st Case. Train is moving down the plane as shown in Fig. 11.38 with a constant velocity and hence it will not have any acceleration. So the net force acting on the train in the direction of motion should be zero. But the forces acting on the train are :

- (i) $W \sin \theta$ in the direction of motion ;
- (ii) Force of friction (F) in the opposite direction of motion ;
- (iii) Force (P) exerted by engine in the direction of motion.

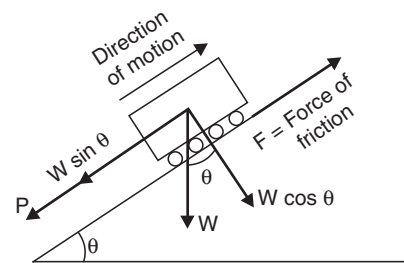


Fig. 11.38

Now, net force acting on the engine in the direction of motion

$$\begin{aligned} &= W \sin \theta - F + P = (2000 \times 1000) \times \frac{1}{150} - F + P \\ &= 13333.3 - F + P \end{aligned} \quad \dots(i)$$

The power developed by engine

$$\begin{aligned} &= (\text{Force exerted by Engine}) \times \text{Velocity} \\ &= P \times 5 \quad \text{or} \quad 35,000 = 5 \times P \end{aligned}$$

$$\therefore P = \frac{35,000}{5} = 7000 \text{ N}$$

Substituting the value of P in equation (i), we get

$$\text{Net force} = 13333.3 - F + 7000$$

But net force in the direction of motion is zero.

$$\begin{aligned} \therefore 13333.3 - F + 7000 &= 0 \quad \text{or} \quad 13333.3 + 7000 = F \quad \text{or} \quad 20333.3 = F \\ \therefore F &= 20333.3 \text{ N} \end{aligned} \quad \dots(ii)$$

2nd Case. The train is moving up the same plane with the same speed as shown in Fig. 11.39.

As the plane is same and hence frictional force will be same.

Let P^* = Force exerted by engine while moving up

Net force in the direction of motion

$$\begin{aligned} &= P^* - W \sin \theta - F \\ &= P^* - (2000 \times 1000) \times \frac{1}{150} - 20333.3 \end{aligned}$$

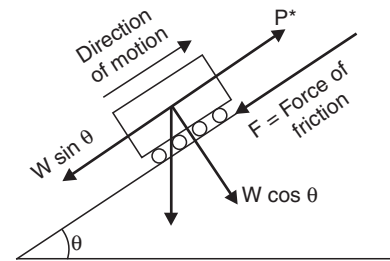


Fig. 11.39

$$= P^* - 13333.3 - 20333.3$$

$$= P^* - 33666.6.$$

The train is moving with constant speed and hence acceleration is zero. This means the net force will also be zero.

$$\therefore P^* - 33666.6 = 0 \quad \text{or} \quad P^* = 33666.6 \text{ N}$$

$\therefore$ Power developed by engine

$$= \text{Force exerted by engine} \times \text{Speed}$$

$$= P^* \times v = 33666.6 \times 5 \text{ Nm/s} = 168333 \text{ W} = \mathbf{168.333 \text{ kW. Ans.}}$$

11.9.4. Work done by a Torque. Fig. 11.40 shows a circular body, which can rotate about the point A. A tangential force P is applied at point B and the body moves through a small angle θ .

Let R = Radius.

Now, the distance moved by the force P in rotating the body through an angle θ is equal to the length of arc BC .

But length of arc $BC = R \times \theta$.

As the body has moved a distance $R \times \theta$, the work will be done by the force on the body.

$\therefore$ Work done

$$= \text{Force} \times \text{Distance moved}$$

$$= P \times R \times \theta = (P \times R) \times \theta$$

But $P \times R$ is equal to torque (T).

$$\therefore \text{Work done} = T \times \theta$$

...(11.20)

In the above equation, θ should be taken in radians.

11.9.5. Power Developed by a Torque

Let T = torque

N = r.p.m. (or revolution per minute)

Consider a body acted upon by a torque (T) and rotating at N r.p.m.

Now, angle turned in one revolution = 2π or 360° .

$$\therefore \text{Angle turned in } N \text{ revolution} = 2\pi \times N.$$

But N revolutions are performed in one minute.

$$\therefore \text{Angle turned in one minute} = 2\pi N$$

$$\text{or} \quad \text{Angle turned in one second} = \frac{2\pi N}{60}$$

Now, using equation (11.20), we get

$$\text{Work done} = T \times \theta$$

$$\text{or} \quad \text{Work done per second} = T \times \text{Angle turned in one second}$$

$$= T \times \frac{2\pi N}{60} \text{ Nm/s}$$

But, power = Work done per second

$$= T \times \frac{2\pi N}{60} = \frac{2\pi NT}{60} \text{ W.} \quad \text{...(11.21)}$$

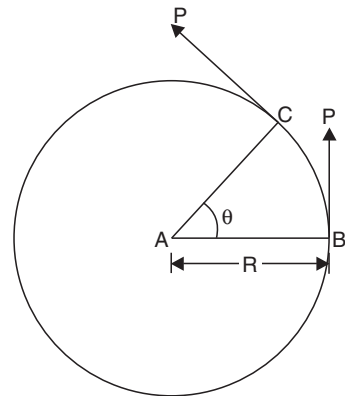


Fig. 11.40

Problem 11.29. A tangential force of 1800 N is acting on a shaft of diameter 10 mm. Find the work done by the force for one revolution of the shaft.

Sol. Given :

Force, $P = 1800 \text{ N}$

Dia. of shaft, $D = 10 \text{ mm} = \frac{10}{1000} = 0.01 \text{ m}$

$\therefore$ Radius of shaft, $R = \frac{0.01}{2} = 0.005 \text{ m}$

Angle turned for one revolution = 2π radians

$\therefore \theta = 2\pi$ radians

Torque, $T = P \times R = 1800 \times 0.005 = 9.0 \text{ Nm}$

Work done is given by equation (11.20).

$\therefore$ Work done = $T \times \theta$
 $= 9 \times 2\pi \text{ Nm} = \mathbf{56.54 \text{ Nm. Ans.}}$

Problem 11.30. If in Problem 11.29, the shaft is rotating at 200 r.p.m., find the power of the shaft.

Sol. Given :

From Problem 11.29, $P = 1800 \text{ N}$, $R = 0.005 \text{ m}$, $T = 9 \text{ Nm}$

$N = 200 \text{ r.p.m.}$

Using equation (11.21), we get

Power = $\frac{2\pi NT}{60} \text{ W} = \frac{2\pi \times 200 \times 9}{60} = \mathbf{188.49 \text{ W. Ans.}}$

11.9.6. Energy. The capacity of doing work is known as energy. It is the product of power and time. The energy is expressed in Nm. It exists in many forms *i.e.*, mechanical, electrical, heat, chemical, light etc. In engineering mechanics, we are only concerned with mechanical energy and the same will be dealt with.

11.9.7. Mechanical Energy. It consists of the following two types :

- (i) Potential energy (or position energy or datum energy)
- (ii) Kinetic energy.

Depending upon the state of rest or that of motion, a body may possess anyone or both of the above energies.

1. Potential Energy. Potential energy is also known as position energy or datum energy. It is the energy by *virtue of position* of a body with respect to any given reference or datum. A weight W lying on the top of a tower of height h possesses a potential energy of $W \times h$ with respect to the ground, as the weight W is capable of doing $W \times h$ work if it falls on the ground.

A compressed spring has potential energy, because it can do work in recovering its original shape. Similarly, compressed air also possesses potential energy because it is capable of doing work when allowed to expand.

2. Kinetic Energy. The energy possessed by a body by virtue of its velocity (or its motion) is known as kinetic energy. It is represented by K.E.

11.10. KINETIC ENERGY

The energy possessed by a body due to its velocity is known as kinetic energy. It is written as K.E.

The expression for the kinetic energy is derived as follows : Consider a body of mass m starting from rest. Let it be subjected to an accelerating force F and after covering a distance S , its velocity becomes v .

$\therefore$ Initial velocity, $u = 0$

$$\begin{aligned}\text{Now, work done on the body} &= \text{Force} \times \text{Distance} \\ &= F \times S\end{aligned}\quad \dots(i)$$

But $\text{Force} = \text{Mass} \times \text{Acceleration}$

$$\therefore F = m \times a$$

Substituting the value of F in equation (i),

$$\begin{aligned}\text{Work done} &= m \times a \times S \\ &= m \times (a \times S)\end{aligned}\quad \dots(ii)$$

But from equation of motion, we have

$$v^2 - u^2 = 2a \times S \quad \text{or} \quad v^2 - 0^2 = 2a \times S \quad (\because u = 0)$$

$$\text{or} \quad a \times S = \frac{v^2}{2}$$

Substituting the value of $a \times S$ in equation (ii),

$$\text{Work done} = m \times \frac{v^2}{2}$$

But work done on the body is equal to K.E. possessed by the body.

$$\therefore \text{K.E.} = \frac{1}{2}mv^2.$$

11.11. PRINCIPLE OF WORK AND ENERGY

It states that the change in kinetic energy of a body during any displacement is equal to the work done by the net force acting on the body. Or we can say that work done is equal to change of kinetic energy of the body.

The principle of work and energy is derived as follows :

$$\text{We know that,} \quad F = m \times a \quad \dots(i)$$

where F = resultant of all forces acting on a body,

m = mass of the body, and

a = acceleration* in the direction of resultant force

$$= v \frac{dv}{ds}$$

Substituting the value of a in equation (i), we get

$$F = m \times \left(v \frac{dv}{ds} \right) \quad \text{or} \quad F \cdot ds = m \times v dv \quad \dots(ii)$$

But $F \cdot ds$ is the work done by the resultant force F in displacing the body by a small distance ds . The total work done by the resultant force F in displacing the body by a distance S is obtained by integrating the above equation (ii).

*Acceleration, $a = \frac{dv}{dt}$. But $v = \frac{ds}{dt}$. Hence $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v$ or $a = v \frac{dv}{ds}$.

Hence, integrating equation (ii) on both sides, we get

$$\int_0^S F \cdot ds = \int_u^v m \times v \times dv$$

where u is the initial velocity when distance is zero and v is the final velocity when distance is S .

$$\therefore F \cdot S = m \cdot \left[\frac{v^2}{2} \right]_u^v = \frac{m}{2} [v^2 - u^2] = \frac{mv^2}{2} - \frac{mu^2}{2} \quad \dots(iii)$$

or Work done by resultant force = change in kinetic energy

Note. All the forces acting on a body do not do work when the body is displaced. These forces are : (i) the forces acting on a fixed point such as support reactions of pin ball and socket connections about which the body rotates, (ii) the forces acting perpendicular to direction of motion such as normal reaction of the surface over which the body is moving and weight of body if body moves in a horizontal direction.

Problem 11.31. A bullet of mass 81 gm and moving with a velocity of 300 m/s is fired into a block of wood and it penetrates to a depth of 10 cm. If the bullet moving with the same velocity, were fired into a similar piece of wood 5 cm thick, with what velocity would it emerge ? Find also the force of resistance, assuming it to be uniform.

Sol. Given :

Mass of bullet, $m = 81 \text{ gm} = 0.081 \text{ kg}$

Initial velocity of bullet, $u = 300 \text{ m/s}$

First Case

Penetration of bullet, $S = 10 \text{ cm} = 0.1 \text{ m}$

Final velocity of bullet, $v = 0$

Let P = Force of resistance in Newton (N)

The force of resistance will be acting in the opposite direction of motion of bullet. The work done by the force of resistance

$$= \text{Force of resistance} \times \text{Depth of Penetration}$$

$$= (-P) \times S = (-P) \times 0.1 \text{ N m} = -0.1P$$

(-ve sign is due to opposite direction of force)

$$\text{Change of K.E. of bullet} = \frac{1}{2} mv^2 - \frac{1}{2} mu^2 = 0 - \frac{1}{2} \times 0.081 \times 300^2 \text{ Nm} = -3645 \text{ Nm}$$

But the change of K.E. of the bullet is equal to work done

$$\text{Change of K.E. of bullet} = \text{Work done by force of resistance}$$

$$\text{or} \quad -3645 = -0.1P$$

$$\therefore P = \frac{3645}{0.1} = 36450 \text{ N}$$

Second Case. Velocity of bullet after 5 cm penetration

The force of resistance will be same i.e., $P = 36450 \text{ N}$.

Now, depth of penetration,

$$S = 5 \text{ cm} = 0.05 \text{ m}$$

$\therefore$ Work done by force of resistance

$$= -P \times S = -36450 \times 0.05 \text{ Nm}$$

(-ve sign is due to opposite direction of force)

$$= -1822.5 \text{ Nm}$$

Let

$$v_1 = \text{Velocity of bullet after 5 cm penetration}$$

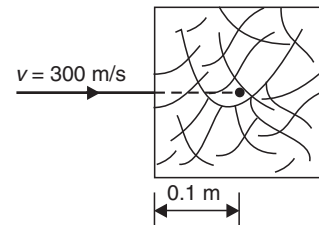


Fig. 11.41

Change of K.E. of bullet after penetration

$$\begin{aligned}
 &= \frac{1}{2} m v_1^2 - \frac{1}{2} m u^2 \\
 &= \frac{1}{2} \times 0.081 \times v_1^2 - \frac{1}{2} \times 0.081 \times 300^2 \quad (\because u = 300 \text{ m/s})
 \end{aligned}$$

But the change of K.E. is equal to work done

$$\therefore \frac{1}{2} \times 0.081 \times v_1^2 - \frac{1}{2} \times 0.081 \times 300^2 = -1822.5$$

$$\text{or} \quad \frac{1}{2} \times 0.081 \times v_1^2 - 3645 = -1822.5$$

$$\text{or} \quad \frac{1}{2} \times 0.081 v_1^2 = 3645 - 1822.5 \quad \text{or} \quad \frac{1}{2} \times 0.081 v_1^2 = 1822.5$$

$$\text{or} \quad v_1 = \left(\frac{1822.5 \times 2}{0.081} \right)^{1/2} = 212.132 \text{ m/s.}$$

11.12. CONSERVATIVE FORCES

Fig. 11.41(a) show a body of weight ' $m \times g$ ' which is moved from position 1 to position 2. The work done by the force on the body will be $mg \times y$ where y is the vertical distance between points 1 and 2. This work done by the force is independent of the path followed by the body. Such a force is known as conservative force.

Hence, if the work of a force in moving a body between two positions is independent of path followed by the body and work can be expressed as a change in its potential energy, then such a force is known as a **conservative force**.

Gravity force is a conservative force whereas the frictional force is a non-conservative force. The work of friction force depends upon the path followed and this work cannot be expressed as a change in potential energy.

So it is a non-conservative force. Force due to gravity, spring force and elastic force are conservative forces.

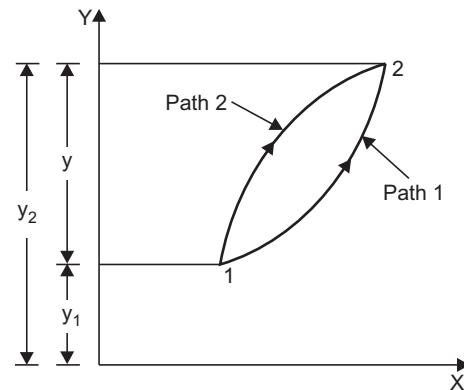


Fig. 11.41(a)

11.13. LAW OF CONSERVATION OF ENERGY

It states that the energy can neither be created nor destroyed though it can be transformed from one form to another form. The second statement of this law is :

“The total energy possessed by a body remains constant provided no energy is added to or taken from it.”

Fig. 11.42 shows a body resting on the top of a tower.

Let W = Weight of body

$$m = \text{Mass of body} = \frac{W}{g}$$

h = Height of tower

Let us find the potential energy and K.E. of the body at the top of the tower.

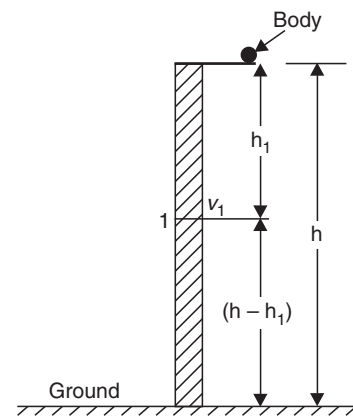


Fig. 11.42

Potential energy of the body with respect to ground

$$= W \times h \text{ Nm} \quad \dots(i)$$

K.E. of the body

$$= \frac{1}{2} mv^2 = 0 \quad (\because v = 0)$$

$\therefore$ Total energy at the top of tower

= Potential energy + Kinetic energy

$$= Wh + 0 = Wh$$

Suppose the body falls down by a height h_1 to the position 1 as shown in Fig. 11.42. Let v_1 is the velocity at position 1.

Using equation $v^2 - u^2 = 2gh$, we get

$$v_1^2 - 0^2 = 2gh_1 \quad (\because u = 0, h = h_1)$$

$$v_1^2 = 2gh_1$$

$\therefore$ K.E. of the body at position 1

$$= \frac{1}{2} mv_1^2 = \frac{1}{2} m \times 2gh_1 \quad (\because v_1^2 = 2gh_1)$$

$$= mgh_1 = Wh_1 \quad (\because W = mg)$$

Potential energy at position 1

$$= W \times \text{Height of body at 1 w.r.t. ground} = W \times (h - h_1)$$

$\therefore$ Total energy at position 1

= Potential energy + Kinetic energy

$$= W(h - h_1) + Wh_1 = Wh - Wh_1 + Wh_1 = Wh$$

And it is the same energy, the body was possessing at the top of the tower.

Similarly, it can be proved that total energy of the body on the ground will be equal to 'Wh'.

Problem 11.32. A body weighing 196.2 N slides up a 30° inclined plane under the action of an applied force 300 N acting parallel to the inclined plane. The co-efficient of friction, μ is equal to 0.2. The body moves from rest. Determine :

- (i) acceleration of the body,
- (ii) distance travelled by body in four seconds,
- (iii) velocity of body after four seconds,
- (iv) kinetic energy of the body after four seconds,
- (v) work done on the body in four seconds,
- (vi) momentum of the body after four seconds,
- (vii) impulse applied in four seconds.

Sol. Given :

Weight, $W = 196.2 \text{ N}$

$$\therefore \text{ Mass, } m = \frac{W}{g} = \frac{196.2}{9.81} = 20 \text{ kg}$$

Applied force $= 300 \text{ N}$

Angle of inclination, $\theta = 30^\circ$

Co-efficient of friction, $\mu = 0.2$

As body moves from rest, hence initial velocity (u) will be zero.

$$\therefore u = 0$$

Fig. 11.43 shows the free-body diagram. The net force in the direction of motion is given by

$$\begin{aligned} F &= \text{Applied force} - W \sin \theta - \mu R \\ &= 300 - 196.2 \times \sin 30^\circ - 0.2 \times W \cos \theta \\ &\quad (\because R = W \cos \theta) \\ &= 300 - 98.1 - 0.2 \times 196.2 \times \cos 30^\circ \\ &= 300 - 98.1 - 33.98 = 167.92 \text{ N} \end{aligned}$$

(i) Let a = acceleration of the body.

But Net force = Mass $\times$ Acceleration

$$\begin{aligned} \text{or} \quad F &= m \times a \\ \text{or} \quad 167.92 &= 20 \times a \end{aligned}$$

$$\therefore a = \frac{167.92}{20} = 8.396 \text{ m/s}^2. \text{ Ans.}$$

(ii) Let s = distance travelled in four seconds

$$\begin{aligned} \text{Using} \quad s &= ut + \frac{1}{2} at^2 \\ &= 0 \times 4 + \frac{1}{2} \times 8.396 \times 4^2 = 67.168 \text{ m. Ans.} \quad (\because u = 0) \end{aligned}$$

(iii) Let v = velocity after 4 seconds

$$\begin{aligned} \therefore v &= u + at \\ &= 0 + 8.396 \times 4 = 33.584 \text{ m/s. Ans.} \end{aligned}$$

(iv) The kinetic energy after 4 second is given by

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} mv^2 \quad \text{where } v = \text{velocity after 4 seconds} \\ &= \frac{1}{2} \times 20 \times 33.584^2 = 11278.8 \text{ Nm. Ans.} \end{aligned}$$

(v) Work done on the body in four seconds

$$\begin{aligned} &= \text{Net force} \times \text{distance moved in four seconds} \\ &= 167.92 \times 67.168 \text{ Nm} = 11278.8 \text{ Nm. Ans.} \end{aligned}$$

The work done on the body is equal to the change of kinetic energy of the body as proved below :

$$\begin{aligned} \text{Change of K.E.} &= \frac{1}{2} mv^2 - \frac{1}{2} mu^2 \\ &= \frac{1}{2} \times 20 \times 33.584^2 - \frac{1}{2} \times 20 \times 0^2 \quad (\because u = 0) \\ &= 11278.8 - 0 = 11278.8 \text{ Nm.} \end{aligned}$$

(vi) Momentum of the body after four seconds

$$= m \times v = 20 \times 33.584 = 671.68 \text{ kg m/s. Ans.}$$

(vii) Impulse* applied in four seconds

$$\begin{aligned} &= \text{Net force} \times \text{time} = F \times 4 \quad (\because \text{time} = 4 \text{ s}) \\ &= 167.92 \times 4 = 671.68 \text{ N s. Ans.} \end{aligned}$$

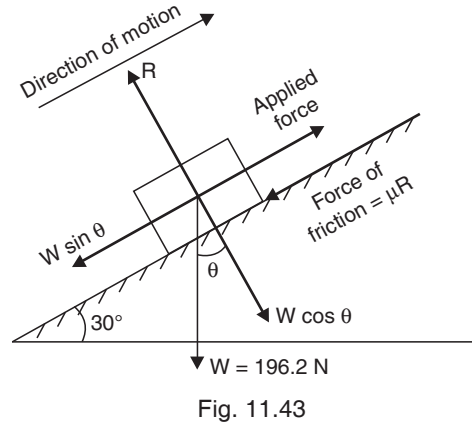


Fig. 11.43

*Impulse = $F \times dt$.

Change of momentum of the body

$$\begin{aligned}
 &= mv - mu = 20 \times 33.584 - 20 \times 0 \\
 &= 671.68 \text{ kg m/s} \\
 &= 671.68 \text{ N s} \quad [\because \text{kg m/s} = \text{kg m/s} \times \text{s/s} = (\text{kg} \times \text{m/s}^2) \times \text{s} \\
 &\quad = \text{N} \times \text{s} = \text{N s}]
 \end{aligned}$$

Change of momentum of the body is equal to the impulse of the body.

Problem 11.33. A bullet of mass 25 gm, moving horizontal with a velocity of 600 m/s strikes a wooden block of mass 5 kg resting on a rough horizontal surface. The bullet after striking the block remains buried in the block and both travels a distance of 90 cm, before coming to rest. Determine (i) average resistance between block and horizontal surface, (ii) co-efficient of friction between block and horizontal surface.

Sol. Given :

$$\begin{aligned}
 \text{Mass of bullet,} & \quad m_1 = 25 \text{ gm} = 0.025 \text{ kg} \\
 \text{Initial velocity of bullet,} & \quad u_1 = 600 \text{ m/s} \\
 \text{Mass of wooden block,} & \quad m_2 = 5 \text{ kg} \\
 \text{Initial velocity of block,} & \quad u_2 = 0 \\
 \text{Total mass of bullet and block,} & \quad M = m_1 + m_2 \\
 & \quad = (0.025 + 5) = 5.025 \text{ kg}
 \end{aligned}$$

Let V = common velocity of bullet and block after impact

The law of conservation of momentum gives :

Total momentum before impact = Total momentum after impact

$$\text{or} \quad (m_1 u_1 + m_2 u_2) = M \times V \quad \text{or} \quad 0.025 \times 600 + 5 \times 0 = 5.025 \times V$$

$$\therefore V = \frac{0.025 \times 600}{5.025} = 2.985 \text{ m/s}$$

(i) Average resistance between block and horizontal surface

If we apply the principle of work and energy to the system of block and bullet, then the initial velocity of block and bullet will be their common velocity after impact and final velocity of the system will be zero.

Let P = Average frictional resistance

S = Distance travelled by block and bullet

$$= 90 \text{ cm} = 0.9 \text{ m (given)}$$

Initial velocity of block and bullet, $V_i = V = 2.985 \text{ m/s}$

After travelling a distance of 90 cm, the block and bullet comes to rest.

Hence, final velocity of block and bullet, $V_f = 0$

Change of K.E. of bullet and block

$$\begin{aligned}
 &= \frac{1}{2} M (V_f^2 - V_i^2) = \frac{1}{2} \times 5.025 (0 - 2.985^2) \\
 &= -\frac{1}{2} (5.025) \times 2.985^2 \text{ Nm} \quad \dots(i)
 \end{aligned}$$

Work done by frictional resistance

= Frictional resistance $\times$ Distance

$$= (-P) \times 0.9 \text{ Nm.} \quad \dots(ii)$$

(-ve sign is due to opposite direction of frictional resistance)

The work done by the frictional resistance is equal to the change of K.E.

Equating the two equations (i) and (ii), we get

$$-P \times 0.9 = -\frac{1}{2} \times 5.025 \times 2.985^2$$

or
$$P = \frac{1}{2} \times \frac{5.025 \times 2.985^2}{0.9}$$

$$= 24.874 \text{ N. Ans.}$$

(ii) To find co-efficient of friction

Let μ = Co-efficient of friction

R = Normal reaction

W = Total weight of bullet and block.

Refer to Fig. 11.44

The frictional force calculated above

$$= P = 24.874 \text{ N}$$

$$W = \text{Total weight} = (\text{Total mass}) \times g = M \times g = 5.025 \times 9.81 \text{ N}$$

$$\therefore R = W = 5.025 \times 9.81 \text{ N}$$

Now using the equation,

$$\text{Force of friction} = \mu \times R$$

$$\therefore \mu = \frac{\text{Force of friction}}{R} = \frac{24.874}{5.025 \times 9.81} = 0.5046. \text{ Ans.}$$

Problem 11.34. A 10 gm bullet is shot horizontally in a wood block of mass 1 kg. The bullet gets embedded in the block and the block is displaced on a rough horizontal table ($\mu = 0.2$) through 1 m. What was the velocity of bullet?

Sol. Given :

Mass of bullet, $m_1 = 10 \text{ gm} = 0.01 \text{ kg}$

Mass of block, $m_2 = 1 \text{ kg}$

Distance travelled by block and bullet, $S = 1 \text{ m}$

Value of $\mu = 0.2$

Let u_1 = Initial velocity of bullet

u_2 = Initial velocity of block which is equal to zero

V = Common velocity of bullet and block after impact

M = Total mass of bullet and block = $0.01 + 1 = 1.01 \text{ kg}$

From the law of conservation of momentum, we get

Total momentum before impact = Total momentum after impact

or $(m_1 u_1 + m_2 u_2) = M \times V$

or $(0.01 \times u_1 + 1 \times 0) = 1.01 \times V$ or $0.01 u_1 = 1.01 V$

$$\therefore V = \frac{0.01 u_1}{1.01} \quad \dots(i)$$

Now, force of friction = μR (R is normal reaction)

$$= \mu W \quad (\text{Here } R = W)$$

$$= \mu \times M \times g \quad (\because W = M \times g)$$

$$= 0.2 \times 1.01 \times 9.81 \text{ N}$$

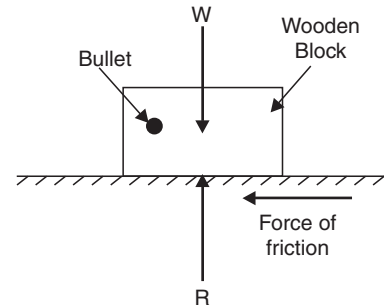


Fig. 11.44

$$\begin{aligned}
 \text{Work done by force of friction} &= \text{Force of friction} \times \text{Distance moved} \\
 &= -0.2 \times 1.01 \times 9.81 \times 1 \text{ Nm} \\
 &= -1.9816 \text{ Nm} \quad \dots(ii) \\
 &\quad (\text{-ve sign is due to opposite direction of force of friction})
 \end{aligned}$$

$$\text{Initial velocity of block and bullet, } V_i = V = \frac{0.01u_1}{1.01}$$

After travelling a distance of 1 m, the block and bullet comes to rest.

Hence final velocity of block of bullet, $V_f = 0$

$$\begin{aligned}
 \text{Change of K.E. of bullet and block} &= \frac{1}{2}M[V_f^2 - V_i^2] = \frac{1}{2} \times 1.01 \left[0 - \left(\frac{0.01u_1}{1.01} \right)^2 \right] \\
 &= -\frac{1}{2} (1.01) \times \left[\left(\frac{0.01u_1}{1.01} \right)^2 \right] \quad \dots(iii)
 \end{aligned}$$

But the work done by force of friction is equal to change of K.E. of block and bullet.

Equating the work done by force of friction to the change of K.E. of bullet and block *i.e.*, equating equations (ii) and (iii), we get

$$\begin{aligned}
 -1.9816 &= -\frac{1}{2} \times 1.01 \times \left(\frac{0.01u_1}{1.01} \right)^2 \\
 \left(\frac{0.01u_1}{1.01} \right)^2 &= \frac{1.9816 \times 2}{1.01} \left(\frac{0.01u_1}{1.01} \right) = \sqrt{\frac{1.9816 \times 2}{1.01}} = 1.98 \text{ m/s} \\
 \text{or } 0.01 u_1 &= 1.01 \times 1.98 \\
 \text{or } u_1 &= \frac{1.01 \times 1.98}{0.01} = 199.98 \approx 200 \text{ m/s. Ans.}
 \end{aligned}$$

Problem 11.35. A mass of 10 kg is dropped through 2 cm on a spring of stiffness $k = 100 \text{ N/cm}$. Working from the first principles, find the maximum displacement of the spring.

Sol. Given :

Mass = 10 kg

Stiffness, $k = 100 \text{ N/cm}$

Height through which mass 10 kg is dropped = 2 cm

Let x = Maximum displacement of the spring when mass 10 kg is dropped as shown in position (1) of Fig. 11.45.

Considering the reference line A-A, let us find the total mechanical energy of the system for position (1) and position (2).

For position (1) (with reference to line A-A)

$$\begin{aligned}
 \text{P.E.} &= m \times g \times h \\
 &= 10 \times 9.81 \times (x + 2) \text{ N cm}
 \end{aligned}$$

$$\text{K.E.} = 0$$

$$\begin{aligned}
 \therefore \text{P.E.} + \text{K.E.} &= 10 \times 9.81 \times (x + 2) + 0 \\
 &= 10 \times 9.81(x + 2) \text{ N cm} \quad \dots(i)
 \end{aligned}$$

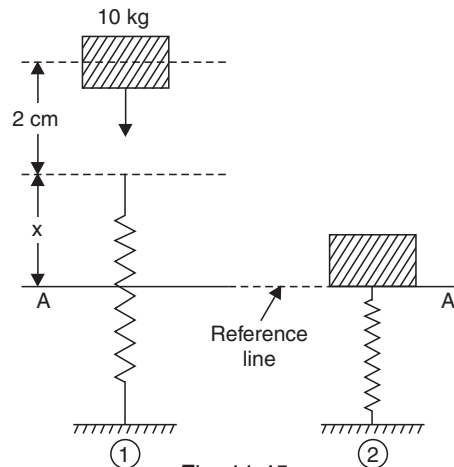


Fig. 11.45

For position (2)

P.E. of mass = 0

$$\text{P.E. of spring} = \frac{1}{2} kx^2 = \frac{1}{2} \times 100 \times x^2 = 50x^2 \text{ N cm}$$

K.E. = 0

 $\therefore$ Total mechanical energy of the system

$$= 0 + 50x^2 + 0 = 50x^2 \text{ N cm} \quad \dots(ii)$$

But according to principle of total mechanical energy, the total energy of a system remains constant.

Equating equations (i) and (ii), we get

$$10 \times 9.81 \times (x + 2) = 50x^2 \quad \text{or} \quad 9.81(x + 2) = 5x^2$$

$$\text{or} \quad 9.81x + 19.62 = 5x^2 \quad \text{or} \quad 5x^2 - 9.81x - 19.62 = 0$$

$$\begin{aligned} \therefore x &= \frac{9.81 \pm \sqrt{9.81^2 + 4 \times 5 \times 19.62}}{2 \times 5} = \frac{9.81 \pm \sqrt{96.236 + 392.4}}{10} \\ &= \frac{9.81 + 22.1}{10} \quad (\text{Neglecting negative value which is not possible}) \\ &= 3.19 \text{ cm.} \quad \text{Ans.} \end{aligned}$$

Problem 11.36. A bullet of mass 10 gm is fired into a body of mass 1 kg, which is suspended by a string 1 m long. The bullet gets embedded in the body and due to the impact, the body swings through an angle of 11.2° . Find the velocity of the bullet.

Sol. Given :Mass of bullet, $m = 10 \text{ gm} = 0.01 \text{ kg}$ Mass of body, $M = 1 \text{ kg}$ Length of string, $L = 1 \text{ m}$ Angle of swing, $\theta = 11.2^\circ$ Let, u = Velocity of bullet V = Common velocity of bullet and body

Initial velocity of the body is zero.

The momentum of the bullet and body before impact

$$\begin{aligned} &= m \times u + M \times \text{Initial velocity of body} \\ &= 0.01 \times u + 1 \times 0 = 0.01u \text{ kg m/s} \quad \dots(i) \end{aligned}$$

Bullet gets embedded in the body. Both of them move together with common velocity.

The momentum of bullet and body after impact

$$= (M + m)V = (1 + 0.01)V = 1.01V \text{ kg m/s} \quad \dots(ii)$$

Equating (i) and (ii), we get

$$0.01u = 1.01V \quad \dots(iii)$$

Now let us apply the law of conservation of energy.

The potential energy of the body and bullet at B

= K.E. of the body and bullet after impact.

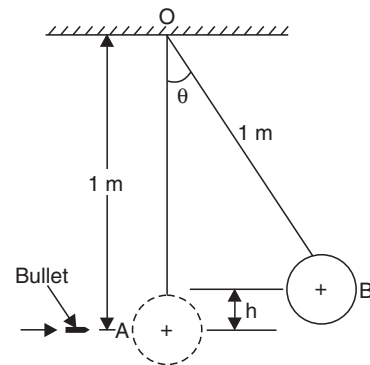
Let us first find the height ' h ' to which the body with bullet rises after impact.

Fig. 11.46

From the geometry of the figure.

$$h = OA - OB \cos \theta \quad (\text{But } OA = OB = 1 \text{ m})$$

$$= 1 - 1 \times \cos 11.2^\circ = 1 - 0.9499 \approx 0.05 \text{ m}$$

The potential energy of the body and bullet at B

$$= \text{Total weight of bullet and body} \times h = [(M + m) \times g] \times h$$

$$= (1 + .01) \times 9.81 \times 0.05 = 0.4954 \text{ Nm} \quad \dots(iv)$$

K.E. of the body and bullet after impact

$$= \frac{1}{2}(M + m) \times V^2 = \frac{1}{2}(1 + .01) \times V^2$$

$$= 0.505 V^2 \text{ Nm} \quad \dots(v)$$

But P.E. of body and bullet at B

$$= \text{K.E. of body and bullet after impact}$$

$$\therefore 0.4954 = 0.505 V^2$$

or
$$V = \sqrt{\frac{0.4954}{0.505}} = 0.99 \text{ m/s}$$

Substituting this value in equation (iii), we get

$$0.01 u = 1.01 \times 0.99$$

$$\therefore u = \frac{1.01 \times 0.99}{0.01} = 99.99 \text{ m/s. Ans.}$$

$$\therefore \text{Velocity of bullet} = 99.99 \text{ m/s.}$$

Alternate Method

This question can also be solved by alternate method. Let us apply the work and energy principle to the system of bullet and a body of given mass suspended by the spring. The initial velocity of the system of bullet and body is the common velocity (V) of the system after impact. The body with bullet rises to a height ' h ' after impact. Hence the final velocity of the system will be zero.

From equation (iii), we have,

$$0.01 u = 1.01 V$$

$$\therefore V = \frac{0.01 u}{1.01}$$

$$\therefore \text{Initial velocity of the system, } V_i = V = \frac{0.01 u}{1.01} \text{ m/s}$$

$$\text{Final velocity of the system, } V_f = 0$$

$$\text{Total mass of the system} = (M + m) = 1 + 0.01 = 1.01 \text{ kg}$$

$$\begin{aligned} \text{Change of K.E. of the system} &= \frac{1}{2}(M + m)[V_f^2 - V_i^2] \\ &= \frac{1}{2}(1 + 0.01) \left[0 - \left(\frac{0.01 u}{1.01} \right)^2 \right] \\ &= -\frac{1}{2} \times 1.01 \left(\frac{0.01}{1.01} u \right)^2 = -\frac{0.0001}{2 \times 1.01} u^2 \quad \dots(A) \end{aligned}$$

The work done will be due to the rise of the weight of the system. As the weight of the system is acting downwards, but the displacement of the weight of the system is upwards. Hence, work done will be -ve.

$$\therefore \text{Work done} = -(\text{Weight of system}) \times \text{rise in height}$$

$$= -(M + m) \times g \times h \quad [\because \text{Weight} = \text{Mass} \times g]$$

But the value of 'h' = 0.05 m (already calculated)

$$\therefore \text{Work done} = -(1 + 0.01) \times 9.81 \times 0.05 \quad \dots(B)$$

But according to the principle of work and energy, the total work done on the system must be equal to the change of K.E. of the system.

Hence, equating the two values given by equation (A) and (B), we get

$$-\frac{0.0001}{2 \times 1.01} u^2 = -1.01 \times 9.81 \times 0.05$$

$$\therefore u^2 = \frac{2 \times 1.01 \times 1.01 \times 9.81 \times 0.05}{0.0001}$$

or $u = \sqrt{\frac{2 \times 1.01 \times 1.01 \times 9.81 \times 0.05}{0.0001}} \approx 100 \text{ m/s. Ans.}$

11.14. LINEAR IMPULSE AND MOMENTUM

To solve kinetic problems of rigid body. Impulse-momentum method is also used. This method which relates force, velocity and time, is based on the integration of equation of motion with respect to time.

In this method also there is no need of determining the acceleration of the body.

Let us first define the impulse and momentum.

Let F = Net force acting on a rigid body in the direction of motion, through the C.G. of the body

m = Mass of the rigid body, and

a = Acceleration of the body.

Then we have

$$F = m \times a = m \times \frac{dv}{dt} \quad \left[\because \text{Acceleration is rate of change of velocity i.e., } a = \frac{dv}{dt} \right]$$

or $F \times dt = m \times dv$

Integrating the above equation, we get

$$\int_{t_1}^{t_2} F \times dt = \int_{v_1}^{v_2} m \times dv$$

$$= m(v_2 - v_1) \quad \dots(11.22)$$

If time-interval is very small, then $\int_{t_1}^{t_2} \mathbf{F} \times d\mathbf{t}$ is known as impulse. Hence impulse

is the product of force and time when time is very small. Momentum is the product of mass and velocity.

$$\therefore \quad \text{Impulse} = \int_{t_1}^{t_2} F \times dt = m(v_2 - v_1) = mv_2 - mv_1$$

or $\text{Impulse} = \text{Change of momentum} \quad [\because \text{Momentum} = \text{Mass} \times \text{Velocity}]$
 $= \text{Final momentum} - \text{Initial momentum}$

or $\text{Initial momentum} + \text{Impulse} = \text{Final momentum}$

The above equation gives the relation between impulse and momentum of a rigid body. The impulse-momentum equation is based on integrating the equation of motion with respect to time. The impulse-momentum equation relates force, velocity and time.

Impulse is the product of force and time. As force is a vector and time is a scalar, their product impulse will be a vector. Similarly, the momentum will be a vector as momentum is the product of mass (a scalar) and velocity (a vector). Also change of momentum will be a vector. Hence, we should take care of the direction in solving the kinetic equation. Hence, equation (11.22) for rectangular co-ordinates will be written as

$$\int_{t_1}^{t_2} F_x \times dt = m (V_{2x} - V_{1x})$$

$$\text{and} \quad \int_{t_1}^{t_2} F_y \times dt = m (V_{2y} - V_{1y})$$

If $t_1 = 0$ and $t_2 = t$, then the above equations are

$$F_x \times t = m (V_{2x} - V_{1x}) \quad \dots(11.23)$$

$$F_y \times t = m (V_{2y} - V_{1y}) \quad \dots(11.24)$$

where F_x = Impulsive force in x -direction

F_y = Impulsive force in y -direction

V_{1x} and V_{2x} = Initial and final velocities in x -direction and

V_{1y} and V_{2y} = Initial and final velocities in y -direction

F = Resultant impulsive force

$$= \sqrt{F_x^2 + F_y^2} \quad \dots(11.25)$$

Note: (i) The impulse-momentum approach is particularly convenient in situations when forces act for very small interval of time as in an impact or sudden blow.

(ii) In satellite motion, a combination of impulse-momentum method and work-energy method is used.

(iii) The unit of impulse is Ns (i.e., Newton second) in S.I. system.

Problem 11.37. A football of mass 200 gm is at rest. A player kicks the ball which moves with a velocity of 20 m/s at an angle of 30° with respect to ground level. Find the force exerted by the player on the ball if duration of strike is 0.02 seconds.

Sol. Given :

Mass, $m = 200 \text{ gm} = 0.2 \text{ kg}$

Duration of strike, $t = 0.02 \text{ s}$

Initially the ball is at rest. Hence, $V_{1x} = 0$ and $V_{1y} = 0$

The ball leaves with a velocity of 20 m/s at an angle of 30° . Refer to Fig. 11.47

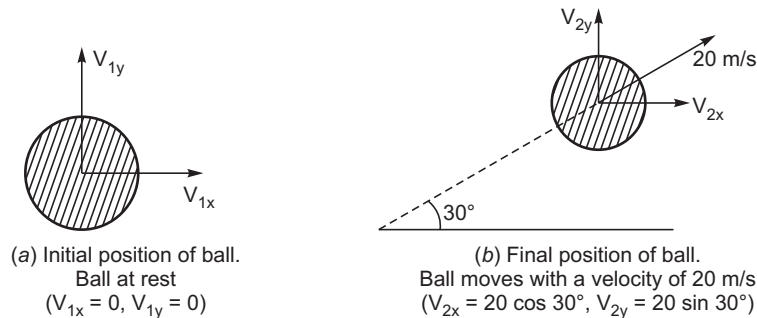


Fig. 11.47

Writing impulse-momentum equation along x - and y -directions, we get
[Refer to equations (11.23) and (11.24)],

For x -direction

$$F_x \times t = m (V_{2x} - V_{1x}), \text{ where } V_{2x} = 20 \cos 30^\circ \text{ and } V_{1x} = 0$$

or $F_x \times 0.02 = 0.2 (20 \cos 30^\circ - 0)$

or $F_x = \frac{0.2(20 \cos 30^\circ)}{0.02} = 173.2 \text{ N}$

For y -direction

$$F_y \times t = m (V_{2y} - V_{1y}), \text{ where } V_{2y} = 20 \sin 30^\circ \text{ and } V_{1y} = 0$$

or $F_y \times 0.02 = 0.2 (20 \sin 30^\circ - 0)$

or $F_y = \frac{0.2 \times 20 \sin 30^\circ}{0.02} = 100 \text{ N}$

Hence, the resultant impulse force exerted by the player on the ball,

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{173.2^2 + 100^2}$$

$$= 199.99 \approx \mathbf{200 \text{ N. Ans.}}$$

Problem 11.38. Fig. 11.48 shows a ball of mass 125 gm, moving in a horizontal direction towards a bat with a velocity of 20 m/s. The ball attains a velocity of 32 m/s at an angle of 30° with respect to x -axis, when hit by a bat. Find the impulsive force exerted by the bat on the ball if duration of strike is 0.025 second.

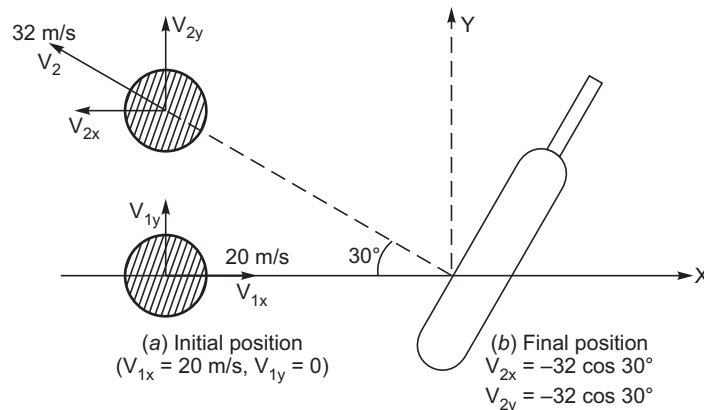


Fig. 11.48

Sol. Given :

Mass, $m = 125 \text{ gm} = 0.125 \text{ kg}$

Initial velocity, $V_1 = 20 \text{ m/s}$ in horizontal direction

$\therefore V_{1x} = 20 \text{ m/s}$ and $V_{1y} = 0$

Final velocity, $V_2 = 32 \text{ m/s}$ at an angle of 30°

$\therefore V_{2x} = -32 \cos 30^\circ$ and $V_{2y} = 32 \sin 30^\circ$

Duration of strike, $t = 0.025 \text{ seconds}$

Writing impulse-momentum equations along x - and y -directions.

For x -direction [equation (11.23)]

$$F_x \times t = m(V_{2x} - V_{1x})$$

$$\text{or } F_x \times 0.025 = 0.125 (-32 \cos 30^\circ - 20) \quad (\because V_{2x} \text{ is in opposite direction hence -ve})$$

$$\text{or } F_x = \frac{+0.125 (-32 \cos 30^\circ - 20)}{0.025} = -238.56 \text{ N}$$

For y -direction

$$F_y \times t = m(V_{2y} - V_{1y})$$

$$\text{or } F_y \times 0.025 = 0.125 (32 \sin 30^\circ - 0)$$

$$\text{or } F_y = \frac{0.125 \times 32 \sin 30^\circ}{0.025} = 80 \text{ N}$$

$\therefore$ Resultant average impulsive force exerted by bat on the ball
[Refer to Fig. 11.48 (a)]

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(-238.56)^2 + 80^2} = 251.6 \text{ N. Ans.} \end{aligned}$$

[equation (11.24)]

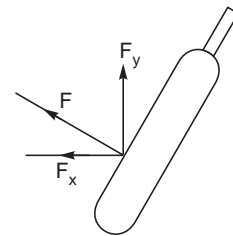


Fig. 11.48 (a)

11.15. CONSERVATION OF LINEAR MOMENTUM

The momentum of a system in linear direction is always conserved. This means the momentum of a system remains constant. Or in other words the total momentum of a system in a direction before collision is equal to the total momentum of the system after collision in that direction.

Problem 11.39. Ball A of mass 1 kg moving with a velocity of 2 m/s, strikes directly on a ball B of mass 2 kg at rest. The ball A, after striking, comes to rest. Find the velocity of ball B after striking.

Sol. Given :

Mass of ball A, $m_1 = 1 \text{ kg}$

Initial velocity of ball A, $u_1 = 2 \text{ m/s}$

Mass of ball B, $m_2 = 2 \text{ kg}$

Initial velocity of ball B, $u_2 = 0$

Final velocity of ball A, $v_1 = 0$

This is a case of direct impact.

Let $v_2 =$ Velocity of ball B after impact, and

$e =$ Co-efficient of restitution.

Total initial momentum $= m_1 u_1 + m_2 u_2 = 1 \times 2 + 2 \times 0 = 2 \text{ kg m/s.}$

Total final momentum $= m_1 v_1 + m_2 v_2 = 1 \times 0 + 2 \times v_2 = 2v_2 \text{ kg m/s.}$

According to the law of conservation of momentum,

Total initial momentum = Total final momentum

$$\therefore 2 = 2 \times v_2$$

$$\therefore v_2 = \frac{2}{2} = 1 \text{ m/s. Ans.}$$

Problem 11.40. A body of mass 50 kg, moving with a velocity of 6 m/s, collides directly with a stationary body of mass 30 kg. If the two bodies become coupled so that they move on together after the impact, what is their common velocity.

Sol. Given :

Mass of first body, $m_1 = 50 \text{ kg}$

Initial velocity of first body, $u_1 = 6 \text{ m/s}$

Mass of second body, $m_2 = 30 \text{ kg}$

Initial velocity of second body, $u_2 = 0$.

Total mass of two bodies $= (m_1 + m_2) = (50 + 30) = 80 \text{ kg}$.

Let $V =$ Common velocity of the two bodies after impact.

Total momentum before impact $= m_1 u_1 + m_2 u_2 = 50 \times 6 + 30 \times 0 = 300 \text{ kg m/s}$

Total momentum after impact $= (m_1 + m_2) \times V = (50 + 30) \times V = 80 V \text{ kg m/s}$

But total momentum before impact = Total momentum after impact

or $300 = 80 \times V$

$$\therefore V = \frac{300}{80} = 3.75 \text{ m/s. Ans.}$$

Problem 11.41. A bullet of mass 50 gm is fired into a freely suspended target to mass 5 kg. On impact, the target moves with a velocity of 7 m/s alongwith the bullet in the direction of firing. Find the velocity of bullet.

Sol. Given :

Mass of bullet, $m_1 = 50 \text{ gm} = \frac{50}{1000} = 0.05 \text{ kg}$

Mass of target, $m_2 = 5 \text{ kg}$

Initial velocity of bullet $= u_1$

Initial velocity of target, $u_2 = 0$

Total mass of bullet and target $= 5 + 0.05 = 5.05 \text{ kg}$

Final velocity of bullet and target $= 7 \text{ m/s}$

Total initial momentum (i.e., momentum before impact)

$$= m_1 \times u_1 + m_2 \times u_2 = 0.05 \times u_1 + 5 \times 0 \\ = 0.05 u_1 \text{ kg m/s}$$

Total final momentum (i.e., momentum after impact)

$$= \text{Total mass} \times \text{Common velocity} = (5.05) \times 7 \text{ kg m/s}$$

Equating the initial momentum to final momentum, we get

$$0.05 u_1 = 5.05 \times 7 \quad \therefore u_1 = \frac{5.05 \times 7}{0.05} = 707 \text{ m/s. Ans.}$$

Problem 11.42. A ball of mass 20 kg moving with a velocity of 5 m/s strikes directly another ball of mass 10 kg moving in the opposite direction with a velocity of 10 m/s. Determine the velocity of first ball in terms of the velocity of second ball after impact.

Sol. Given :

Mass of first ball, $m_1 = 20 \text{ kg}$

Initial velocity of first ball, $u_1 = 5 \text{ m/s}$

Mass of second ball, $m_2 = 10 \text{ kg}$

Initial velocity of second ball, $u_2 = -10 \text{ m/s}$

(Negative sign is due to opposite direction)

Let $v_1 =$ Velocity of first ball after impact

$v_2 =$ Velocity of second ball after impact

Total momentum before impact $= m_1 u_1 + m_2 u_2 = 20 \times 5 + 10 \times (-10) = 100 - 100 = 0$

Total momentum after impact $= m_1 v_1 + m_2 v_2 = 20 v_1 + 10 v_2$

Equating the total momentum after impact and before impact, we get

$$20v_1 + 10v_2 = 0 \quad \text{or} \quad v_1 = \frac{-10v_2}{20} = -\frac{1}{2} v_2. \quad \text{Ans.}$$

Hence, the velocity of first ball will be half of the velocity of second ball in opposite direction after impact.

(Minus sign indicates that the velocity of first ball after impact will be in opposite direction).

Problem 11.43. Due to residual stresses, a body at rest is suddenly burst into two pieces of mass 4 kg and 2 kg respectively. Assuming that the pieces fly apart in opposite directions with a relative velocity of 25 m/s, determine the speed of each. Neglect air resistance.

(U.P. Tech. University, 2002–2003)

Sol. Given :

Initial velocity of body, $V = 0$

Mass of body, $M = 4 + 2 = 6 \text{ kg}$

After explosion,

Mass of first piece, $m_1 = 4 \text{ kg}$

Mass of second piece, $m_2 = 2 \text{ kg}$

Let $v_1 =$ Velocity of first piece

$v_2 =$ Velocity of second piece

Relative velocity $= 25 \text{ m/s}$

As the two pieces are moving in opposite direction, hence relative velocity $= v_1 + v_2$

$$\therefore v_1 + v_2 = 25 \text{ m/s} \quad \dots(i)$$

Now applying the principle of conservation of momentum, we get

Total momentum before explosion

$=$ Total momentum after explosion

$$M \times V = m_1 v_1 + m_2 v_2$$

$$6 \times 0 = 4 \times v_1 + 2 \times (-v_2) \quad (v_2 \text{ is in opposite direction hence } -ve)$$

$$\text{or} \quad 0 = 4v_1 - 2v_2 \quad \text{or} \quad 2v_1 - v_2 = 0 \quad \dots(ii)$$

Adding equations (i) and (ii), we get

$$3v_1 = 25$$

$$\text{or} \quad v_1 = \frac{25}{3} = 8.33 \text{ m/s. Ans.}$$

Substituting this value of v_1 in equation (i),

$$8.33 + v_2 = 25$$

$$\text{or} \quad v_2 = 25 - 8.33 = 16.67 \text{ m/s. Ans.}$$

HIGHLIGHTS

1. The quantity of matter contained in a body known as mass of the body.
2. The weight of a body is defined as the force by which the body is attracted towards the centre of Earth. Weight of the body is given by

$$\text{Weight} = \text{Mass} \times g.$$
3. Momentum of a body is the product of the mass and its velocity.
4. The external force acting on a body is directly proportional to the rate of change of momentum in the same direction.

$$\therefore F = m \times a$$
 where m = Mass of the body, and
 a = Acceleration of the body.
5. A unit force is one, which produces unit acceleration on unit mass.
6. Newton is that force which acts on a body of mass one kilogram and produces an acceleration of 1 m/s^2 in the direction of force.
7. Dyne is that force which acts on a body of mass one gramme and produces an acceleration of 1 cm/s^2 in the direction of force.
8. The relation between newton and dyne is given by

$$1 \text{ N} = 10^5 \text{ dyne}.$$
9. The product of moment of inertia (I) and angular velocity (ω) of a rotating body is known as angular momentum.
 Mathematically, angular momentum = $I \times \omega$.
10. Moment of inertia about an axis of rotation is given by

$$I = \frac{MR^2}{2} \quad \text{.....For a circular section and solid cylinder}$$

$$= \frac{2}{5} MR^2 \quad \text{.....For a solid sphere}$$
 where M = Mass of a circular section, or a solid cylinder or a solid sphere.
11. Volume of a solid sphere is given by,

$$\text{Volume} = \frac{4}{3} \pi R^3.$$
12. Kinetic energy due to rotation of a body is given by

$$\text{K.E.} = \frac{1}{2} I\omega^2.$$
13. If a body is having motion of translation as well as motion of rotation, then total kinetic energy is given by,

$$\text{Total K.E.} = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2.$$
14. Work is the product of force and distance. Energy is the capacity to do work. The work done is given by

$$\begin{aligned} \text{Work done} &= P \times S \\ &= P \cos \theta \times S \end{aligned} \quad \dots \text{ When force and distance are in the same direction}$$
 When force acts at an angle θ with the direction of displacement.
15. The units of work is Nm (or Joule). Hence, one Joule is the work done by a force of 1 N when displacement is 1 m.
16. The area of force-distance curve represents the work done.

17. Power is the rate of doing work. Or power is the work done per second. The unit of power is Nm/s or Watt (W)

$$\text{Power} = \text{Force} \times \text{Velocity}$$

The force and velocity should be in the same direction.

18. Work done on a rotating body, by a torque (T) is given by

$$\text{Work done} = T \times \theta$$

where θ = Angular displacement in radians.

And the power is given by

$$\text{Power} = \frac{2\pi NT}{60} \text{ watts.}$$

19. The Product of power and time is known as Energy. The unit of energy is Nm. Energy is also defined as capacity to do work. Though there are many forms of energies, yet potential energy and kinetic energy are important from the subject point of view.
20. The energy, by virtue of position of a body with respect to any given reference, is known as potential energy and it is equal to $W \times h$, where W = weight of body and h is height of body from the reference line.
21. The energy possessed by a body by virtue of its velocity (or motion) is known as kinetic energy. K.E. is given by

$$\text{K.E.} = \frac{1}{2} mv^2.$$

22. Law of conservation of energy states, "Energy can neither be created nor destroyed, though it can be transformed from one form into another form".

EXERCISE 11

(A) Theoretical Questions

- Define the terms : Mass of a body and weight of a body. What is the relationship between the two ?
- Explain the terms : Momentum of a body and angular momentum of a body.
- Define and explain the Newton's Laws of motion for linear motion and rotational motion.
- State Newton's law of motion and explain them by giving an example in each case.
- (a) State D' Alembert's Principle giving equations expressing the above principle for a rigid body in plane motion.
(b) Write brief note on D' Alembert's Principle and its applications.

(U.P. Tech. University, 2000–2001)

6. Deduce the relation,

$$\text{Kinetic energy} = \frac{1}{2} I\omega^2$$

where I = Moment of inertia about the axis of rotation, and

ω = Angular velocity.

- Define the following terms :
Work, energy, momentum, impulse and moment of momentum.
- What are the units of work ? What is the relation between work done and power ? Also mention the unit of power.
- What is energy ? Explain the various forms of mechanical energies.

(U.P. Tech. University, May 2009)

10. (a) Explain the principle of conservation of energy.
(b) Write brief note on Principle of Conservation of Energy and its applications.
(U.P. Tech. University, 2000–2001)
11. Mention the various forms of mechanical energy known to you and define them.
12. State the law of conservation of momentum.
13. What do you understand by 'Potential energy' and Kinetic energy of a body ?
14. Prove that the K.E. of a moving body is expressed as :

$$\text{K.E.} = \frac{1}{2}mv^2$$

where m = mass of the body and
 v = velocity of the body.

15. State the law of conservation of energy. Give the proof of this law taking mechanical energy only into account.
16. Write short note on :
 (i) Principle of virtual work and its applications.
 (ii) Principle of conservation of momentum and its applications.

(U.P. Tech. University, 2000–2001)

(B) Numerical Problems

1. A block of wood of weight 1200 N is placed on a smooth inclined plane which makes an angle of 30° with the horizontal. Find the work done in pulling the block up for a length of 8 m.
[Ans. 4800 Nm]
2. If in question 1, the inclined plane is rough and co-efficient of friction between the wooden block and plane is 0.3, then find the work done in pulling up for a length of 8 m. [Ans. 7293.92 Nm]
3. A body of weight 1500 N is placed on a rough horizontal surface. An initial force of 600 N applied horizontally is sufficient just to move the body. The applied force is not constant but varies uniformly with the distance moved by the body. If the body moves a distance of 20 m and force applied at any distance x from the initial position is given by the relation $P = P_0 + 10x$, find the work done by the applied force, where P_0 = initial force and x is the distance moved from initial position.
[Ans. 14000 Nm]
4. Determine the work done by an electric motor in winding up a uniform cable which hangs from a hoisting drum if its free length is 20 m and weighs 800 N. The drum is rotated by the motor.
[Ans. 8000 Nm]
5. A train of weight 2000 kN, is ascending a slope of 1 in 100 with a uniform speed of 36 km/hr. Find the power exerted by the engine, if road resistance is 8 N per kN weight of the train.
[Ans. 288 kW]
6. A train of weight 2000 kN moves down a slope of 1 in 100 at 36 km/hr and engine develops a power of 70 kW. If the train is pulled up the same speed, what power will be required to pull the train?
[Ans. 470 kW]
7. A tangential force of 2700 N is acting on a shaft of diameter 20 mm. Find the work done by the force for one revolution of shaft.
[Ans 169.62 Nm]
8. If in the above question, the shaft is rotating at 400 r.p.m., find the power of the shaft.
[Ans. 1130.94 W]
9. A bullet of mass 25 gm is fired with a velocity of 400 m/s. What is the K.E. of the bullet ? If the bullet can penetrate 20 cm in a block of wood, what is the average resistance of wood ? What will be the exit velocity of the bullet, if fired into a similar block of wood 10 cm thick ?
[Ans. 2000 Nm ; 1000 N ; 282.8 m/s]

10. A truck of mass 3000 kg is travelling at 36 km/hr on a level road. It is brought to rest in 15 m, what is the average force of resistance acting on the truck ? **[Ans. 10,000 N]**
11. A car of weight 19620 N acceleration from rest to a speed of 36 km/hr in a distance of 40 m against a resistance of 500 N. Find the average driving force acting on the car. Using that average force, find the greatest power developed by the engine. **[Ans. 3000 N, 30 kW]**
12. A bullet travelling horizontally with a velocity of 600 m/s and weighing 0.25 N strikes a wooden block of weight 50 N resting on a rough horizontal floor. The co-efficient of friction between the floor and the block $\mu = 0.5$. Find the distance through which the block is displaced from its initial position. **[Ans. 0.90 m]**
13. A hammer of mass 750 kg drops from a height of 1.2 m on a pile of mass 200 kg. Find : (i) the common velocity after impact assuming plastic impact, and (ii) the average resistance of the ground if the pile comes to rest after penetrating 10 cm into the ground. **[Ans. (i) 3.829 m/s ; (ii) 79 kN]**

UNIT V

MECHANICS OF DEFORMABLE SOLIDS

Chapters

- 12. Simple Stresses and Strains
- 13. Elastic Constants
- 14. Strain Energy and Impact Loading
- 15. Bending of Beams
- 16. Theory of Torsion

12

Simple Stresses and Strains

12.1. INTRODUCTION

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. This resistance by which material of the body opposes the deformation is known as **strength of material**. Within a certain limit (*i.e.*, in the elastic stage) the resistance offered by the material is proportional to the deformation brought out on the material by the external force. Also within this limit the resistance is equal to the external force (or applied load). But beyond the elastic stage, the resistance offered by the material is less than the applied load. In such a case, the deformation continues, until failure takes place.

Within elastic stage, the resisting force equals applied load. This resisting force per unit area is called stress or intensity of stress.

12.2. SIMPLE STRESS AND STRAIN

The force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called the *load or force*. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

Mathematically stress is written as, $\sigma = \frac{P}{A}$

where σ = Stress (also called intensity of stress),

P = External force or load, and

A = Cross-sectional area.

12.2.1. Units of Stress. The unit of stress depends upon the unit of load (or force) and unit of area. In M.K.S. units, the force is expressed in kgf and area in metre square (*i.e.*, m^2). Hence unit of stress becomes as kgf/m^2 . If area is expressed in centimetre square (*i.e.*, cm^2), the stress is expressed as kgf/cm^2 .

In the S.I. units, the force is expressed in newtons (written as N) and area is expressed as m^2 . Hence unit of stress becomes as N/m^2 . The area is also expressed in millimetre square then unit of force becomes as N/mm^2 .

$$1 \text{ N}/\text{m}^2 = 1 \text{ N}/(100 \text{ cm})^2 = 1 \text{ N}/10^4 \text{ cm}^2$$

$$= 10^{-4} \text{ N}/\text{cm}^2 \text{ or } 10^{-6} \text{ N}/\text{mm}^2$$

$$\left(\because \frac{1}{\text{cm}^2} = \frac{1}{10^2 \text{ mm}^2} \right)$$

$$\therefore 1 \text{ N/mm}^2 = 10^6 \text{ N/m}^2.$$

$$\text{Also } 1 \text{ N/m}^2 = 1 \text{ Pascal} = 1 \text{ Pa}.$$

The large quantities are represented by kilo, mega, giga and terra. They stand for :

$$\text{Kilo} = 10^3 \text{ and represented by k}$$

$$\text{Mega} = 10^6 \text{ and represented by M}$$

$$\text{Giga} = 10^9 \text{ and represented by G}$$

$$\text{Terra} = 10^{12} \text{ and represented by T}.$$

Thus mega newton means 10^6 newtons and is represented by MN. The symbol 1 MPa stands for 1 mega pascal which is equal to 10^6 pascal (or 10^6 N/m^2).

The small quantities are represented by milli, micro, nana and pica. They are equal to

$$\text{Milli} = 10^{-3} \text{ and represented by m}$$

$$\text{Micro} = 10^{-6} \text{ and represented by } \mu$$

$$\text{Nana} = 10^{-9} \text{ and represented by } \eta$$

$$\text{Pica} = 10^{-12} \text{ and represented by p}.$$

Notes 1. Newton is a force acting on a mass of one kg and produces an acceleration of 1 m/s^2 i.e.,

$$1 \text{ N} = 1 (\text{kg}) \times 1 \text{ m/s}^2.$$

2. The stress in S.I. units is expressed in N/m^2 or N/mm^2 .

3. The stress $1 \text{ N/mm}^2 = 10^6 \text{ N/m}^2 = \text{MN/m}^2$. Thus one N/mm^2 is equal to one MN/m^2 .

4. One pascal is written as 1 Pa and is equal to 1 N/m^2 .

12.2.2. Strain. When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless.

Strain may be :

1. Tensile strain,	2. Compressive strain,
3. Volumetric strain, and	4. Shear strain.

If there is some increase in length of a body due to external force, then the ratio of increase of length to the original length of the body is known as *tensile strain*. But if there is some decrease in length of the body, then the ratio of decrease of the length of the body to the original length is known as *compressive strain*. The ratio of change of volume of the body to the original volume is known as *volumetric strain*. The strain produced by shear stress is known as *shear strain*.

12.3. NORMAL AND SHEAR STRESSES

The stress may be normal stress or a shear stress.

Normal stress is the stress which acts in a direction perpendicular to the area. It is represented by σ (sigma). The normal stress is further divided into tensile stress and compressive stress.

12.3.1. Tensile Stress. The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig. 12.1 (a) as a result of which there is an increase in length, is known as tensile stress. The ratio of increase in length to the original length is known as *tensile strain*. The tensile stress acts normal to the area and it pulls on the area.

Let P = Pull (or force) acting on the body,

A = Cross-sectional area of the body,

- L = Original length of the body,
 dL = Increase in length due to pull P acting on the body,
 σ = Stress induced in the body, and
 e = Strain (*i.e.*, tensile strain).

Fig. 12.1 (a) shows a bar subjected to a tensile force P at its ends. Consider a section $X-X$, which divides the bar into two parts. The part left to the section $X-X$, will be in equilibrium if P = Resisting force (R). This is shown in Fig. 12.1 (b). Similarly the part right to the section $X-X$, will be in equilibrium if P = Resisting force as shown in Fig. 12.1 (c). This resisting force per unit area is known as stress or intensity of stress.

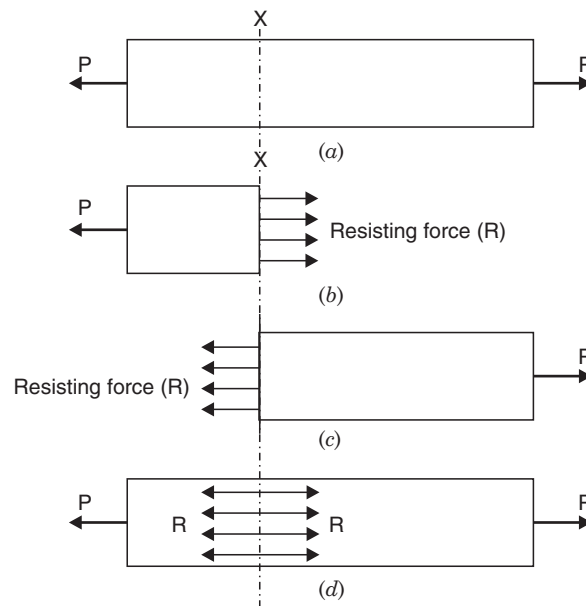


Fig. 12.1

$$\therefore \text{ Tensile stress, } \sigma = \frac{\text{Resisting force (R)}}{\text{Cross-sectional area}} = \frac{\text{Tensile load (P)}}{A} \quad (\because P = R)$$

or
$$\sigma = \frac{P}{A} \quad \dots(12.1)$$

And tensile strain is given by,

$$e = \frac{\text{Increase in length}}{\text{Original length}} = \frac{dL}{L} \quad \dots(12.2)$$

12.3.2. Compressive Stress. The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig. 12.2 (a) as a result of which there is a decrease in length of the body, is known as compressive stress. And the ratio of decrease in length to the original length is known as *compressive strain*. The compressive stress acts normal to the area and it pushes on the area.

Let an axial push P is acting on a body of cross-sectional area A . Due to external push P , let the original length L of the body decreases by dL .

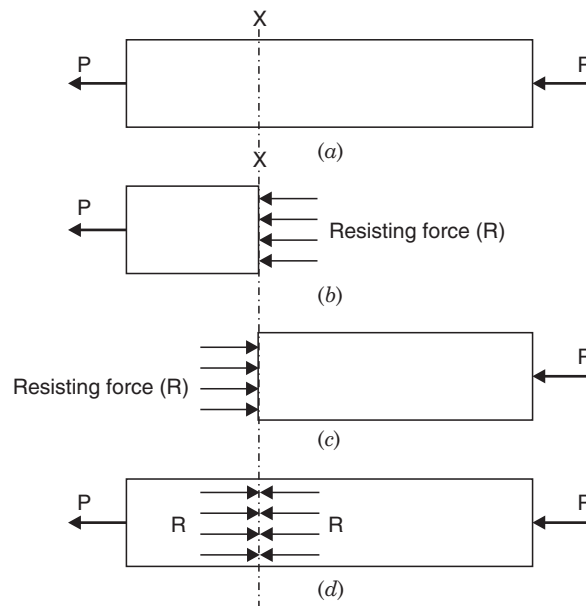


Fig. 12.2

Then compressive stress is given by,

$$\sigma = \frac{\text{Resisting Force (R)}}{\text{Area (A)}} = \frac{\text{Push (P)}}{\text{Area (A)}} = \frac{P}{A}.$$

And compressive strain is given by,

$$e = \frac{\text{Decrease in length}}{\text{Original length}} = \frac{dL}{L}.$$

12.3.3. Shear Stress. The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as shown in Fig. 12.3 as a result of which the body tends to shear off across the section, is known as shear stress. The corresponding strain is known as *shear strain*. The shear stress is the stress which acts tangential to the area. It is represented by τ .

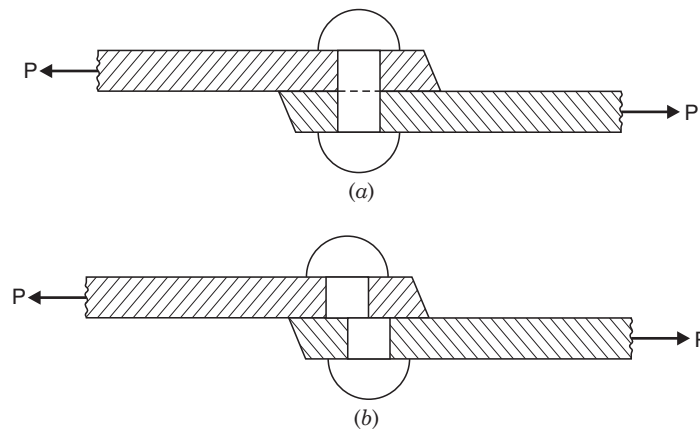


Fig. 12.3

Consider a rectangular block of height h , length L and width unity. Let the bottom face AB of the block be fixed to the surface as shown in Fig. 12.4 (a). Let a force P be applied tangentially along the top face CD of the block. Such a force acting tangentially along a surface is known as shear force. For the equilibrium of the block, the surface AB will offer a tangential reaction P equal and opposite to the applied tangential force P .

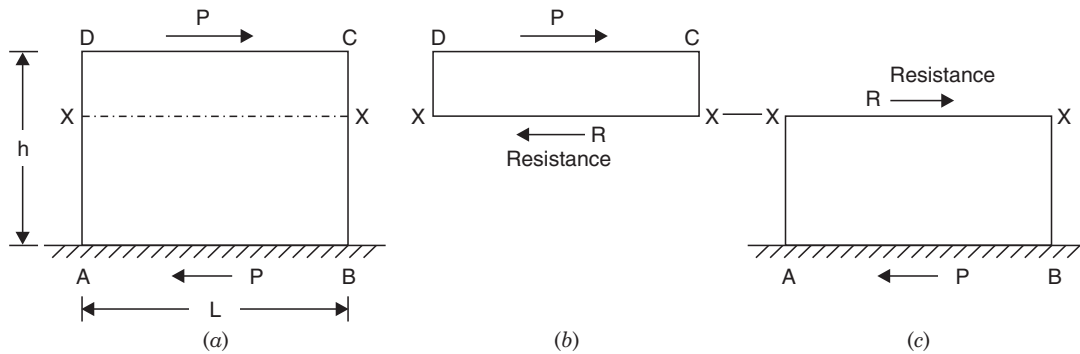


Fig. 12.4

Consider a section $x-x$ (parallel to the applied force), which divides the block into two parts. The upper part will be in equilibrium if $P = \text{Resistance } (R)$. This is shown in Fig. 12.4 (b). Similarly the lower part will be in equilibrium if $P = \text{Resistance } (R)$ as shown in Fig. 12.4 (c). This resistance is known as shear resistance. And the shear resistance per unit area is known as shear stress which is represented by τ .

$$\begin{aligned} \therefore \text{Shear stress, } \tau &= \frac{\text{Shear resistance}}{\text{Shear area}} = \frac{R}{A} \\ &= \frac{P}{L \times 1} \quad (\because R = P \text{ and } A = L \times 1) \quad \dots(12.3) \end{aligned}$$

Note that shear stress is tangential to the area over which it acts.

As the bottom face of the block is fixed, the face $ABCD$ will be distorted to ABC_1D_1 through an angle ϕ as a result of force P as shown in Fig. 12.4 (d).

And shear strain (ϕ) is given by,

$$\begin{aligned} \phi &= \frac{\text{Transversal displacement}}{\text{Distance AD}} \\ \text{or} \quad \phi &= \frac{DD_1}{AD} = \frac{dl}{h} \quad \dots(12.4) \end{aligned}$$

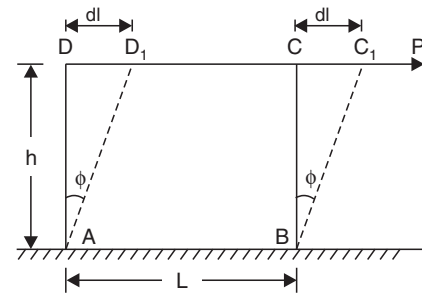


Fig. 12.4 (d)

12.4. ELASTICITY AND ELASTIC LIMIT

When an external force acts on a body, the body tends to undergo some deformation. If the external force is removed and the body comes back to its origin shape and size (which means the deformation disappears completely), the body is known as *elastic body*. This property, by virtue of which certain materials return back to their original position after the removal of the external force, is called *elasticity*.

The body will regain its previous shape and size only when the deformation caused by the external force, is within a certain limit. Thus there is a limiting value of force up to and within which, the deformation completely disappears on the removal of the force. The value of stress corresponding to this limiting force is known as the *elastic limit* of the material.

If the external force is so large that the stress exceeds the elastic limit, the material loses to some extent its property of elasticity. If now the force is removed, the material will not return to its origin shape and size and there will be a residual deformation in the material.

12.4.1. Hooke's Law and Elastic Moduli. Hooke's Law states that when a material is loaded within elastic limit, the stress is proportional to the strain produced by the stress. This means the ratio of the stress to the corresponding strain is a constant within the elastic limit. This constant is known as Modulus of Elasticity or Modulus of Rigidity or Elastic Moduli.

12.4.2. Modulus of Elasticity (or Young's modulus). The ratio of tensile stress or compressive stress to the corresponding strain is a constant. This ratio is known as Young's Modulus or Modulus of Elasticity and is denoted by E .

$$\therefore E = \frac{\text{Tensile stress}}{\text{Tensile strain}} \quad \text{or} \quad \frac{\text{Compressive stress}}{\text{Compressive strain}}$$

or
$$E = \frac{\sigma}{e} \quad \dots(12.5)$$

12.4.3. Modulus of Rigidity or Shear Modulus. The ratio of shear stress to the corresponding shear strain within the elastic limit, is known as Modulus of Rigidity or Shear Modulus. This is denoted by C or G or N .

$$\therefore C \text{ (or } G \text{ or } N) = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\phi} \quad \dots(12.6)$$

Let us define factor of safety also.

12.4.4. Factor of Safety. It is defined as the ratio of ultimate tensile stress to the working (or permissible) stress. Mathematically it is written as

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Permissible stress}} \quad \dots(12.7)$$

12.5. ONE DIMENSIONAL LOADING

The load acting in one direction only is known as one dimensional loading. The following problems illustrates one dimensional loading.

Problem 12.1. A rod 150 cm long and of diameter 2.0 cm is subjected to an axial pull of 20 kN. If the modulus of elasticity of the material of the rod is $2 \times 10^5 \text{ N/mm}^2$; determine :

- (i) the stress,
- (ii) the strain, and
- (iii) the elongation of the rod.

Sol. Given : Length of the rod, $L = 150 \text{ cm}$

Diameter of the rod, $D = 2.0 \text{ cm} = 20 \text{ mm}$

$$\therefore \text{Area, } A = \frac{\pi}{4} (20)^2 = 100\pi \text{ mm}^2$$

Axial pull, $P = 20 \text{ kN} = 20,000 \text{ N}$

Modulus of elasticity, $E = 2.0 \times 10^5 \text{ N/mm}^2$

(i) The stress (σ) is given by equation (12.1) as

$$\sigma = \frac{P}{A} = \frac{20000}{100\pi} = 63.662 \text{ N/mm}^2. \quad \text{Ans.}$$

(ii) Using equation (12.5), the strain is obtained as

$$E = \frac{\sigma}{e}.$$

$$\therefore \text{Strain, } e = \frac{\sigma}{E} = \frac{63.662}{2 \times 10^5} = 0.000318. \quad \text{Ans.}$$

(iii) Elongation is obtained by using equation (12.2) as

$$e = \frac{dL}{L}.$$

$$\therefore \text{Elongation, } dL = e \times L \\ = 0.000318 \times 150 = 0.0477 \text{ cm.} \quad \text{Ans.}$$

Problem 12.2. Find the minimum diameter of a steel wire, which is used to raise a load of 4000 N if the stress in the rod is not to exceed 95 MN/m².

Sol. Given : Load, $P = 4000 \text{ N}$

Stress, $\sigma = 95 \text{ MN/m}^2 = 95 \times 10^6 \text{ N/m}^2 \quad (\because \text{M} = \text{Mega} = 10^6)$
 $= 95 \text{ N/mm}^2 \quad (\because 10^6 \text{ N/m}^2 = 1 \text{ N/mm}^2)$

Let $D = \text{Diameter of wire in mm}$

$$\therefore \text{Area, } A = \frac{\pi}{4} D^2$$

$$\text{Now, stress} = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

$$95 = \frac{4000}{\frac{\pi}{4} D^2} = \frac{4000 \times 4}{\pi D^2} \quad \text{or} \quad D^2 = \frac{4000 \times 4}{\pi \times 95} = 53.61$$

$$\therefore D = 7.32 \text{ mm.} \quad \text{Ans.}$$

Problem 12.3. Find the Young's Modulus of a brass rod of diameter 25 mm and of length 250 mm which is subjected to a tensile load of 50 kN when the extension of the rod is equal to 0.3 mm.

Sol. Given : Dia. of rod, $D = 25 \text{ mm}$

$$\therefore \text{Area of rod, } A = \frac{\pi}{4} (25)^2 = 490.87 \text{ mm}^2$$

Tensile load, $P = 50 \text{ kN} = 50 \times 1000 = 50,000 \text{ N}$

Extension of rod, $dL = 0.3 \text{ mm}$

Length of rod, $L = 250 \text{ mm}$

Stress (σ) is given by equation (12.1), as

$$\sigma = \frac{P}{A} = \frac{50,000}{490.87} = 101.86 \text{ N/mm}^2.$$

Strain (e) is given by equation (12.2), as

$$e = \frac{dL}{L} = \frac{0.3}{250} = 0.0012.$$

Using equation (12.5), the Young's Modulus (E) is obtained, as

$$\begin{aligned} E &= \frac{\text{Stress}}{\text{Strain}} = \frac{101.86 \text{ N/mm}^2}{0.0012} = 84883.33 \text{ N/mm}^2 \\ &= 84883.33 \times 10^6 \text{ N/m}^2. \quad \text{Ans.} \quad (\because 1 \text{ N/mm}^2 = 10^6 \text{ N/m}^2) \\ &= 84.883 \times 10^9 \text{ N/m}^2 = 84.883 \text{ GN/m}^2. \quad \text{Ans.} \quad (\because 10^9 = \text{G}) \end{aligned}$$

Problem 12.4. A tensile test was conducted on a mild steel bar. The following data was obtained from the test :

- (i) Diameter of the steel bar = 3 cm
- (ii) Gauge length of the bar = 20 cm
- (iii) Load at elastic limit = 250 kN
- (iv) Extension at a load of 150 kN = 0.21 mm
- (v) Maximum load = 380 kN
- (vi) Total extension = 60 mm
- (vii) Diameter of the rod at the failure = 2.25 cm.

Determine : (a) the Young's modulus, (b) the stress at elastic limit,
(c) the percentage elongation, and (d) the percentage decrease in area.

Sol. Area of the rod, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4} (3)^2 \text{ cm}^2$

$$= 7.0685 \text{ cm}^2 = 7.0685 \times 10^{-4} \text{ m}^2. \quad \left[\because \text{cm}^2 = \left(\frac{1}{100} \text{ m} \right)^2 \right]$$

(a) To find Young's modulus, first calculate the value of stress and strain within elastic limit. The load at elastic limit is given but the extension corresponding to the load at elastic limit is not given. But a load of 150 kN (which is within elastic limit) and corresponding extension of 0.21 mm are given. Hence these values are used for stress and strain within elastic limit

$$\begin{aligned} \therefore \text{Stress} &= \frac{\text{Load}}{\text{Area}} = \frac{150 \times 1000}{7.0685 \times 10^{-4}} \text{ N/m}^2 \quad (\because 1 \text{ kN} = 1000 \text{ N}) \\ &= 21220.9 \times 10^4 \text{ N/m}^2 \end{aligned}$$

and
$$\begin{aligned} \text{Strain} &= \frac{\text{Increase in length (or Extension)}}{\text{Original length (or Gauge length)}} \\ &= \frac{0.21 \text{ mm}}{20 \times 10 \text{ mm}} = 0.00105 \end{aligned}$$

$\therefore$ Young's Modulus,

$$\begin{aligned} E &= \frac{\text{Stress}}{\text{Strain}} = \frac{21220.9 \times 10^4}{0.00105} = 20209523 \times 10^4 \text{ N/m}^2 \\ &= 202.095 \times 10^9 \text{ N/m}^2 \quad (\because 10^9 = \text{Giga} = \text{G}) \\ &= 202.095 \text{ GN/m}^2. \quad \text{Ans.} \end{aligned}$$

(b) The stress at the elastic limit is given by,

$$\begin{aligned}\text{Stress} &= \frac{\text{Load at elastic limit}}{\text{Area}} = \frac{250 \times 1000}{7.0685 \times 10^{-4}} \\ &= 35368 \times 10^4 \text{ N/m}^2 \\ &= 353.68 \times 10^6 \text{ N/m}^2 \quad (\because 10^6 = \text{Mega} = \text{M}) \\ &= \mathbf{353.68 \text{ MN/m}^2. \text{ Ans.}}\end{aligned}$$

(c) The percentage elongation is obtained as,

Percentage elongation

$$\begin{aligned}&= \frac{\text{Total increase in length}}{\text{Original length (or Gauge length)}} \times 100 \\ &= \frac{60 \text{ mm}}{20 \times 10 \text{ mm}} \times 100 = \mathbf{30\% \text{ Ans.}}\end{aligned}$$

(d) The percentage decrease in area is obtained as,

Percentage decrease in area

$$\begin{aligned}&= \frac{(\text{Original area} - \text{Area at the failure})}{\text{Original area}} \times 100 \\ &= \frac{\left(\frac{\pi}{4} \times 3^2 - \frac{\pi}{4} \times 2.25^2\right)}{\frac{\pi}{4} \times 3^2} \times 100 \\ &= \left(\frac{3^2 - 2.25^2}{3^2}\right) \times 100 = \frac{(9 - 5.0625)}{9} \times 100 = \mathbf{43.75\% \text{ Ans.}}\end{aligned}$$

Problem 12.5. The safe stress, for a hollow steel column which carries an axial load of $2.1 \times 10^3 \text{ kN}$ is 125 MN/m^2 . If the external diameter of the column is 30 cm , determine the internal diameter.

Sol. Given :

Safe stress*, $\sigma = 125 \text{ MN/m}^2 = 125 \times 10^6 \text{ N/m}^2$

Axial load, $P = 2.1 \times 10^3 \text{ kN} = 2.1 \times 10^6 \text{ N}$

External diameter, $D = 30 \text{ cm} = 0.30 \text{ m}$

Let $d = \text{Internal diameter}$

$\therefore$ Area of cross-section of the column,

$$A = \frac{\pi}{4} (D^2 - d^2) = \frac{\pi}{4} (.30^2 - d^2) \text{ m}^2$$

Using equation (12.1), $\sigma = \frac{P}{A}$

*Safe stress is a stress which is within elastic limit.

$$\text{or} \quad 125 \times 10^6 = \frac{2.1 \times 10^6}{\frac{\pi}{4} (.30^2 - d^2)} \quad \text{or} \quad (.30^2 - d^2) = \frac{4 \times 2.1 \times 10^6}{\pi \times 125 \times 10^6}$$

$$\text{or} \quad 0.09 - d^2 = 213.9 \quad \text{or} \quad 0.09 - 0.02139 = d^2$$

$$\therefore d = \sqrt{0.09 - 0.02139} = 0.2619 \text{ m} = \mathbf{26.19 \text{ cm.} \quad \text{Ans.}}$$

Problem 12.6. The ultimate stress, for a hollow steel column which carries an axial load of 1.9 MN is 480 N/mm². If the external diameter of the column is 200 mm, determine the internal diameter. Take the factor of safety as 4.

Sol. Given :

Ultimate stress, = 480 N/mm²

Axial load, $P = 1.9 \text{ MN} = 1.9 \times 10^6 \text{ N}$ ($\because M = 10^6$)
= 1900000 N

External dia., $D = 200 \text{ mm}$

Factor of safety = 4

Let d = Internal diameter in mm

$\therefore$ Area of cross-section of the column,

$$A = \frac{\pi}{4} (D^2 - d^2) = \frac{\pi}{4} (200^2 - d^2) \text{ mm}^2$$

Using equation (12.7), we get

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working stress or Permissible stress}}$$

$$\therefore 4 = \frac{480}{\text{Working stress}}$$

$$\text{or} \quad \text{Working stress} = \frac{480}{4} = 120 \text{ N/mm}^2$$

$$\therefore \sigma = 120 \text{ N/mm}^2$$

Now using equation (12.1), we get

$$\sigma = \frac{P}{A} \quad \text{or} \quad 120 = \frac{1900000}{\frac{\pi}{4} (200^2 - d^2)} = \frac{1900000 \times 4}{\pi (40000 - d^2)}$$

$$\text{or} \quad 40000 - d^2 = \frac{1900000 \times 4}{\pi \times 120} = 20159.6$$

$$\text{or} \quad d^2 = 40000 - 20159.6 = 19840.4$$

$$\therefore d = \mathbf{140.85 \text{ mm.} \quad \text{Ans.}}$$

Problem 12.7. A stepped bar shown in Fig. 12.5 is subjected to an axially applied compressive load of 35 kN. Find the maximum and minimum stresses produced.

Sol. Given :

Axial load, $P = 35 \text{ kN} = 35 \times 10^3 \text{ N}$

Dia. of upper part, $D_1 = 2 \text{ cm} = 20 \text{ mm}$

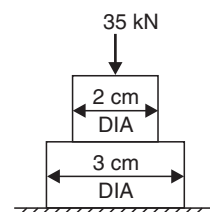


Fig. 12.5

$$\therefore \text{Area of upper part, } A_1 = \frac{\pi}{4} (20^2) = 100 \pi \text{ mm}^2$$

$$\text{Area of lower part, } A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (30^2) = 225 \pi \text{ mm}^2$$

The stress is equal to load divided by area. Load is same for both the bars as they are connected in series. Hence stress will be maximum where area is minimum. Hence stress will be maximum in upper part and minimum in lower part.

$$\therefore \text{Maximum stress} = \frac{\text{Load}}{A_1} = \frac{35 \times 10^3}{100 \times \pi} = 111.408 \text{ N/mm}^2. \quad \text{Ans.}$$

$$\text{Minimum stress} = \frac{\text{Load}}{A_2} = \frac{35 \times 10^3}{225 \times \pi} = 49.5146 \text{ N/mm}^2. \quad \text{Ans.}$$

12.6. STRESS ON AN INCLINED PLANE

Fig. 12.6 (a) shows a rectangular member of uniform cross-sectional area A and of unit thickness.

Let P = Axial force acting on the member.

A = Area of cross-section, which is perpendicular to the line of action of the force P .

$$\text{The stress along } x\text{-axis, } \sigma = \frac{P}{A} \quad \dots(12.8)$$

Hence, the member is subjected to a stress along x -axis.

Consider a cross-section EF which is perpendicular to the line of action of the force P .

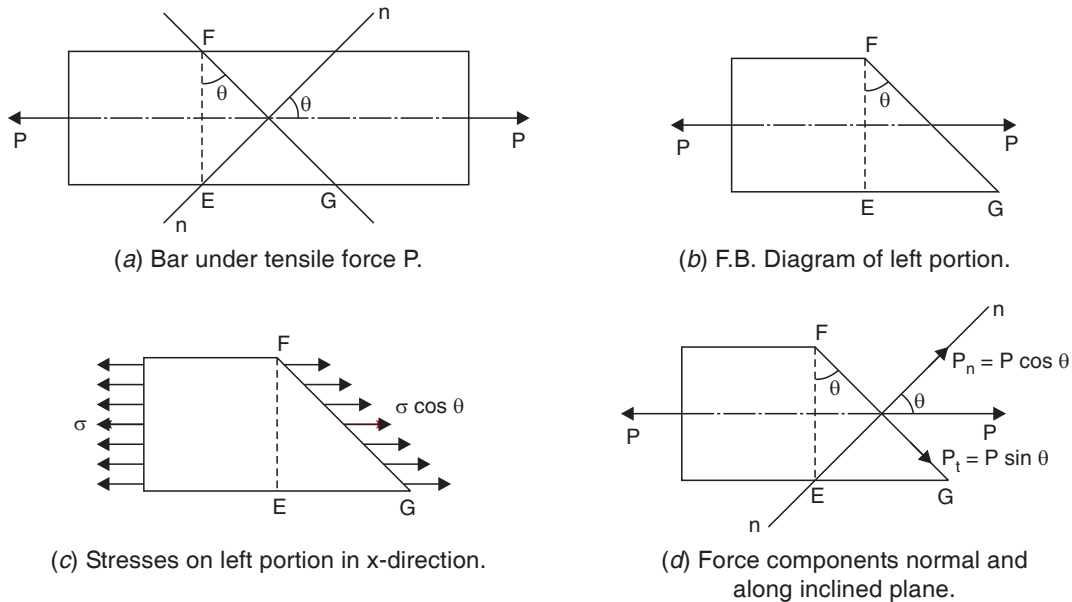


Fig. 12.6

Then area of section EF , $A = EF \times 1$

Now consider a section FG at an angle θ with the normal cross-section EF as shown in Fig. 12.6 (a). This is also the angle between normal $n-n$ of inclined plane and x -axis.

Area of section $FG = FG \times 1$ (member is having unit thickness)

$$\begin{aligned}
 &= \frac{EF}{\cos \theta} \times 1 \quad \left(\because \text{In } \Delta EFG, \frac{EF}{FG} = \cos \theta \therefore FG = \frac{EF}{\cos \theta} \right) \\
 &= \frac{A}{\cos \theta} \quad (\because EF \times 1 = A)
 \end{aligned}$$

Let us find the stresses on the inclined section FG .

First, isolate the portion of the member to the left of the inclined section as a free-body. This portion will be in equilibrium, if a force equal, opposite and collinear with the original force P is acting on the inclined section as shown in Fig. 12.6 (b).

The stress on inclined section FG along x -axis will be equal to force along x -axis divided by area of inclined section.

$\therefore$ Stress on section FG along x -axis

$$\begin{aligned}
 &= \frac{\text{Force along } x\text{-axis}}{\text{Area of section } FG} \\
 &= \frac{P}{\left(\frac{A}{\cos \theta} \right)} = \frac{P}{A} \cos \theta \\
 &= \sigma \cos \theta \quad \left(\because \frac{P}{A} = \sigma \right)
 \end{aligned}$$

Refer to Fig. 12.6 (c).

The above equation shows that stress on inclined plane in x -direction is less than that the stress on the plane perpendicular to the line of action of the force P .

Now let us find stresses normal to inclined plane and along the inclined plane.

Let P_n = The component of the force P , normal to section FG [Refer to Fig. 12.6(d)]
 $= P \cos \theta$

P_t = The component of force P , along the surface of the section FG (or tangential to the surface FG)
 $= P \sin \theta$

σ_n = Normal stress across the section FG

σ_t = Tangential stress (*i.e.*, shear stress) across the section FG .

$\therefore$ Normal stress and tangential stress across the section FG are obtained as,

$$\begin{aligned}
 \text{Normal stress, } \sigma_n &= \frac{\text{Force normal to section } FG}{\text{Area of section } FG} \\
 &= \frac{P_n}{\left(\frac{A}{\cos \theta} \right)} = \frac{P \cos \theta}{\left(\frac{A}{\cos \theta} \right)} \quad (\because P_n = P \cos \theta) \\
 &= \frac{P}{A} \cos \theta \cdot \cos \theta = \frac{P}{A} \cos^2 \theta \\
 &= \sigma \cos^2 \theta \quad \left(\because \frac{P}{A} = \sigma \right) \quad \dots(12.9)
 \end{aligned}$$

Tangential stress (*i.e.*, shear stress),

$$\begin{aligned}
 \sigma_t &= \frac{\text{Tangential force across-section } FG}{\text{Area of section } FG} \\
 &= \frac{P_t}{\left(\frac{A}{\cos \theta}\right)} = \frac{P \sin \theta}{\left(\frac{A}{\cos \theta}\right)} \quad (\because P_t = P \sin \theta) \\
 &= \frac{P}{A} \sin \theta \cdot \cos \theta = \sigma \sin \theta \cdot \cos \theta \\
 &= \frac{\sigma}{2} \times 2 \sin \theta \cos \theta \quad [\text{Multiplying and dividing by 2}] \\
 &= \frac{\sigma}{2} \sin 2\theta \quad (\because 2 \sin \theta \cos \theta = \sin 2\theta) \quad \dots(12.10)
 \end{aligned}$$

From equation (12.9), it is seen that the normal stress (σ_n) on the section FG will be maximum, when $\cos^2 \theta$ or $\cos \theta$ is maximum. And $\cos \theta$ will be maximum when $\theta = 0^\circ$ as $\cos 0^\circ = 1$. But when $\theta = 0^\circ$, the section FG will coincide with section EF . But the section EF is normal to the line of action of the loading. This means the plane normal to the axis of loading will carry the maximum normal stress.

$$\therefore \text{Maximum normal stress, } = \sigma \cos^2 \theta = \sigma \cos^2 0^\circ = \sigma \quad \dots(12.11)$$

From equation (12.10), it is observed that the tangential stress (*i.e.*, shear stress) across the section FG will be maximum when $\sin 2\theta$ is maximum. And $\sin 2\theta$ will be maximum when $\sin 2\theta = 1$ or $2\theta = 90^\circ$ or 270°

or $\theta = 45^\circ$ or 135° .

This means the shear stress will be maximum on two planes inclined at 45° and 135° to the normal section EF .

$$\therefore \text{Max. value of shear stress} = \frac{\sigma}{2} \sin 2\theta = \frac{\sigma}{2} \sin 90^\circ = \frac{\sigma}{2}. \quad \dots(12.12)$$

Problem 12.8. A rectangular bar of cross-sectional area 10000 mm^2 is subjected to an axial load of 20 kN . Determine the normal and shear stresses on a section which is inclined at an angle of 30° with normal cross-section of the bar.

Sol. Given:

Cross-sectional area of the rectangular bar,

$$A = 10000 \text{ mm}^2$$

Axial load, $P = 20 \text{ kN} = 20,000 \text{ N}$

Angle of oblique plane with the normal cross-section of the bar,

$$\theta = 30^\circ$$

$$\text{Now direct stress } \sigma = \frac{P}{A} = \frac{20000}{10000} = 2 \text{ N/mm}^2$$

Let σ_n = Normal stress on the oblique plane.

σ_t = Shear stress on the oblique plane.

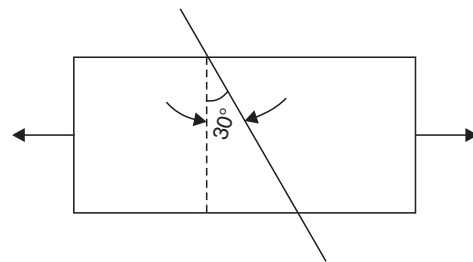


Fig. 12.7

Using equation (12.9) for normal stress, we get

$$\begin{aligned}\sigma_n &= \sigma \cos^2 \theta \\ &= 2 \times \cos^2 30^\circ \quad (\because \sigma = 2 \text{ N/mm}^2) \\ &= 2 \times 0.866^2 \quad (\because \cos 30^\circ = 0.866) \\ &= 1.5 \text{ N/mm}^2. \quad \text{Ans.}\end{aligned}$$

Using equation (12.10) for shear stress, we get

$$\begin{aligned}\sigma_t &= \frac{\sigma}{2} \sin 2\theta = \frac{2}{2} \times \sin (2 \times 30^\circ) \\ &= 1 \times \sin 60^\circ = 0.866 \text{ N/mm}^2. \quad \text{Ans.}\end{aligned}$$

Problem 12.9. Find the diameter of a circular bar which is subjected to an axial pull of 160 kN, if the maximum allowable shear stress on any section is 65 N/mm².

Sol. Given :

Axial pull, $P = 160 \text{ kN} = 160000 \text{ N}$

Maximum shear stress = 65 N/mm²

Let $D = \text{Diameter of the bar}$

$$\therefore \text{Area of the bar} = \frac{\pi}{4} D^2$$

$$\therefore \text{Direct stress, } \sigma = \frac{P}{A} = \frac{160000}{\frac{\pi}{4} D^2} = \frac{640000}{\pi D^2} \text{ N/mm}^2$$

Maximum shear stress is given by equation (12.12)

$$\therefore \text{Maximum shear stress} = \frac{\sigma}{2} = \frac{640000}{2 \times \pi D^2}.$$

But maximum shear stress is given as = 65 N/mm².

Hence equating the two values of maximum shear, we get

$$\begin{aligned}\therefore 65 &= \frac{640000}{2 \times \pi D^2} \\ D^2 &= \frac{640000}{2 \times \pi \times 65} = 1567 \\ D &= 39.58 \text{ mm.} \quad \text{Ans.}\end{aligned}$$

Problem 12.10. A rectangular bar of cross-sectional area of 11000 mm² is subjected to a tensile load P as shown in Fig. 12.8. The permissible normal and shear stresses on the oblique plane BC are given as 7 N/mm² and 3.5 N/mm² respectively. Determine the safe value of P .

Sol. Given :

Area of cross-section, $A = 11000 \text{ mm}^2$

Normal stress, $\sigma_n = 7 \text{ N/mm}^2$

Shear stress, $\sigma_t = 3.5 \text{ N/mm}^2$

Angle of oblique plane with the axis of bar = 60°.

$\therefore$ Angle of oblique plane BC with the normal cross-section of the bar,

$$\theta = 90^\circ - 60^\circ = 30^\circ$$

Let $P = \text{Safe value of axial pull}$

$\sigma = \text{Safe stress in the member.}$

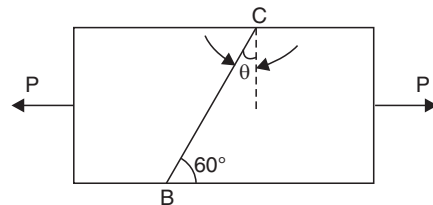


Fig. 12.8

Using equation (12.9)

$$\sigma_n = \sigma \cos^2 \theta \quad \text{or} \quad 7 = \sigma \cos^2 30^\circ$$

$$= \sigma(0.866)^2. \quad (\because \cos 30^\circ = 0.866)$$

$$\therefore \sigma = \frac{7}{0.866 \times 0.866} = 9.334 \text{ N/mm}^2$$

Using equation (12.10)

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or $3.5 = \frac{\sigma}{2} \sin 2 \times 30^\circ = \frac{\sigma}{2} \sin 60^\circ = \frac{\sigma}{2} \times 0.866$

$$\therefore \sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2.$$

The safe stress is the least of the two, i.e., 8.083 N/mm².

$\therefore$ Safe value of axial pull,

$$P = \text{Safe stress} \times \text{Area of cross-section}$$

$$= 8.083 \times 11000 = 88913 \text{ N} = \mathbf{88.913 \text{ kN.} \quad \text{Ans.}}$$

Problem 12.11. Two wooden pieces 10 cm $\times$ 10 cm in cross-section are glued together along line AB as shown in Fig. 12.9 below. What maximum axial force P can be applied if the allowable shearing stress along AB is 1.2 N/mm²?

Sol. Given :

$$\text{Area of cross-section} = 10 \times 10 = 100 \text{ cm}^2$$

$$= 100 \times 100 \text{ mm}^2 = 10000 \text{ mm}^2$$

$$\text{Allowable shear stress, } \sigma_t = 1.2 \text{ N/mm}^2$$

$$\text{Angle of line AB with the axis of axial force} = 30^\circ$$

$\therefore$ Angle of line AB with the normal cross-section,

$$\theta = 90^\circ - 30^\circ = 60^\circ$$

Let

$$P = \text{Maximum axial force}$$

$$\sigma = \text{Maximum allowable stress in the direction of } P.$$

Using equation (12.10)

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or $1.2 = \frac{\sigma}{2} \times \sin (2 \times 60^\circ) = \frac{\sigma}{2} \times \sin 120^\circ$

$$\therefore \sigma = \frac{1.2 \times 2}{\sin 120^\circ} = \frac{2.4}{0.866} = 2.771 \text{ N/mm}^2$$

$\therefore$ Maximum axial force,

$$P = \text{Stress in the direction of } P \times \text{Area of cross-section}$$

$$= \sigma \times 10000 = 2.771 \times 10000 = 27710 \text{ N} = \mathbf{27.71 \text{ kN.} \quad \text{Ans.}}$$

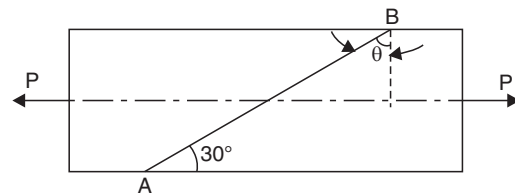


Fig. 12.9

12.7. MEMBERS OF VARYING CROSS-SECTIONS

A bar of different lengths and of different diameters (and hence of different cross-sectional areas) is shown in Fig. 12.10. Let this bar is subjected to an axial load P .

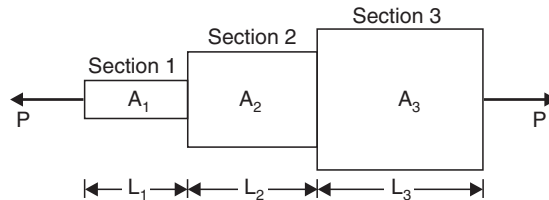


Fig. 12.10

Though each section is subjected to the same axial load P , yet the stresses, strains and change in lengths will be different. The total change in length will be obtained by adding the changes in length of individual section.

Let P = Axial load acting on the bar,
 L_1 = Length of section 1,
 A_1 = Cross-sectional area of section 1,
 L_2, A_2 = Length and cross-sectional area of section 2,
 L_3, A_3 = Length and cross-sectional area of section 3, and
 E = Young's modulus for the bar.

Then stress for the section 1,

$$\sigma_1 = \frac{\text{Load}}{\text{Area of section 1}} = \frac{P}{A_1}.$$

Similarly stresses for the section 2 and section 3 are given as,

$$\sigma_2 = \frac{P}{A_2} \quad \text{and} \quad \sigma_3 = \frac{P}{A_3}$$

Using equation (12.5), the strains in different sections are obtained.

$$\therefore \text{Strain of section 1, } e_1 = \frac{\sigma_1}{E} = \frac{P}{A_1 E} \quad \left(\because \sigma_1 = \frac{P}{A_1} \right)$$

Similarly the strains of section 2 and of section 3 are,

$$e_2 = \frac{\sigma_2}{E} = \frac{P}{A_2 E} \quad \text{and} \quad e_3 = \frac{\sigma_3}{E} = \frac{P}{A_3 E}.$$

$$\text{But strain in section 1} = \frac{\text{Change in length of section 1}}{\text{Length of section 1}}$$

$$\text{or} \quad e_1 = \frac{dL_1}{L_1}$$

where dL_1 = Change in length of section 1.

$$\therefore \text{Change in length of section 1, } dL_1 = e_1 L_1$$

$$= \frac{P L_1}{A_1 E} \quad \left(\because e_1 = \frac{P}{A_1 E} \right)$$

Similarly changes in length of section 2 and of section 3 are obtained as :

Change in length of section 2, $dL_2 = e_2 L_2$

$$= \frac{PL_2}{A_2 E} \quad \left(\because e_2 = \frac{P}{A_2 E} \right)$$

and change in length of section 3, $dL_3 = e_3 L_3$

$$= \frac{PL_3}{A_3 E} \quad \left(\because e_3 = \frac{P}{A_3 E} \right)$$

$\therefore$ Total change in the length of the bar,

$$\begin{aligned} dL &= dL_1 + dL_2 + dL_3 = \frac{PL_1}{A_1 E} + \frac{PL_2}{A_2 E} + \frac{PL_3}{A_3 E} \\ &= \frac{P}{E} \left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3} \right] \end{aligned} \quad \dots(12.13)$$

Equation (12.13) is used when the Young's modulus of different sections is same. If the Young's modulus of different sections is different, then total change in length of the bar is given by,

$$dL = P \left[\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} + \frac{L_3}{E_3 A_3} \right] \quad \dots(12.14)$$

Problem 12.12. An axial pull of 35000 N is acting on a bar consisting of three lengths as shown in Fig. 12.11. If the Young's modulus = 2.1×10^5 N/mm², determine :

- stresses in each section and
- total extension of the bar.

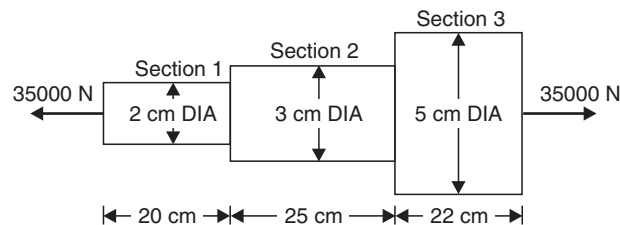


Fig. 12.11

Sol. Given :

Axial pull,	$P = 35000$ N
Length of section 1,	$L_1 = 20$ cm = 200 mm
Dia. of section 1,	$D_1 = 2$ cm = 20 mm
$\therefore$ Area of section 1,	$A_1 = \frac{\pi}{4} (20^2) = 100 \pi$ mm ²
Length of section 2,	$L_2 = 25$ cm = 250 mm
Dia. of section 2,	$D_2 = 3$ cm = 30 mm
$\therefore$ Area of section 2,	$A_2 = \frac{\pi}{4} (30^2) = 225 \pi$ mm ²
Length of section 3,	$L_3 = 22$ cm = 220 mm
Dia. of section 3,	$D_3 = 5$ cm = 50 mm

$$\therefore \text{Area of section 3, } A_3 = \frac{\pi}{4} (50^2) = 625 \pi \text{ mm}^2$$

$$\text{Young's modulus, } E = 2.1 \times 10^5 \text{ N/mm}^2.$$

(i) *Stresses in each section*

$$\begin{aligned} \text{Stress in section 1, } \sigma_1 &= \frac{\text{Axial load}}{\text{Area of section 1}} \\ &= \frac{P}{A_1} = \frac{35000}{100 \pi} = 111.408 \text{ N/mm}^2. \quad \text{Ans.} \end{aligned}$$

$$\text{Stress in section 2, } \sigma_2 = \frac{P}{A_2} = \frac{35000}{225 \times \pi} = 49.5146 \text{ N/mm}^2. \quad \text{Ans.}$$

$$\text{Stress in section 3, } \sigma_3 = \frac{P}{A_3} = \frac{35000}{625 \times \pi} = 17.825 \text{ N/mm}^2. \quad \text{Ans.}$$

(ii) *Total extension of the bar*

Using equation (12.13), we get

$$\begin{aligned} \text{Total extension} &= \frac{P}{E} \left(\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3} \right) \\ &= \frac{35000}{2.1 \times 10^5} \left(\frac{200}{100 \pi} + \frac{250}{225 \times \pi} + \frac{220}{625 \times \pi} \right) \\ &= \frac{35000}{2.1 \times 10^5} (6.366 + 3.536 + 1.120) = 0.183 \text{ mm.} \quad \text{Ans.} \end{aligned}$$

12.7.1. Principle of Superposition. When a number of loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body (For the equilibrium of the body, the forces along the axis of the body acting towards right should be equal to the forces acting towards left.), first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained. The total deformation of the body will be then equal to the algebraic sum of deformations of the individual sections.

Problem 12.13. A brass bar, having cross-sectional area of 1000 mm^2 , is subjected to axial forces as shown in Fig. 12.12.

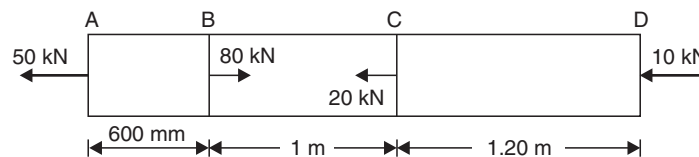


Fig. 12.12

Find the total elongation of the bar. Take $E = 1.05 \times 10^5 \text{ N/mm}^2$.

Sol. Given :

$$\text{Area, } A = 1000 \text{ mm}^2$$

$$\text{Value of } E = 1.05 \times 10^5 \text{ N/mm}^2$$

$$\text{Let } dL = \text{Total elongation of the bar.}$$

For the equilibrium of the bar, the forces along the axis of the bar acting towards right should be equal to forces acting towards left.

$$\therefore 80 = 50 + 20 + 10$$

The force of 80 kN acting at B is split up into three forces of 50 kN, 20 kN and 10 kN. Then the part AB of the bar will be subjected to a tensile load of 50 kN, part BC is subjected to a compressive load of 20 kN and part BD is subjected to a compressive load of 10 kN as shown in Fig. 12.12 (a).

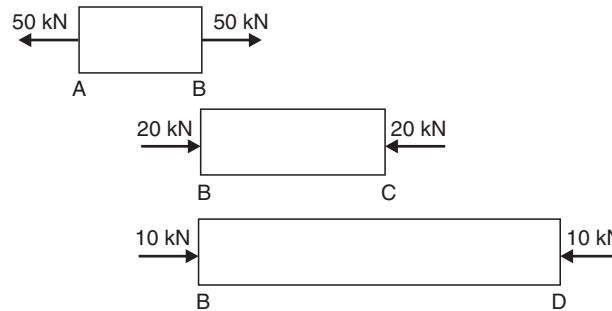


Fig. 12.12 (a)

Part AB. This part is subjected to a tensile load of 50 kN. Hence there will be increase in length of this part.

$\therefore$ Increase in the length of AB

$$\begin{aligned} &= \frac{P_1}{AE} \times L_1 \\ &= \frac{50 \times 1000}{1000 \times 1.05 \times 10^5} \times 600 \quad (\because P_1 = 50,000 \text{ N}, L_1 = 600 \text{ mm}) \\ &= 0.2857. \end{aligned}$$

Part BC. This part is subjected to a compressive load of 20 kN or 20,000 N. Hence there will be decrease in length of this part.

$\therefore$ Decrease in the length of BC

$$\begin{aligned} &= \frac{P_2}{AE} \times L_2 = \frac{20,000}{1000 \times 1.05 \times 10^5} \times 1000 \quad (\because L_2 = 1 \text{ m} = 1000 \text{ mm}) \\ &= 0.1904. \end{aligned}$$

Part BD. This part is subjected to a compressive load of 10 kN or 10,000 N.

Hence there will be decrease in length of this part.

$\therefore$ Decrease in the length of BD

$$\begin{aligned} &= \frac{P_3}{AE} \times L_3 = \frac{10000}{1000 \times 1.05 \times 10^5} = 2200 \\ & \quad (\because L_3 = 1.2 + 1 = 2.2 \text{ m or } 2200 \text{ mm}) \\ &= 0.2095. \end{aligned}$$

$\therefore$ Total elongation of bar = $0.2857 - 0.1904 - 0.2095$

(Taking +ve sign for increase in length and
-ve sign for decrease in length)

$$= -0.1142 \text{ mm. Ans.}$$

Negative sign shows, that there will be decrease in length of the bar.

Problem 12.14. A member ABCD is subjected to point loads P_1 , P_2 , P_3 and P_4 as shown in Fig. 12.13.

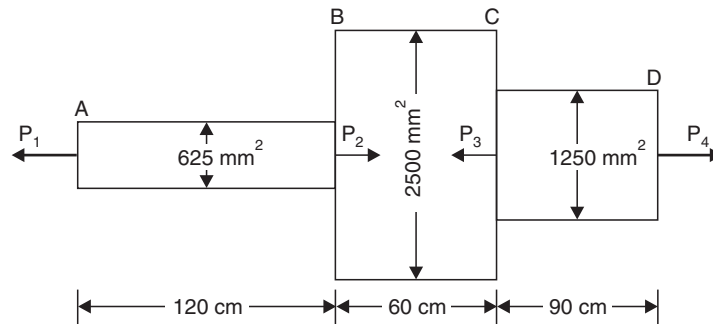


Fig. 12.13

Calculate the force P_2 necessary for equilibrium, if $P_1 = 45 \text{ kN}$, $P_3 = 450 \text{ kN}$ and $P_4 = 130 \text{ kN}$. Determine the total elongation of the member, assuming the modulus of elasticity to be $2.1 \times 10^5 \text{ N/mm}^2$. (Similar Question U.P. Tech. University, May 2009)

Sol. Given :

Part AB :	Area,	$A_1 = 625 \text{ mm}^2$ and
	Length,	$L_1 = 120 \text{ cm} = 1200 \text{ mm}$
Part BC :	Area,	$A_2 = 2500 \text{ mm}^2$ and
	Length,	$L_2 = 60 \text{ cm} = 600 \text{ mm}$
Part CD :	Area,	$A_3 = 12.0 \text{ mm}^2$ and
	Length,	$L_3 = 90 \text{ cm} = 900 \text{ mm}$
Value of		$E = 2.1 \times 10^5 \text{ N/mm}^2$.

Value of P_2 necessary for equilibrium

Resolving the forces on the rod along its axis (*i.e.*, equating the forces acting towards right to those acting towards left), we get

$$P_1 + P_3 = P_2 + P_4$$

But

$$P_1 = 45 \text{ kN},$$

$$P_3 = 450 \text{ kN} \text{ and } P_4 = 130 \text{ kN}$$

$$\therefore 45 + 450 = P_2 + 130 \quad \text{or} \quad P_2 = 495 - 130 = 365 \text{ kN}$$

The force of 365 kN acting at B is split into two forces of 45 kN and 320 kN (*i.e.*, $365 - 45 = 320 \text{ kN}$).

The force of 450 kN acting at C is split into two forces of 320 kN and 130 kN (*i.e.*, $450 - 320 = 130 \text{ kN}$) as shown in Fig. 12.13 (a).

From Fig. 12.13 (a), it is clear that part AB is subjected to a tensile load of 45 kN, part BC is subjected to a compressive load of 320 kN and part CD is subjected to a tensile load 130 kN.

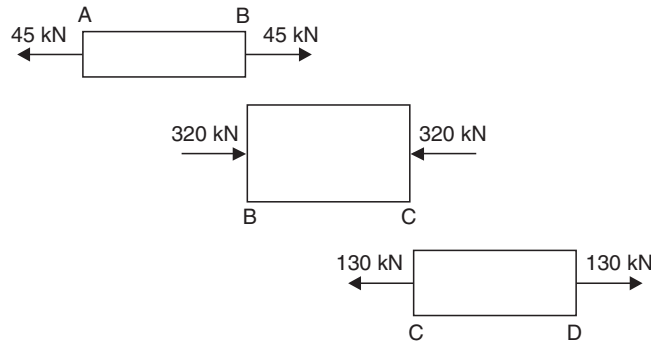


Fig. 12.13 (a)

Hence for part AB , there will be increase in length ; for part BC there will be decrease in length and for part CD there will be increase in length.

∴ Increase in length of AB

$$= \frac{P}{A_1 E} \times L_1 = \frac{45000}{625 \times 2.1 \times 10^5} \times 1200 \quad (\because P = 45 \text{ kN} = 45000 \text{ N})$$

$$= 0.4114 \text{ mm}$$

Decrease in length of BC

$$= \frac{P}{A_2 E} \times L_2 = \frac{320,000}{2500 \times 2.1 \times 10^5} \times 600 \quad (\because P = 320 \text{ kN} = 320000 \text{ N})$$

$$= 0.3657 \text{ mm}$$

Increase in length of CD

$$= \frac{P}{A_3 E} \times L_3 = \frac{130,000}{1250 \times 2.1 \times 10^5} \times 900 \quad (\because P = 130 \text{ kN} = 130000 \text{ N})$$

$$= 0.4457 \text{ mm}$$

Total change in the length of member

$$= 0.4114 - 0.3657 + 0.4457$$

(Taking +ve sign for increase in length and
-ve sign for decrease in length)

$$= 0.4914 \text{ mm (extension). Ans.}$$

12.8. BARS IN SERIES

When bars are connected in series, then each bar will be subjected to the same axial load. And $E = \frac{\text{Stress}}{\text{Strain}} = \frac{(P/A)}{(dL/L)} = \frac{P \times L}{A \times dL}$ or $dL = \frac{P \times L}{EA}$ will be used.

Problem 12.15. A member formed by connecting a steel bar to an aluminium bar is shown in Fig. 12.14. Assuming that the bars are prevented from buckling sideways, calculate the magnitude of force P that will cause the total length of the member to decrease 0.25 mm. The value of elastic modulus for steel and aluminium are $2.1 \times 10^5 \text{ N/mm}^2$ and $7 \times 10^4 \text{ N/mm}^2$ respectively.

Sol. Given :

Length of steel bar, $L_1 = 30 \text{ cm} = 300 \text{ mm}$

Area of steel bar, $A_1 = 5 \times 5 = 25 \text{ cm}^2 = 2500 \text{ mm}^2$

Elastic modulus for steel bar,

$$E_1 = 2.1 \times 10^5 \text{ N/mm}^2$$

Length of aluminium bar,

$$L_2 = 38 \text{ cm} = 380 \text{ mm}$$

Area of aluminium bar,

$$A_2 = 10 \times 10 = 100 \text{ cm}^2 = 10000 \text{ mm}^2$$

Elastic modulus for aluminium bar,

$$E_2 = 7 \times 10^4 \text{ N/mm}^2$$

Total decrease in length, $dL = 0.25 \text{ mm}$

Let $P = \text{Required force.}$

As bars are in series, the axial force P will be same for both bars.

As both the bars are made of different materials, hence total change in the lengths of the bar is given by equation (12.14).

$$\begin{aligned} dL &= P \left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right) \\ 0.25 &= P \left(\frac{300}{2.1 \times 10^5 \times 2500} + \frac{380}{7 \times 10^4 \times 10000} \right) \\ &= P (5.714 \times 10^{-7} + 5.428 \times 10^{-7}) = P \times 11.142 \times 10^{-7} \\ P &= \frac{0.25}{11.142 \times 10^{-7}} = \frac{0.025 \times 10^7}{11.142} \\ &= 2.2437 \times 10^5 = \mathbf{224.37 \text{ kN. Ans.}} \end{aligned}$$

Problem 12.16. A member is formed by connecting a steel bar to an aluminium bar in series. These bars are subjected to an axial load of 200 kN. The length, area and modulus of elasticity for steel bar are 500 mm, 2500 mm² and $2 \times 10^5 \text{ N/mm}^2$ respectively whereas for aluminium bar these values are 280 mm, 4000 mm² and $7 \times 10^4 \text{ N/mm}^2$. Determine the total change in length.

Sol. Given :

For Steel Bar

$$L_1 = 500 \text{ mm}$$

$$A_1 = 2500 \text{ mm}^2$$

$$E_1 = 2 \times 10^5 \text{ N/mm}^2$$

For Aluminium Bar

$$L_2 = 280 \text{ mm}$$

$$A_2 = 4000 \text{ mm}^2$$

$$E_2 = 7 \times 10^4 \text{ N/mm}^2$$

As bars are in series, the axial load will be same and is equal to

$$P = 200 \text{ kN} = 200 \times 10^3 \text{ N} = 2 \times 10^5 \text{ N}$$

Total change in length is given by equation (12.14),

$$\begin{aligned} dL &= P \left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right) \\ &= 2 \times 10^5 \left(\frac{500}{2 \times 10^5 \times 2500} + \frac{280}{7 \times 10^4 \times 4000} \right) = \frac{2 \times 10^5 \times 500}{2 \times 10^5 \times 2500} + \frac{2 \times 10^5 \times 280}{7 \times 10^4 \times 4000} \\ &= \frac{1}{5} + \frac{1}{5} = 0.2 + 0.2 = \mathbf{0.4 \text{ mm. Ans.}} \end{aligned}$$

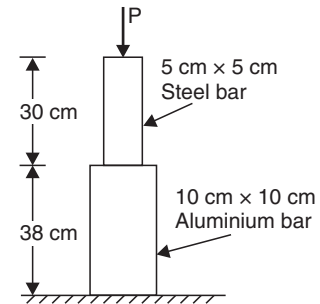


Fig. 12.14

12.9. TENSILE TEST DIAGRAMS FOR DUCTILE AND BRITTLE MATERIALS

A material is said to be ductile, if it elongates appreciably before fracture. One such material is mild steel.

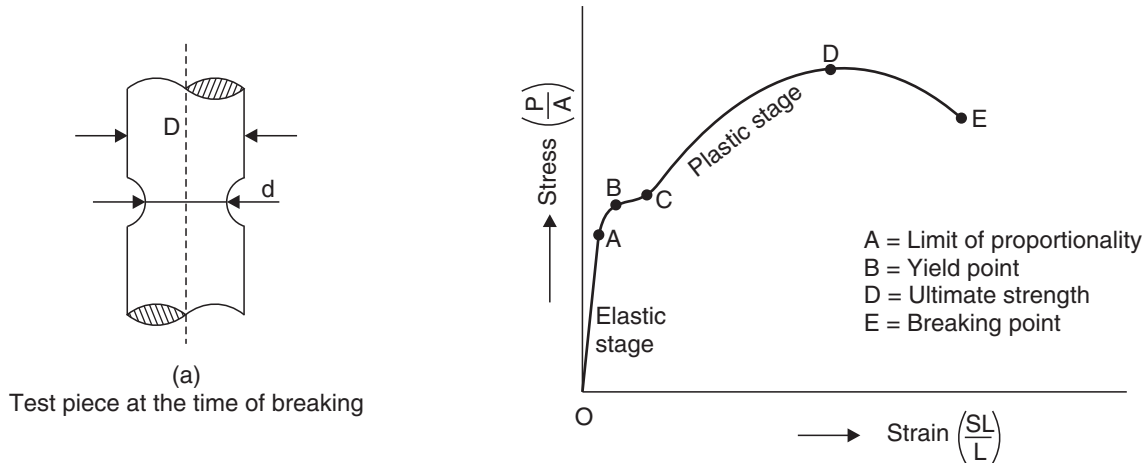


Fig.12.15

Fig.12.15 shows a stress-strain diagram for a mild steel, in which the axial strain $\left(\frac{\delta L}{L}\right)$ are plotted along x -axis and corresponding stresses $\left(\frac{P}{A} = \frac{\text{Load}}{\text{Area}}\right)$ are plotted along y -axis. This diagram is obtained by performing a tensile test on the specimen. The end of the test piece is fixed into grips connected to a straining device and load measuring device. The load on the test piece is increased slowly and corresponding extension is measured. From these readings, the curve $OABCDE$ is obtained.

In this curve, from O to A the stress is proportional to the strain and this is known as elastic stage. Upto point A , Hooke's law is applicable. The stress at A (*i.e.*, load at A divided by area of the specimen) represents the proportional limit. If the load is increased further the elongation becomes more rapid and diagram becomes curved. The stress will not be proportional to the strain. Beyond point B , a sudden elongation of bar takes place without increase in load. This phenomenon is known as yielding of the material and it is shown by a horizontal line BC . The point B is known as yield point. The load divided by initial area gives the corresponding stress, the stress at yield point. Here the material becomes plastic. If load is removed, the specimen will not return to its original shape. At point C , the material begins to strain harden and recovers some of its elastic property. If the load is increased beyond point C , the stress strain curve climbs to point D . The load at D divided by initial area gives the ultimate strength. Beyond point D , further stretching of the bar is accompanied by a decrease in load and fracture takes place at point E , suddenly.

12.9.1. Ultimate Stress. It is the stress obtained by dividing maximum load (*i.e.*, load corresponding to point D) by the initial cross-sectional area of the specimen.

12.9.2. Breaking Stress. It is the stress obtained by dividing the breaking load (*i.e.*, load corresponding to point E) by the initial cross-sectional area.

12.9.3. Tensile Test Diagram for Brittle Materials. The materials which show very small elongation before they fracture, are known as brittle materials. Cast iron, concrete and high carbon steel are brittle materials. Hence brittle materials break easily when subjected to shock.

Fig. 12.16 shows the stress-strain diagram for brittle materials, in which the axial strain $\left(\frac{\delta L}{L}\right)$ are plotted along x -axis and corresponding stresses $\left(\frac{P}{A} = \frac{\text{Load}}{\text{Area}}\right)$ are plotted along y -axis. This diagram is obtained by performing a tensile test on the specimen. The end of the test piece is fixed into grips connected to a straining device and load measuring device. The load on the test piece is increased slowly and corresponding extension is measured. From these readings, the curve OAB is obtained.

In this curve, from O to A the stress is proportional to the strain and this is known as elastic stage. Upto point A , Hooke's law is applicable. The stress at A (*i.e.*, load at A divided by area of the specimen) represents the proportional limit. If the load is increased further the elongation becomes more rapid and diagram becomes curved. The stress will not be proportional to the strain. At point B , suddenly the specimen breaks into pieces. The total elongation OC before fracture is very small as compared to total elongation of ductile materials.

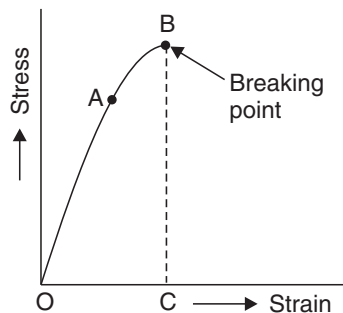


Fig. 12.16. Stress-strain curve for brittle materials.

HIGHLIGHTS

1. The resistance per unit area, offered by a body against deformation is known as stress. The stress is given by

$$\sigma = \frac{P}{A}$$

where P = External force or load ; A = Cross-sectional area.

2. Stress is expressed as kgf/m^2 , kgf/cm^2 , N/m^2 and N/mm^2 .
3. $1 \text{ N/m}^2 = 10^{-4} \text{ N/cm}^2$ or 10^{-6} N/mm^2 .
4. The ratio of change of dimension of the body to the original dimension is known as strain.
5. The stress induced in a body, which is subjected to two equal and opposite pulls, is known as tensile stress.
6. The stress induced in a body, which is subjected to two equal and opposite pushes, is known as compressive stress.
7. Elasticity is the property by virtue of which certain materials return back to their original position after the removal of the external force.
8. Hooke's law states that within elastic limit, the stress is proportional to the strain.

9. The ratio of tensile stress (or compressive stress) to the corresponding strain is known as Young's modulus or modulus of elasticity and is denoted by E .

$$\therefore E = \frac{\text{Tensile or compressive stress}}{\text{Corresponding strain}}.$$

10. The ratio of shear stress to the corresponding shear strain within the elastic limit, is known as modulus of rigidity or shear modulus. It is denoted by C (or G or N).
11. Total change in the length of a bar of different lengths and of different diameters when subjected to an axial load P , is given by

$$dL = \frac{P}{E} \left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3} + \dots \right] \quad \text{..... when } E \text{ is same}$$

$$= P \left[\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} + \frac{L_3}{E_3 A_3} + \dots \right] \quad \text{..... when } E \text{ is different.}$$

EXERCISE 12

(A) Theoretical Questions

1. Define stress and strain. Write down the S.I. and M.K.S. units of stress and strain.
2. Explain clearly the different types of stresses and strains.
3. Define the terms : Elasticity, elastic limit, Young's modulus and modulus of rigidity.
4. State Hooke's law.
5. Three sections of a bar are having different lengths and different diameters. The bar is subjected to an axial load P . Determine the total change in length of the bar. Take Young's modulus of different sections same.
6. Distinguish between the following, giving due explanation :
 - (i) Stress and strain,
 - (ii) Force and stress, and
 - (iii) Tensile stress and compressive stress.

(B) Numerical Problems

1. A rod 200 cm long and of diameter 3.0 cm is subjected to an axial pull of 30 kN. If the Young's modulus of the material of the rod is $2 \times 10^5 \text{ N/mm}^2$, determine : (i) stress, (ii) strain, and (iii) the elongation of the rod.
[Ans. (i) 42.44 N/mm^2 , (ii) 0.000212 , (iii) 0.0424 cm]
2. Find the Young's modulus of a rod of diameter 30 mm and of length 300 mm which is subjected to a tensile load of 60 kN and the extension of the rod is equal to 0.4 mm. [Ans. 63.6 GN/m^2]
3. The safe stress, for a hollow steel column which carries an axial load of $2.2 \times 10^3 \text{ kN}$ is 120 MN/m^2 . If the external diameter of the column is 25 cm, determine the internal diameter.
[Ans. 19.79 cm]
4. An axial pull of 40000 N is acting on a bar consisting of three sections of length 30 cm, 25 cm and 20 cm and of diameters 2 cm, 4 cm and 5 cm respectively. If the Young's modulus $= 2 \times 10^5 \text{ N/mm}^2$, determine :
 - (i) stress in each section and
 - (ii) total extension of the bar.
 [Ans. (i) $127.32, 31.8, 20.37 \text{ N/mm}^2$, (ii) 0.025 cm]
5. The ultimate stress for a hollow steel column which carries an axial load of 2 MN is 500 N/mm^2 . If the external diameter of the column is 250 mm, determine the internal diameter. Take the factor of safety as 4.0.
[Ans. 205.25 mm]

6. A member formed by connecting a steel bar to an aluminium bar is shown in Fig. 12.17. Assuming that the bars are prevented from buckling sideways, calculate the magnitude of force P , that will cause the total length of the member to decrease 0.30 mm. The values of elastic modulus for steel and aluminium are $2 \times 10^5 \text{ N/mm}^2$ and $6.5 \times 10^4 \text{ N/mm}^2$ respectively. [Ans. 406.22 kN]

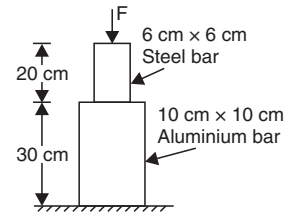


Fig. 12.17

7. A brass bar, having cross-section area of 900 mm^2 , is subjected to axial forces as shown in Fig. 12.18 in which $AB = 0.6 \text{ m}$, $BC = 0.8 \text{ m}$ and $CD = 1.0 \text{ m}$.

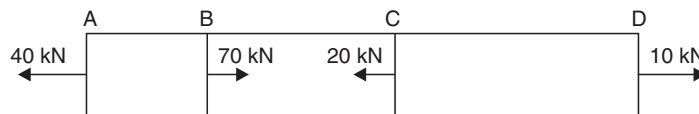


Fig. 12.18

Find the total elongation of the bar. Take $E = 1 \times 10^5 \text{ N/mm}^2$.

[Ans. - 0.111 mm]

8. A member $ABCD$ is subjected to point loads P_1, P_2, P_3 and P_4 as shown in Fig. 12.19. Calculate the force P_3 necessary for equilibrium if $P_1 = 120 \text{ kN}$, $P_2 = 220 \text{ kN}$ and $P_4 = 160 \text{ kN}$. Determine also the net change in the length of the member. Take $E = 200 \text{ GN/m}^2$. [Ans. 0.55 mm]

(U.P. Tech. University, May 2009)

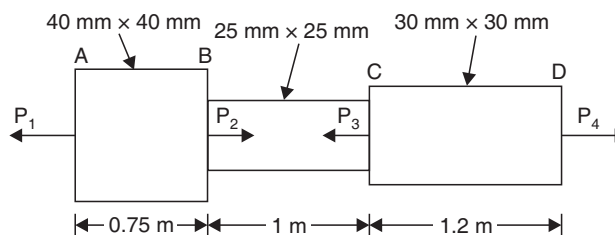


Fig. 12.19

[Hint. For equilibrium,

$$P_1 + P_3 = P_2 + P_4$$

or

$$120 + P_3 = 220 + 160$$

$\therefore$

$$P_3 = 220 + 160 - 120 = 260 \text{ kN. Ans.}$$

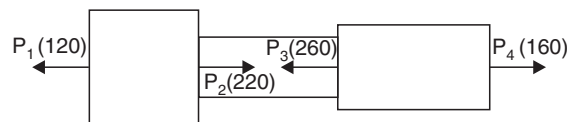


Fig. 12.20

$$E = 200 \text{ GN/m}^2 = \frac{200 \times 10^9}{10^6} \text{ N/mm}^2 = 200 \times 10^3 \text{ N/mm}^2$$

The various parts are subjected to forces as shown in Fig. 12.20 (a).

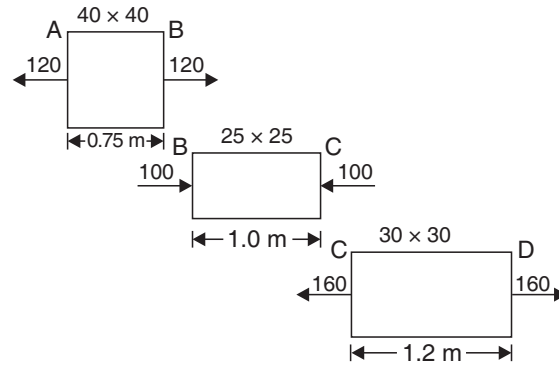


Fig. 12.20 (a)

$$\text{Net change in the length} = dL_1 - dL_2 + dL_3$$

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{\left(\frac{P}{A}\right)}{\left(\frac{\delta L}{L}\right)} = \frac{P}{A} \times \frac{L}{\delta L}$$

$$\therefore dL = \frac{P \times L}{AE}$$

$$dL_1 = \frac{P_1 \times L_1}{A_1 \times E}, \quad dL_2 = \frac{P_2 \times L_2}{A_2 \times E}, \quad dL_3 = \frac{P_3 \times L_3}{A_3 \times E}$$

$\therefore$ Net change in length

$$= dL_1 - dL_2 + dL_3$$

$$= \frac{P_1 \times L_1}{A_1 \times E} - \frac{P_2 \times L_2}{A_2 \times E} + \frac{P_3 \times L_3}{A_3 \times E}$$

$$= \frac{(120 \times 10^3) \times (0.75 \times 10^3)}{(40 \times 40) \times 200 \times 10^3} - \frac{(100 \times 10^3) \times (1 \times 1000)}{(25 \times 25) \times 200 \times 10^3} + \frac{(160 \times 10^3) \times 1200}{(30 \times 30) \times 200 \times 10^3}$$

$$= 0.28125 - 0.8 + 1.0666 = 0.548 \text{ mm. Ans.}]$$

13

Elastic Constants

13.1. INTRODUCTION

When a body is subjected to an axial tensile load, there is an increase in the length of the body. But at the same time there is a decrease in other dimensions of the body at right angles to the line of action of the applied load. Thus the body is having axial deformation and also deformation at right angles to the line of action of the applied load (*i.e.*, lateral deformation). This chapter deals with these deformations, Poisson's ratio, volumetric strains, bulk modulus, relation between Young's modulus and modulus of rigidity and relation between Young's modulus and bulk modulus.

13.2. LONGITUDINAL STRAIN

When a body is subjected to an axial tensile or compressive load, there is an axial deformation in the length of the body. The ratio of axial deformation to the original length of the body is known as longitudinal (or linear) strain. The longitudinal strain is also defined as the deformation of the body per unit length in the direction of the applied load.

Let L = Length of the body,

P = Tensile force acting on the body,

δL = Increase in the length of the body in the direction of P .

Then, longitudinal strain = $\frac{\delta L}{L}$.

13.3. LATERAL STRAIN

The strain at right angles to the direction of applied load is known as lateral strain. Let a rectangular bar of length L , breadth b and depth d is subjected to an axial tensile load P as shown in Fig. 13.1. The length of the bar will increase while the breadth and depth will decrease.

Let δL = Increase in length,

δb = Decrease in breadth, and

δd = Decrease in depth.

Then longitudinal strain = $\frac{\delta L}{L}$...(13.1)

and lateral strain = $\frac{\delta b}{b}$ or $\frac{\delta d}{d}$...(13.2)

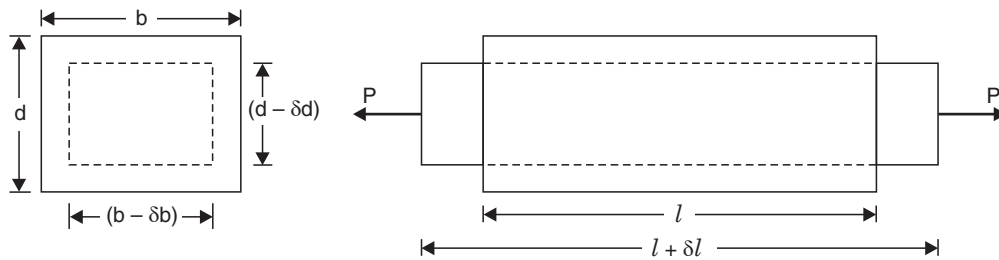


Fig. 13.1

Note. (i) If longitudinal strain is tensile, the lateral strains will be compressive.

(ii) If longitudinal strain is compressive then lateral strains will be tensile.

(iii) Hence every longitudinal strain in the direction of load is accompanied by lateral strains of the opposite kind in all directions perpendicular to the load.

13.4. POISSON'S RATIO

The ratio of lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called **Poisson's ratio** and it is generally denoted by μ . Hence mathematically,

$$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} \quad \dots(13.3)$$

or $\text{Lateral strain} = \mu \times \text{longitudinal strain}$

As lateral strain is opposite in sign to longitudinal strain, hence algebraically, the lateral strain is written as

$$\text{Lateral strain} = -\mu \times \text{longitudinal strain} \quad \dots[13.3 (A)]$$

The value of Poisson's ratio varies from 0.25 to 0.33. For rubber, its value ranges from 0.45 to 0.50.

Problem 13.1. Determine the changes in length, breadth and thickness of a steel bar which is 4 m long, 30 mm wide and 20 mm thick and is subjected to an axial pull of 30 kN in the direction of its length. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio = 0.3.

Sol. Given :

Length of the bar, $L = 4 \text{ m} = 4000 \text{ mm}$

Breadth of the bar, $b = 30 \text{ mm}$

Thickness of the bar, $t = 20 \text{ mm}$

$\therefore$ Area of cross-section, $A = b \times t = 30 \times 20 = 600 \text{ mm}^2$

Axial pull, $P = 30 \text{ kN} = 30000 \text{ N}$

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Poisson's ratio, $\mu = 0.3$.

Now strain in the direction of load (or longitudinal strain),

$$\begin{aligned} &= \frac{\text{Stress}}{E} = \frac{\text{Load}}{\text{Area} \times E} & \left(\because \text{Stress} = \frac{\text{Load}}{\text{Area}} \right) \\ &= \frac{P}{A \cdot E} = \frac{30000}{600 \times 2 \times 10^5} = 0.00025. \end{aligned}$$

But longitudinal strain $= \frac{\delta L}{L}$.

$$\therefore \frac{\delta L}{L} = 0.00025.$$

$$\therefore \delta L \text{ (or change in length)} = 0.00025 \times L \\ = 0.00025 \times 4000 = \mathbf{1.0 \text{ mm. Ans.}}$$

Using equation (13.3),

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$\text{or } 0.3 = \frac{\text{Lateral strain}}{0.00025}$$

$$\therefore \text{Lateral strain} = 0.3 \times 0.00025 = 0.000075.$$

We know that

$$\text{Lateral strain} = \frac{\delta b}{b} \quad \text{or} \quad \frac{\delta d}{d} \left(\text{or } \frac{\delta t}{t} \right)$$

$$\therefore \delta b = b \times \text{Lateral strain} \\ = 30 \times 0.000075 = \mathbf{0.00225 \text{ mm. Ans.}}$$

$$\text{Similarly, } \delta t = t \times \text{Lateral strain} \\ = 20 \times 0.000075 = \mathbf{0.0015 \text{ mm. Ans.}}$$

Problem 13.2. Determine the value of Young's modulus and Poisson's ratio of a metallic bar of length 30 cm, breadth 4 cm and depth 4 cm when the bar is subjected to an axial compressive load of 400 kN. The decrease in length is given as 0.075 cm and increase in breadth is 0.003 cm.

Sol. Given :

Length, $L = 30 \text{ cm}$; Breadth, $b = 4 \text{ cm}$; and Depth, $d = 4 \text{ cm}$.

$$\therefore \text{Area of cross-section, } A = b \times d = 4 \times 4 \\ = 16 \text{ cm}^2 = 16 \times 100 = 1600 \text{ mm}^2$$

$$\text{Axial compressive load, } P = 400 \text{ kN} = 400 \times 1000 \text{ N}$$

$$\text{Decrease in length, } \delta L = 0.075 \text{ cm}$$

$$\text{Increase in breadth, } \delta b = 0.003 \text{ cm}$$

$$\text{Longitudinal strain} = \frac{\delta L}{L} = \frac{0.075}{30} = 0.0025$$

$$\text{Lateral strain} = \frac{\delta b}{b} = \frac{0.003}{4} = 0.00075.$$

Using equation (13.3),

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{0.00075}{0.0025} = \mathbf{0.3. Ans.}$$

$$\text{Longitudinal strain} = \frac{\text{Stress}}{E} = \frac{P}{A \times E} \quad \left(\because \text{Stress} = \frac{\text{Load}}{\text{Area}} = \frac{P}{A} \right)$$

$$\text{or } 0.0025 = \frac{400000}{1600 \times E} \\ E = \frac{400000}{1600 \times 0.0025} = \mathbf{1 \times 10^5 \text{ N/mm}^2. Ans.}$$

13.5. VOLUMETRIC STRAIN

The ratio of change in volume to the original volume of a body (when the body is subjected to a single force or a system of forces) is called volumetric strain. It is denoted by e_v .

Mathematically, volumetric strain is given by

$$e_v = \frac{\delta V}{V}$$

where δV = Change in volume, and

V = Original volume.

13.5.1. Volumetric Strain of a Rectangular Bar which is Subjected to an Axial Load P in the Direction of its Length. Consider a rectangular bar of length L , width b and depth d which is subjected to an axial load P in the direction of its length as shown in Fig. 13.2.

Let δL = Change in length,

δb = Change in width,

and δd = Change in depth.

$\therefore$ Final length of the bar $= L + \delta L$

Final width of the bar $= b + \delta b$

Final depth of the bar $= d + \delta d$

Now original volume of the bar, $V = L.b.d$

Final volume $= (L + \delta L)(b + \delta b)(d + \delta d)$

$$= L.b.d. + bd\delta L + Lb\delta d + Ld\delta b$$

(Ignoring products of small quantities)

$\therefore$ Change in volume,

$$\begin{aligned}\delta V &= \text{Final volume} - \text{Original volume} \\ &= (Lbd + bd\delta L + Lb\delta d + Ld\delta b) - Lbd \\ &= bd\delta L + Lb\delta d + Ld\delta b\end{aligned}$$

$\therefore$ Volumetric strain,

$$\begin{aligned}e_v &= \frac{\delta V}{V} \\ &= \frac{bd\delta L + Lb\delta d + Ld\delta b}{Lbd} \\ &= \frac{\delta L}{L} + \frac{\delta d}{d} + \frac{\delta b}{b}\end{aligned}\quad \dots(13.4)$$

But $\frac{\delta L}{L}$ = Longitudinal strain and $\frac{\delta d}{d}$ or $\frac{\delta b}{b}$ are lateral strains.

Substituting these values in the above equation, we get

$$e_v = \text{Longitudinal strain} + 2 \times \text{Lateral strain} \quad \dots(i)$$

From equation (13.3 A), we have

$\therefore$ Lateral strain $= -\mu \times \text{Longitudinal strain}$.

Substituting the value of lateral strain in equation (i), we get

$$e_v = \text{Longitudinal strain} - 2 \times \mu \text{ longitudinal strain}$$



Fig. 13.2

$$\begin{aligned}
 &= \text{Longitudinal strain } (1 - 2\mu) \\
 &= \frac{\delta L}{L} (1 - 2\mu) \quad \dots(13.5)
 \end{aligned}$$

Problem 13.3. For the problem 13.1, determine the volumetric strain and final volume of the given steel bar.

Sol. Given :

The following data is given in problem 13.1. :

$$L = 4000 \text{ mm}, b = 30 \text{ mm}, t \text{ or } d = 20 \text{ mm}, \mu = 0.3.$$

$$\text{Original volume, } V = L.b.d = 4000 \times 30 \times 20 = 2400000 \text{ mm}^3$$

The value of longitudinal strain $\left(i.e., \frac{\delta L}{L}\right)$ in problem 13.1 is calculated

$$\text{as, } \frac{\delta L}{L} = 0.00025$$

Now using equation (13.5), we have

$$\begin{aligned}
 \text{Volumetric strain, } e_v &= \frac{\delta L}{L} (1 - 2\mu) \\
 &= 0.00025(1 - 2 \times 0.3) = \mathbf{0.0001. \quad Ans.}
 \end{aligned}$$

$$\text{or } \frac{\delta V}{V} = 0.0001 \quad \left(\because e_v = \frac{\delta V}{V} \right)$$

$$\begin{aligned}
 \therefore \delta V &= 0.0001 \times V \\
 &= 0.0001 \times 2400000 = 240 \text{ mm}^3
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Final volume} &= \text{Original volume} + \delta V \\
 &= 2400000 + 240 \text{ mm}^3 \\
 &= \mathbf{2400240 \text{ mm}^3. \quad Ans.}
 \end{aligned}$$

Problem 13.4. A steel bar 300 mm long, 50 mm wide and 40 mm thick is subjected to a pull of 300 kN in the direction of its length. Determine the change in volume. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $\mu = 0.25$.

Sol. Given :

$$\text{Length, } L = 300 \text{ mm}$$

$$\text{Width, } b = 50 \text{ mm}$$

$$\text{Thickness, } t = 40 \text{ mm}$$

$$\text{Pull, } P = 300 \text{ kN} = 300 \times 10^3 \text{ N}$$

$$\text{Value of } E = 2 \times 10^5 \text{ N/mm}^2$$

$$\text{Value of } \mu = 0.25$$

$$\begin{aligned}
 \text{Original volume, } V &= L \times b \times t \\
 &= 300 \times 50 \times 40 \text{ mm}^3 = 600000 \text{ mm}^3
 \end{aligned}$$

The longitudinal strain (i.e., the strain in the direction of load) is given by

$$\frac{dL}{L} = \frac{\text{Stress in the direction of load}}{E}$$

But stress in the direction of load

$$= \frac{P}{\text{Area}} = \frac{P}{b \times t}$$

$$= \frac{300 \times 10^3}{50 \times 40} = 150 \text{ N/mm}^2$$

$$\therefore \frac{dL}{L} = \frac{150}{2 \times 10^5} = 0.00075$$

Now volumetric strain is given by equation (13.5) as

$$\begin{aligned} e_v &= \frac{dL}{L} (1 - 2\mu) \\ &= 0.00075 (1 - 2 \times 0.25) = 0.000375 \end{aligned}$$

Let δV = Change in volume. Then $\frac{dV}{V}$ represents volumetric strain.

$$\therefore \frac{dV}{V} = 0.000375$$

or

$$\begin{aligned} dV &= 0.000375 \times V \\ &= 0.000375 \times 600000 = \mathbf{225 \text{ mm}^3}. \quad \text{Ans.} \end{aligned}$$

13.5.2. Volumetric Strain of a Rectangular Bar Subjected to Three Forces which are Mutually Perpendicular. Consider a rectangular block of dimensions x , y and z subjected to three direct tensile stresses along three mutually perpendicular axis as shown in Fig. 13.3.

Then volume of block, $V = xyz$.

Taking logarithm to both sides, we have

$$\log V = \log x + \log y + \log z.$$

Differentiating the above equation, we get

$$\frac{1}{V} dV = \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$$

or

$$\frac{dV}{V} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \quad \dots(13.6)$$

$$\text{But } \frac{dV}{V} = \frac{\text{Change of volume}}{\text{Original volume}} = \text{Volumetric strain}$$

$$\begin{aligned} \frac{dx}{x} &= \frac{\text{Change of dimension } x}{\text{Original dimension } x} \\ &= \text{Strain in the } x\text{-direction} = e_x \end{aligned}$$

$$\text{Similarly, } \frac{dy}{y} = \text{Strain in } y\text{-direction} = e_y$$

$$\text{and } \frac{dz}{z} = \text{Strain in } z\text{-direction} = e_z$$

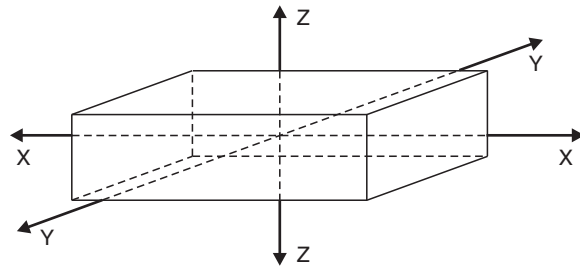


Fig. 13.3

Substituting these values in equation (13.6), we get

$$\frac{dV}{V} = e_x + e_y + e_z$$

Now, Let σ_x = Tensile stress in x - x direction,
 σ_y = Tensile stress in y - y direction, and
 σ_z = Tensile stress in z - z direction.
 E = Young's modulus
 μ = Poisson's ratio.

Now σ_x will produce a tensile strain equal to $\frac{\sigma_x}{E}$ in the direction of x , and a compressive strain equal to $\frac{\mu \times \sigma_x}{E}$ in the direction of y and z . Similarly, σ_y will produce a tensile strain equal to $\frac{\sigma_y}{E}$ in the direction of y and a compressive strain equal to $\frac{\mu \times \sigma_y}{E}$ in the direction of x and z . Similarly σ_z will produce a tensile strain equal to $\frac{\sigma_z}{E}$ in the direction of z and a compressive strain equal to $\frac{\mu \times \sigma_z}{E}$ in the direction of x and y . Hence σ_y and σ_z will produce compressive strains equal to $\frac{\mu \times \sigma_y}{E}$ and $\frac{\mu \times \sigma_z}{E}$ in the direction of x .

$\therefore$ Net tensile strain along x -direction is given by

$$e_x = \frac{\sigma_x}{E} - \frac{\mu \times \sigma_y}{E} - \frac{\mu \times \sigma_z}{E} = \frac{\sigma_x}{E} - \mu \left(\frac{\sigma_y + \sigma_z}{E} \right).$$

Similarly,
$$e_y = \frac{\sigma_y}{E} - \mu \left(\frac{\sigma_x + \sigma_z}{E} \right)$$

and
$$e_z = \frac{\sigma_z}{E} - \mu \left(\frac{\sigma_x + \sigma_y}{E} \right)$$

Adding all the strains, we get

$$\begin{aligned} e_x + e_y + e_z &= \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z) - \frac{2\mu}{E} (\sigma_x + \sigma_y + \sigma_z) \\ &= \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu). \end{aligned}$$

But
$$e_x + e_y + e_z = \text{Volumetric strain} = \frac{dV}{V}.$$

$$\therefore \frac{dV}{V} = \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu) \quad \dots(13.7)$$

Equation (13.7) gives the volumetric strain. In this equation the stresses σ_x , σ_y and σ_z are all tensile. If any of the stresses is compressive, it may be regarded as negative, and the above equation will hold good. If the value of $\frac{dV}{V}$ is positive, it represents increase in volume whereas the negative value of $\frac{dV}{V}$ represents a decrease in volume.

Problem 13.5. A metallic bar $300 \text{ mm} \times 100 \text{ mm} \times 40 \text{ mm}$ is subjected to a force of 5 kN (tensile), 6 kN (tensile) and 4 kN (tensile) along x , y and z directions respectively. Determine the change in the volume of the block. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio $= 0.25$.

Sol. Given :

Dimensions of bar $= 300 \text{ mm} \times 100 \text{ mm} \times 40 \text{ mm}$

$\therefore x = 300 \text{ mm}, y = 100 \text{ mm}$ and $z = 40 \text{ mm}$

$\therefore$ Volume, $V = x \times y \times z = 300 \times 100 \times 40$
 $= 1200000 \text{ mm}^3$

Load in the direction of $x = 5 \text{ kN} = 5000 \text{ N}$

Load in the direction of $y = 6 \text{ kN} = 6000 \text{ N}$

Load in the direction of $z = 4 \text{ kN} = 4000 \text{ N}$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$

Poisson's ratio, $\mu = 0.25$

$\therefore$ Stress in the x -direction,

$$\sigma_x = \frac{\text{Load in } x\text{-direction}}{y \times z}$$

$$= \frac{5000}{100 \times 40} = 1.25 \text{ N/mm}^2$$

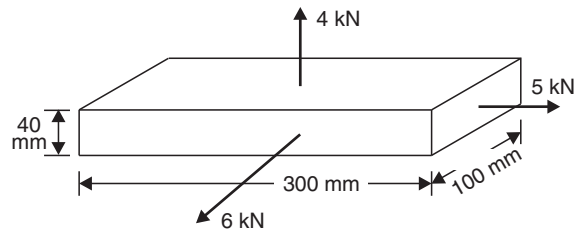


Fig. 13.4

Similarly the stress in y -direction is given by,

$$\sigma_y = \frac{\text{Load in } y\text{-direction}}{x \times z}$$

$$= \frac{6000}{300 \times 40} = 0.5 \text{ N/mm}^2$$

And stress in z -direction $= \frac{\text{Load in } z\text{-direction}}{x \times y}$

or $\sigma_z = \frac{4000}{300 \times 100}$
 $= 0.133 \text{ N/mm}^2$

Using equation (13.9), we get

$$\frac{dV}{V} = \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu)$$

$$= \frac{1}{2 \times 10^5} (1.25 + 0.5 + 0.113)(1 - 2 \times 0.25)$$

$$= \frac{1.883}{2 \times 10^5 \times 2}$$

$$\therefore dV = \frac{1.883}{4 \times 10^5} \times V$$

$$= \frac{1.883}{4 \times 10^5} \times 1200000$$

$$= 5.649 \text{ mm}^3. \text{ Ans.}$$

Problem 13.6. A metallic bar $250 \text{ mm} \times 100 \text{ mm} \times 50 \text{ mm}$ is loaded as shown in Fig. 13.5.

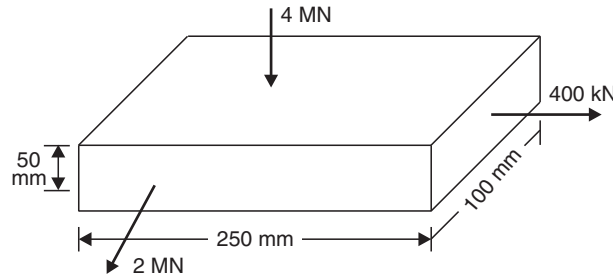


Fig. 13.5

Find the change in volume. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio $= 0.25$.

Sol. Given :

Length, $x = 250 \text{ mm}$, $y = 100 \text{ mm}$ and $z = 50 \text{ mm}$

$\therefore$ Volume, $V = xyz = 250 \times 100 \times 50 = 1250000 \text{ mm}^3$

Load in x -direction $= 400 \text{ kN} = 400000 \text{ N}$ (tensile)

Load in y -direction $= 2 \text{ MN} = 2 \times 10^6 \text{ N}$ (tensile)

Load in z -direction $= 4 \text{ MN} = 4 \times 10^6 \text{ N}$ (compressive)

Modulus of elasticity, $E = 2 \times 10^5 \text{ N/mm}^2$

Poisson's ratio, $\mu = 0.25$.

Now,

$\sigma_x =$ Stress in x -direction

$$= \frac{\text{Load in } x\text{-direction}}{\text{Area of cross-section}}$$

$$= \frac{400000}{y \times z} = \frac{400000}{100 \times 50} = 80 \text{ N/mm}^2 \text{ (tension).}$$

Similarly,

$$\sigma_y = \frac{\text{Load in } y\text{-direction}}{x \times z}$$

$$= \frac{2 \times 10^6}{250 \times 50} = 160 \text{ N/mm}^2$$

and

$$\sigma_z = \frac{4000000}{250 \times 100}$$

$$= 160 \text{ N/mm}^2 \text{ (compression).}$$

Using equation (13.7) and taking tensile stresses positive and compressive stresses negative, we get

$$\frac{dV}{V} = \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu)$$

or

$$\frac{dV}{V} = \frac{1}{2 \times 10^5} (80 + 160 - 160)(1 - 2 \times 0.25)$$

$$= \frac{80}{2 \times 10^5} \times 0.5 = 0.0002.$$

∴ Change in volume,

$$\begin{aligned} dV &= 0.0002 \times V \\ &= 0.0002 \times 1250000 \\ &= 250 \text{ mm}^3. \text{ Ans.} \end{aligned}$$

13.6. VOLUMETRIC STRAIN OF A CYLINDRICAL ROD

Consider a cylindrical rod which is subjected to an axial tensile load P .

Let d = Diameter of the rod

L = Length of the rod

Due to tensile load P , there will be an increase in the length of the rod, but the diameter of the rod will decrease as shown in Fig. 13.6.

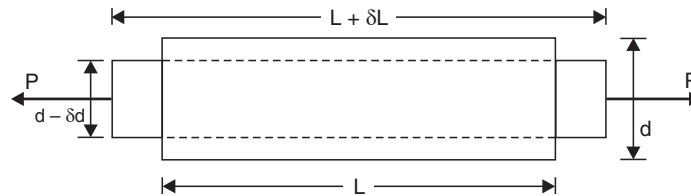


Fig. 13.6

∴ Final length $= L + \delta L$

∴ Final diameter $= d - \delta d$

Now original volume of the rod,

$$V = \frac{\pi}{4} d^2 \times L$$

$$\begin{aligned} \text{Final volume} &= \frac{\pi}{4} (d - \delta d)^2 (L + \delta L) \\ &= \frac{\pi}{4} (d^2 + \delta d^2 - 2d \times \delta d)(L + \delta L) \\ &= \frac{\pi}{4} (d^2 \times L + \delta d^2 \times L - 2d \times L \times \delta d + d^2 \times \delta L \\ &\quad + \delta d^2 \times \delta L - 2d \times \delta d \times \delta L) \\ &= \frac{\pi}{4} (d^2 \times L - 2d \times L \times \delta d + d^2 \times \delta L) \end{aligned}$$

Neglecting the products and higher powers of two small quantities.

∴ Change in volume, $\delta V = \text{Final volume} - \text{Original volume}$

$$\begin{aligned} &= \frac{\pi}{4} (d^2 \times L - 2d \times L \times \delta d + d^2 \times \delta L) - \frac{\pi}{4} d^2 \times L \\ &= \frac{\pi}{4} (d^2 \times \delta L - 2d \times L \times \delta d) \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Volumetric strain, } e_v &= \frac{\text{Change in volume}}{\text{Original volume}} = \frac{\delta V}{V} \\
 &= \frac{\frac{\pi}{4} (d^2 \times \delta L - 2d \times L \times \delta d)}{\frac{\pi}{4} d^2 \times L} = \frac{\delta L}{L} - \frac{2\delta d}{d} \quad \dots(13.8)
 \end{aligned}$$

where $\frac{\delta L}{L}$ is the strain of length and $\frac{\delta d}{d}$ is the strain of diameter.

$\therefore$ Volumetric strain = Strain in length – Twice the strain of diameter.

13.7. BULK MODULUS

When a body is subjected to the mutually perpendicular like and equal direct stresses, the ratio of direct stress to the corresponding volumetric strain is found to be constant for a given material when the deformation is within a certain limit. This ratio is known as bulk modulus and is usually denoted by K . Mathematically bulk modulus is given by

$$K = \frac{\text{Direct stress}}{\text{Volumetric strain}} = \frac{\sigma}{\left(\frac{dV}{V}\right)} \quad \dots(13.9)$$

13.8. EXPRESSION FOR YOUNG'S MODULUS IN TERMS OF BULK MODULUS

Fig. 13.7 shows a cube $A B C D E F G H$ which is subjected to three mutually perpendicular tensile stresses of equal intensity.

Let L = Length of cube

dL = Change in length of the cube

E = Young's modulus of the material of the cube

σ = Tensile stress acting on the faces

μ = Poisson's ratio.

Then volume of cube, $V = L^3$

Now let us consider the strain of one of the sides of the cube (say AB) under the action of the three mutually perpendicular stresses. This side will suffer the following three strains :

1. Strain of AB due to stresses on the faces $AEHD$ and $BFGC$. This strain is tensile and is equal to $\frac{\sigma}{E}$.

2. Strain of AB due to stresses on the faces $AEFB$ and $DHGC$. This is compressive lateral strain and is equal to $-\mu \frac{\sigma}{E}$.

3. Strain of AB due to stresses on the faces $ABCD$ and $EFGH$. This is also compressive lateral strain and is equal to $-\mu \frac{\sigma}{E}$.

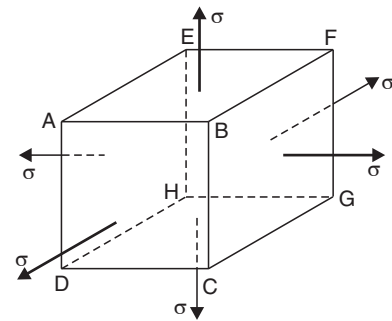


Fig. 13.7

Hence the total strain of AB is given by

$$\frac{dL}{L} = \frac{\sigma}{E} - \mu \times \frac{\sigma}{E} - \mu \times \frac{\sigma}{E} = \frac{\sigma}{E} (1 - 2\mu) \quad \dots(i)$$

Now original volume of cube, $V = L^3$...(ii)

If dL is the change in length, then dV is the change in volume.

Differentiating equation (ii), with respect to L ,

$$dV = 3L^2 \times dL \quad \dots(iii)$$

Dividing equation (iii) by equation (ii), we get

$$\frac{dV}{V} = \frac{3L^2 \times dL}{L^3} = \frac{3dL}{L}$$

Substituting the value of $\frac{dL}{L}$ from equation (i), in the above equation, we get

$$\frac{dV}{V} = \frac{3\sigma}{E} (1 - 2\mu)$$

From equation (13.9), bulk modulus is given by

$$\begin{aligned} K &= \frac{\sigma}{\left(\frac{dV}{V}\right)} = \frac{\sigma}{\frac{3\sigma}{E} (1 - 2\mu)} \quad \left[\because \frac{dV}{V} = \frac{3\sigma}{E} (1 - 2\mu) \right] \\ &= \frac{E}{3(1 - 2\mu)} \quad \dots(13.10) \end{aligned}$$

or

$$E = 3K (1 - 2\mu) \quad \dots(13.11)$$

From equation (13.11), the expression for Poisson's ratio (μ) is obtained as $\mu = \frac{3K - E}{6K}$.

Problem 13.7. For a material, Young's modulus is given as $1.2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio $\frac{1}{4}$. Calculate the Bulk modulus.

Sol. Given : Young's modulus, $E = 1.2 \times 10^5 \text{ N/mm}^2$

Poisson's ratio, $\mu = \frac{1}{4}$

Let K = Bulk modulus

Using equation (13.10),

$$\begin{aligned} K &= \frac{E}{3(1 - 2\mu)} = \frac{1.2 \times 10^5}{3\left(1 - \frac{2}{4}\right)} = \frac{1.2 \times 10^5}{3 \times \frac{1}{2}} \\ &= \frac{2 \times 1.2 \times 10^5}{3} = 0.8 \times 10^5 \text{ N/mm}^2. \quad \text{Ans.} \end{aligned}$$

Problem 13.8. A bar of 30 mm diameter is subjected to a pull of 60 kN. The measured extension on gauge length of 200 mm is 0.1 mm and change in diameter is 0.004 mm. Calculate :

- (i) Young's modulus
- (ii) Poisson's ratio and
- (iii) Bulk modulus.

Sol. Given : Dia. of bar, $d = 30$ mm

$$\therefore \text{Area of bar, } A = \frac{\pi}{4} (30)^2 = 225\pi \text{ mm}^2$$

$$\text{Pull, } P = 60 \text{ kN} = 60 \times 1000 \text{ N}$$

$$\text{Gauge length, } L = 200 \text{ mm}$$

$$\text{Extension, } \delta L = 0.1 \text{ mm}$$

$$\text{Change in dia., } \delta d = 0.004 \text{ mm}$$

(i) *Young's modulus (E)*

$$\text{Tensile stress, } \sigma = \frac{P}{A} = \frac{60000}{225\pi} = 84.87 \text{ N/mm}^2$$

$$\text{Longitudinal strain} = \frac{\delta L}{L} = \frac{0.1}{200} = 0.0005$$

$$\begin{aligned} \therefore \text{Young's modulus, } E &= \frac{\text{Tensile stress}}{\text{Longitudinal strain}} \\ &= \frac{84.87}{0.0005} = 16.975 \times 10^4 \text{ N/mm}^2 \\ &= 1.6975 \times 10^5 \text{ N/mm}^2. \quad \text{Ans.} \end{aligned}$$

(ii) *Poisson's ratio (μ)*

Poisson's ratio is given by equation (13.3) as

$$\text{Poisson's ratio } (\mu) = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= \frac{\left(\frac{\delta d}{d}\right)}{0.0005} \quad \left(\because \text{Lateral strain} = \frac{\delta d}{d}\right)$$

$$= \frac{\left(\frac{0.004}{30}\right)}{0.0005} = \frac{0.000133}{0.0005} = 0.266. \quad \text{Ans.}$$

(iii) *Bulk modulus (K)*

Using equation (13.10), we get

$$\begin{aligned} K &= \frac{E}{3(1-2\mu)} = \frac{1.6975 \times 10^5}{3(1-0.266 \times 2)} \\ &= 1.209 \times 10^5 \text{ N/mm}^2. \quad \text{Ans.} \end{aligned}$$

13.9. RELATIONSHIP BETWEEN MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

We know that when a square block $ABCD$ of unit thickness is subjected to a set of shear stresses of magnitude τ on the faces AB , CD and the faces AD and CB , then the diagonal strain due to shear stress τ is equal to $\frac{\tau}{E} (1 + \mu)$.

∴ Total tensile strain* along diagonal $BD = \frac{\tau}{E} (1 + \mu)$

Also we have total tensile strain* in diagonal BD

$$= \frac{1}{2} \text{ shear strain} = \frac{1}{2} \times \frac{\text{Shear stress}}{C} \left(\frac{\text{Shear stress}}{\text{Shear strain}} = \text{modulus of rigidity} = C \right)$$

$$= \frac{1}{2} \times \frac{\tau}{C} \quad (\because \text{Shear stress} = \tau)$$

∴ Equating the two tensile strain along diagonal BD , we get

$$\frac{\tau}{E} (1 + \mu) = \frac{1}{2} \times \frac{\tau}{C}$$

or $\frac{\tau}{E} (1 + \mu) = \frac{1}{2C}$ (Cancelling τ to both sides)

$$\therefore E = 2C (1 + \mu) \quad \dots(13.12)$$

$$\text{or} \quad C = \frac{E}{2(1 + \mu)} \quad \dots(13.13)$$

Problem 13.9. Determine the Poisson's ratio and bulk modulus of a material, for which Young's modulus is $1.2 \times 10^5 \text{ N/mm}^2$ and modulus of rigidity is $4.8 \times 10^4 \text{ N/mm}^2$.

Sol. Given :

Young's modulus, $E = 1.2 \times 10^5 \text{ N/mm}^2$

Modulus of rigidity, $C = 4.8 \times 10^4 \text{ N/mm}^2$

Let the Poisson's ratio $= \mu$

Using equation (13.12), we get

$$E = 2C (1 + \mu)$$

or $1.2 \times 10^5 = 2 \times 4.8 \times 10^4 (1 + \mu)$

$$\text{or} \quad (1 + \mu) = \frac{1.2 \times 10^5}{2 \times 4.8 \times 10^4} = 1.25 \quad \text{or} \quad \mu = 1.25 - 1.0 = \mathbf{0.25. \quad Ans.}$$

Bulk modulus is given by equation (13.10) as

$$K = \frac{E}{3(1 - 2\mu)} = \frac{1.2 \times 10^5}{3(1 - 0.25 \times 2)} \quad (\because \mu = 0.25)$$

$$= \mathbf{8 \times 10^4 \text{ N/mm}^2. \quad Ans.}$$

Problem 13.10. A bar of cross-section $8 \text{ mm} \times 8 \text{ mm}$ is subjected to an axial pull of 7000 N . The lateral dimension of the bar is found to be changed to $7.9985 \text{ mm} \times 7.9985 \text{ mm}$. If the modulus of rigidity of the material is $0.8 \times 10^5 \text{ N/mm}^2$, determine the Poisson's ratio and modulus of elasticity.

Sol. Given :

Area of section $= 8 \times 8 = 64 \text{ mm}^2$

Axial pull, $P = 7000 \text{ N}$

Lateral dimensions $= 7.9985 \text{ mm} \times 7.9985 \text{ mm}$

Volume of $C = 0.8 \times 10^5 \text{ N/mm}^2$

Let $\mu =$ Poisson's ratio and

$E =$ Modulus of elasticity.

*For derivation of total tensile strain along any diagonal due to shear stress, please refer to any standard book Strength of Material.

$$\begin{aligned}\text{Now lateral strain} &= \frac{\text{Change in lateral dimension}}{\text{Original lateral dimension}} \\ &= \frac{8 - 7.9985}{8} = \frac{0.0015}{8} = 0.0001875.\end{aligned}$$

To find the value of Poisson's ratio, we must know the value of longitudinal strain. But in this problem, the length of bar and the axial extension is not given. Hence longitudinal strain cannot be calculated. But axial stress can be calculated. Then longitudinal strain will be equal to axial stress divided by E .

$$\therefore \text{Axial stress, } \sigma = \frac{P}{\text{Area}} = \frac{7000}{64} = 109.375 \text{ N/mm}^2 \text{ and longitudinal strain} = \frac{\sigma}{E}$$

$$\text{But lateral strain} = \mu \times \text{longitudinal strain} = \mu \times \frac{\sigma}{E}$$

$$\text{or} \quad 0.0001875 = \frac{\mu \times 109.375}{E} \quad (\because \text{Lateral strain} = 0.0001875)$$

$$\therefore \quad \frac{E}{\mu} = \frac{109.375}{0.0001875} = 583333.33$$

$$\text{or} \quad E = 583333.33\mu \quad \dots(i)$$

Using equation (13.13), we get

$$C = \frac{E}{2(1 + \mu)} \quad \text{or} \quad E = 2C(1 + \mu)$$

$$= 2 \times 0.8 \times 10^5 (1 + \mu) \quad (\because C = 0.8 \times 10^5)$$

$$\text{or} \quad 583333.33\mu = 2 \times 0.8 \times 10^5 (1 + \mu) \quad (\because E = 583333.33\mu)$$

$$\text{or} \quad 1 + \mu = \frac{583333.33\mu}{2 \times 0.8 \times 10^5} = 3.6458\mu$$

$$\therefore \quad 1 = 3.6458\mu - \mu = 2.6458\mu$$

$$\therefore \text{Poisson's ratio} \quad \mu = \frac{1}{2.6458} = \mathbf{0.378. \quad \text{Ans.}}$$

Modulus of elasticity (E) is obtained by substituting the value of μ in equation (i).

$$\therefore \quad E = 583333.33\mu$$

$$\therefore \quad E = \frac{583333.33}{2.6458} = \mathbf{2.2047 \times 10^5 \text{ N/mm}^2. \quad \text{Ans.}}$$

Problem 13.11. Calculate the modulus of rigidity and bulk modulus of a cylindrical bar of diameter 30 mm and of length 1.5 m if the longitudinal strain in a bar during a tensile stress is four times the lateral strain. Find the change in volume, when the bar is subjected to a hydrostatic pressure of 100 N/mm². Take $E = 1 \times 10^5 \text{ N/mm}^2$.

Sol. Given :

$$\text{Dia. of bar,} \quad d = 30 \text{ mm}$$

$$\text{Length of bar,} \quad L = 1.5 \text{ m} = 1.5 \times 1000 = 1500 \text{ mm}$$

$$\begin{aligned}\therefore \text{Volume of bar,} \quad V &= \frac{\pi}{4} d^2 \times L = \frac{\pi}{4} \times 30 \times 1500 \\ &= 1060287.52 \text{ mm}^3\end{aligned}$$

$$\text{Longitudinal strain} = 4 \times \text{Lateral strain}$$

$$\text{Hydrostatic pressure, } p = 100 \text{ N/mm}^2$$

$$\therefore \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{1}{4} = 0.25$$

or Poisson's ratio, $\mu = 0.25$

Let $C = \text{Modulus of rigidity}$
 $K = \text{Bulk modulus}$
 $E = \text{Young's modulus} = 1 \times 10^5 \text{ N/mm}^2$

Using equation (13.12), we get

$$E = 2C(1 + \mu)$$

or $1 \times 10^5 = 2C(1 + 0.25)$

$$\therefore C = \frac{1 \times 10^5}{2 \times 1.25}$$

$$= 4 \times 10^4 \text{ N/mm}^2. \text{ Ans.}$$

For bulk modulus, using equation (13.11), we get

$$E = 3K(1 - 2\mu)$$

or $1 \times 10^5 = 3K(1 - 2 \times 0.25)$ ($\because \mu = 0.25$)

$$\therefore K = \frac{1 \times 10^5}{3 \times 0.5}$$

$$= 0.667 \times 10^5 \text{ N/mm}^2. \text{ Ans.}$$

Now using equation (13.9), we get

$$K = \frac{p}{\text{Volumetric strain}} = \frac{p}{\left(\frac{dV}{V}\right)}$$

where $p = 100 \text{ N/mm}^2$

$$\therefore 0.667 \times 10^5 = \frac{100}{\left(\frac{dV}{V}\right)}$$

or $\frac{dV}{V} = \frac{100}{0.667 \times 10^5} = 1.5 \times 10^{-3}$

$$\therefore dV = V \times 1.5 \times 10^{-3}$$

$$= 1060287.52 \times 1.5 \times 10^{-3}$$

$$= 1590.43 \text{ mm}^3. \text{ Ans.}$$

HIGHLIGHTS

1. Poisson's ratio is the ratio of lateral strain to longitudinal strain. It is generally denoted by μ .
2. The tensile longitudinal stress produces compressive lateral strains.
3. If a load acts in the direction of length of a rectangular bar, then longitudinal strain = $\frac{\delta l}{l}$ and
 Lateral strain = $\frac{\delta b}{b}$ or $\frac{\delta d}{d}$
 where δl = Change in length, δb = Change in width,
 δd = Change in depth.

4. The ratio of change in volume to original volume is known as volumetric strain.
5. Volumetric strain (e_v) for a rectangular bar subjected to an axial load P , is given by

$$e_v = \frac{\delta l}{l} (1 - 2\mu).$$

6. Volumetric strain for a rectangular bar subjected to three mutually perpendicular stresses is given by,

$$e_v = \frac{1}{E} (\sigma_x + \sigma_y + \sigma_z)(1 - 2\mu)$$

where σ_x , σ_y and σ_z are stresses in x , y and z direction respectively.

7. Principle of complementary shear stresses states that a set of shear stresses across a plane is always accompanied by a set of balancing shear stresses (*i.e.*, of the same intensity) across the plane and normal to it.
8. Volumetric strain of a cylindrical rod, subjected to an axial tensile load is given by,

$$\begin{aligned} e_v &= \text{Longitudinal strain} - 2 \times \text{strain of diameter} \\ &= \frac{\delta l}{l} - 2 \frac{\delta d}{d}. \end{aligned}$$

9. Bulk modulus K is given by,

$$K = \frac{\sigma}{\left(\frac{\delta V}{V}\right)}.$$

10. The relation between Young's modulus and bulk modulus is given by,

$$E = 3K(1 - 2\mu).$$

11. The relation between modulus of elasticity and modulus of rigidity is given by

$$E = 2C(1 + \mu) \quad \text{or} \quad C = \frac{E}{2(1 + \mu)}.$$

EXERCISE 13

(A) Theoretical Questions

1. Define and explain the terms : Longitudinal strain, lateral strain and Poisson's ratio.
2. Prove that the volumetric strain of a cylindrical rod which is subjected to an axial tensile load is equal to strain in the length minus twice the strain of diameter.
3. What is a bulk modulus ? Derive an expression for Young's modulus in terms of bulk modulus and Poisson's ratio.
4. Define volumetric strain. Prove that the volumetric strain for a rectangular bar subjected to an axial load P in the direction of its length is given by

$$e_v = \frac{\delta l}{l} (1 - 2\mu)$$

where μ = Poisson's ratio and $\frac{\delta l}{l}$ = Longitudinal strain.

5. (a) Derive an expression for volumetric strain for a rectangular bar which is subjected to three mutually perpendicular tensile stresses.
(b) A test element is subjected to three mutually perpendicular unequal stresses. Find the change in volume of the element, if the algebraic sum of these stresses is equal to zero.
6. Derive an expression between modulus of elasticity and modulus of rigidity.

(B) Numerical Problems

1. Determine the changes in length, breadth and thickness of a steel bar which is 5 m long, 40 mm wide and 30 mm thick and is subjected to an axial pull of 35 kN in the direction of its length. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio = 0.32.
[Ans. 0.0729 cm, 0.000186 cm, 0.000139 cm]
2. For the above problem, determine the volumetric strain and the final volume of the given steel bar.
[Ans. 0.0000525, 6000317 mm³]
3. Determine the value of Young's modulus and Poisson's ratio of a metallic bar of length 25 cm, breadth 3 cm and depth 2 cm when the bar is subjected to an axial compressive load of 240 kN. The decrease in length is given as 0.05 cm and increase in breadth is 0.002.
[Ans. $2 \times 10^5 \text{ N/mm}^2$ and 0.33]
4. A steel bar 320 mm long, 40 mm wide and 30 mm thick is subjected to a pull of 250 kN in the direction of its length. Determine the change in volume. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $m = 4$.
[Ans. 200 mm³]
5. A metallic bar 250 mm × 80 mm × 30 mm is subjected to a force of 20 kN (tensile), 30 kN (tensile) and 15 kN (tensile) along x , y and z directions respectively. Determine the change in the volume of the block. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio = 0.25.
[Ans. 19.62 mm³]
6. A metallic bar 300 mm × 120 mm × 50 mm is loaded as shown in Fig. 13.8. Find the change in volume. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio = 0.30.

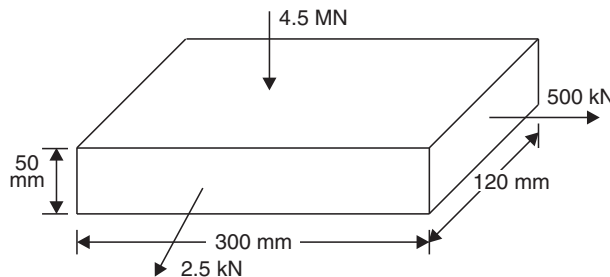


Fig. 13.8

Also find the change that should be made in 4.5 MN load, in order that there should be no change in the volume of the bar.
[Ans. 450 mm², 4.5 MN]

7. A steel rod 4 m long and 20 mm diameter is subjected to an axial tensile load of 40 kN. Determine the change in length, diameter and volume of the rod. Take $E = 2 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio = 0.25.
[Ans. 2.5464, 0.05092, 5598 mm³]
8. For a material, Young's modulus is given as $1.4 \times 10^5 \text{ N/mm}^2$ and Poisson's ratio 0.28. Calculate the bulk modulus.
[Ans. $1.06 \times 10^5 \text{ N/mm}^2$]
9. A bar of 20 mm diameter subjected to a pull of 50 kN. The measured extension on gauge length of 250 mm is 0.12 mm and change in diameter is 0.00375 mm. Calculate :
(i) Young's modulus (ii) Poisson's ratio and (iii) Bulk modulus.
[Ans. (i) $1.989 \times 10^5 \text{ N/mm}^2$, (ii) 0.234, (iii) $1.2465 \times 10^5 \text{ N/mm}^2$]
10. Determine the Poisson's ratio and bulk modulus of a material, for which Young's modulus is $1.2 \times 10^5 \text{ N/mm}^2$ and modulus of rigidity is $4.5 \times 10^4 \text{ N/mm}^2$.
[Ans. 0.33, $1.2 \times 10^5 \text{ N/mm}^2$]

11. A bar of cross-section $10 \text{ mm} \times 10 \text{ mm}$ is subjected to an axial pull of 8000 N . The lateral dimension of the bar is found to be changed to $9.9985 \text{ mm} \times 9.9985 \text{ mm}$. If the modulus of rigidity of the material is $0.8 \times 10^5 \text{ N/mm}^2$, determine the Poisson's ratio and modulus of elasticity.
[Ans. 0.45 , $2.4 \times 10^5 \text{ N/mm}^2$]
12. Calculate the modulus of rigidity and bulk modulus of a cylindrical bar of diameter of 25 mm and of length 1.6 m , if the longitudinal strain in a bar during a tensile test is four times the lateral strain. Find the change in volume, when the bar is subjected to a hydrostatic pressure of 100 N/mm^2 . Take $E = 1 \times 10^5 \text{ N/mm}^2$.
[Ans. $4 \times 10^4 \text{ N/mm}^2$, $0.667 \times 10^5 \text{ N/mm}^2$, 1178 mm^3]
13. A bar 30 mm in diameter was subjected to tensile load of 54 kN and the measured extension on 300 mm gauge length was 0.112 mm and change in diameter was 0.00366 mm . Calculate Poisson's ratio and values of three moduli.
[Ans. $\mu = 0.326$, $E = 204.6 \text{ kN/mm}^2$, $C = 77.2 \text{ kN/mm}^2$, $K = 196 \text{ kN/mm}^2$]
14. Derive the relation between E and C . Using the derived relationship, estimate the Young's modulus (E) when the modulus of rigidity (C) is $0.80 \times 10^5 \text{ N/mm}^2$ and the Poisson's ratio is 0.3 .
[Hint. $E = 2C (1 + \mu) = 2 \times 0.80 \times 10^5 (1 + 0.3) = 2.08 \times 10^5 \text{ N/mm}^2$.]

Strain Energy and Impact Loading

14.1. INTRODUCTION

Whenever a body is strained, the energy is absorbed in the body. The energy, which is absorbed in the body due to straining effect is known as *strain energy*. The straining effect may be due to gradually applied load or suddenly applied load or load with impact. Hence the strain energy will be stored in the body when the load is applied gradually or suddenly or with an impact. The strain energy stored in the body is equal to the work done by the applied load in stretching the body.

14.2. SOME DEFINITIONS

Before deriving the expressions for the strain energy stored in a body due to gradually applied load or suddenly applied load or load with an impact, the following terms will be defined :

1. Resilience
2. Proof resilience, and
3. Modulus of resilience.

14.2.1. Resilience. The total strain energy stored in a body is commonly known as resilience. Whenever the straining force is removed from the strained body, the body is capable of doing work. Hence the resilience is also defined as the capacity of a strained body for doing work on the removal of the straining force.

14.2.2. Proof Resilience. The maximum strain energy, stored in a body, is known as proof resilience. The strain energy stored in the body will be maximum when the body is stressed upto elastic limit. Hence the proof resilience is the quantity of strain energy stored in a body when strained up to elastic limit.

14.2.3. Modulus of Resilience. It is defined as the proof resilience of a material per unit volume. It is an important property of a material. Mathematically,

$$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Volume of the body}}.$$

14.3. EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED GRADUALLY

In Art. 14.1, we have mentioned that the strain energy stored in a body is equal to the work done by the applied load in stretching the body.

Fig. 14.1 shows load extension diagram of a body under tensile test upto elastic limit. The tensile load P increases gradually from zero to the value of P and the extension of the body increases from zero to the value of x .

The load P performs work in stretching the body. This work will be stored in the body as *strain energy* which is recoverable after the load P is removed.

Let P = Gradually applied load,
 x = Extension of the body,
 A = Cross-sectional area,
 L = Length of the body,
 V = Volume of the body,
 E = Young's modulus,
 U = Strain energy stored in the body, and
 σ = Stress induced in the body.

Now work done by the load = Area of load extension curve (Shaded area in Fig. 14.1)

= Area of triangle ONM

$$= \frac{1}{2} \times P \times x. \quad \dots(i)$$

But load, P = Stress $\times$ Area = $\sigma \times A$

and extension, x = Strain $\times$ Length $\left(\because \text{Strain} = \frac{\text{Extension}}{\text{Length}} \therefore \text{Extension} = \text{Strain} \times L \right)$

$$= \frac{\text{Stress}}{E} \times L \quad \left(\because \text{Strain} = \frac{\text{Stress}}{E} \right)$$

$$= \frac{\sigma}{E} \times L. \quad \dots(14.1)$$

Substituting the values of P and x in equation (i), we get

$$\begin{aligned} \text{Work done by the load} &= \frac{1}{2} \times (\sigma \times A) \times \left(\frac{\sigma}{E} \times L \right) = \frac{1}{2} \frac{\sigma^2}{E} \times A \times L \\ &= \frac{\sigma^2}{2E} \times V \quad (\because \text{Volume } V = A \times L) \end{aligned}$$

But the work done by the load in stretching the body is equal to the strain energy stored in the body.

$\therefore$ Energy stored in the body,

$$U = \frac{\sigma^2}{2E} \times V. \quad \dots(14.2)$$

Proof resilience. The maximum energy stored in the body without permanent deformation (*i.e.*, upto elastic limit) is known as proof resilience. Hence if in equation (14.2), the stress σ is taken at the elastic limit, we will get proof resilience.

$$\therefore \text{Proof resilience} = \frac{\sigma^{*2}}{2E} \times \text{Volume} \quad \dots(14.3)$$

where σ^* = Stress at the elastic limit.

Modulus of resilience = Strain energy per unit volume

$$= \frac{\text{Total strain energy}}{\text{Volume}} = \frac{\frac{\sigma^{*2}}{2E} \times V}{V} = \frac{\sigma^{*2}}{2E} \quad \dots(14.4)$$

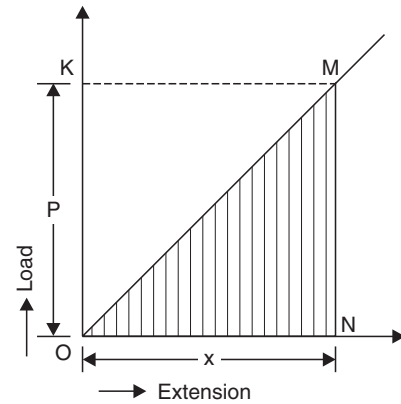


Fig. 14.1

14.4. EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED SUDDENLY

When the load is applied suddenly to a body, the load is constant throughout the process of the deformation of the body.

Consider a bar subjected to a sudden load.

Let P = Load applied suddenly,
 L = Length of the bar,
 A = Area of the cross-section,
 V = Volume of the bar = $A \times L$,
 E = Young's modulus,
 x = Extension of the bar,
 σ = Stress induced by the suddenly applied load, and
 U = Strain energy stored.

As the load is applied suddenly, the load P is constant when the extension of the bar takes place.

$\therefore$ Work done by the load = Load $\times$ Extension = $P \times x$.

The strain energy stored (*i.e.*, energy stored upto elastic limit) in a body is given by equation (14.2) as

$$\begin{aligned} U &= \frac{\sigma^2}{2E} \times \text{Volume of the body} \\ &= \frac{\sigma^2}{2E} \times A \times L. \end{aligned} \quad (\because \text{Volume} = A \times L)$$

Equating the strain energy stored in the body to the work done, we get

$$\frac{\sigma^2}{2E} \times A \times L = P \times x = P \times \frac{\sigma}{E} \times L. \quad \left[\because \text{From equation (14.1), } x = \frac{\sigma}{E} \times L \right]$$

Cancelling $\frac{\sigma \times L}{E}$ to both sides, we get

$$\frac{\sigma \times A}{2} = P \quad \text{or} \quad \sigma = 2 \times \frac{P}{A}. \quad \dots(14.5)$$

From the above equation it is clear that the maximum stress induced due to suddenly applied load is twice the stress induced when the same load is applied gradually.

After obtaining the value of stress (σ), the values of extension (x) and the strain energy stored in the body may be calculated easily.

Problem 14.1. A tensile load of 60 kN is gradually applied to a circular bar of 4 cm diameter and 5 m long. If the value of $E = 2.0 \times 10^5 \text{ N/mm}^2$, determine :

- (i) stretch in the rod,
- (ii) stress in the rod,
- (iii) strain energy absorbed by the rod.

Sol. Given :

Gradually applied load,

$$P = 60 \text{ kN} = 60 \times 1000 \text{ N}$$

$$\text{Dia. of rod, } d = 4 \text{ cm} = 40 \text{ mm}$$

$$\therefore \text{ Area, } A = \frac{\pi}{4} \times 40^2 = 400 \pi \text{ mm}^2$$

Length of rod, $L = 5 \text{ m} = 500 \text{ cm} = 5000 \text{ mm}$
 $\therefore$ Volume of rod, $V = A \times L = 400 \pi \times 5000 = 2 \times 10^6 \pi \text{ mm}^3$
 Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$.
 Let $x =$ stretch or extension in the rod,
 $\sigma =$ stress in the rod, and
 $U =$ strain energy absorbed by the rod.

Now stress, $\sigma = \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{60000}{400\pi} = 47.746 \text{ N/mm}^2$. **Ans.**

The stretch or extension is given by equation (14.1),

$$x = \frac{\sigma}{E} \times L = \frac{47.746}{2 \times 10^5} \times 5000 = 1.19 \text{ mm. Ans.}$$

The strain energy absorbed by the rod is given by equation (14.2),

$$U = \frac{\sigma^2}{2E} \times V = \frac{47.746^2}{2 \times 2 \times 10^5} \times 2 \times 10^6 \pi = 35810 \text{ Nmm} = 35.81 \text{ Nm. Ans.}$$

Problem 14.2. If in problem 14.1, the tensile load of 60 kN is applied suddenly determine :

- (i) maximum instantaneous stress induced,
- (ii) instantaneous elongation in the rod, and
- (iii) strain energy absorbed in the rod.

Sol. Given :

The data given in problem 14.1 is $d = 40 \text{ mm}$, $\text{Area} = 400 \pi \text{ mm}^2$, $L = 5000 \text{ mm}$, $\text{Volume} = 2 \times 10^6 \pi \text{ mm}^3$, $E = 2 \times 10^5 \text{ N/mm}^2$ and suddenly applied load, $P = 60000 \text{ N}$.

- (i) Maximum instantaneous stress induced

Using equation (14.5),

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{60000}{400\pi} = 95.493 \text{ N/mm}^2. \text{ Ans.}$$

- (ii) Instantaneous elongation in the rod

Let $x =$ Instantaneous elongation

$$\begin{aligned} \text{Then } x &= \frac{\sigma}{E} \times L = \frac{95.493}{2 \times 10^5} \times 5000 && [\text{see equation (14.1)}] \\ &= 2.38 \text{ mm. Ans.} \end{aligned}$$

- (iii) Strain energy is given by,

$$\begin{aligned} U &= \frac{\sigma^2}{2E} \times V = \frac{95.493^2}{2 \times 2 \times 10^5} \times 2 \times 10^6 \pi = 143238 \text{ Nmm} \\ &= 143.238 \text{ Nm. Ans.} \end{aligned}$$

Problem 14.3. Calculate instantaneous stress produced in a bar 10 cm^2 in area and 3 m long by the sudden application of a tensile load of unknown magnitude, if the extension of the bar due to suddenly applied load is 1.5 mm . Also determine the suddenly applied load. Take $E = 2 \times 10^5 \text{ N/mm}^2$.

Sol. Given :

Area of bar, $A = 10 \text{ cm}^2 = 1000 \text{ mm}^2$
 Length of bar, $L = 3 \text{ m} = 3000 \text{ mm}$

Extension due to suddenly applied load,

$$x = 1.5 \text{ mm}$$

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$.

Let σ = Instantaneous stress due to sudden load, and
 P = Suddenly applied load.

The extension x is given by equation (14.1),

$$x = \frac{\sigma}{E} \times L \quad \text{or} \quad 1.5 = \frac{\sigma}{2 \times 10^5} \times 3000$$

$$\therefore \sigma = \frac{1.5 \times 2 \times 10^5}{3000} = 100 \text{ N/mm}^2. \quad \text{Ans.}$$

Suddenly applied load

The instantaneous stress produced by a sudden load is given by equation (14.5) as

$$\sigma = 2 \times \frac{P}{A} \quad \text{or} \quad 100 = 2 \times \frac{P}{1000}$$

$$\therefore P = \frac{1000 \times 100}{2} = 50000 \text{ N} = 50 \text{ kN}. \quad \text{Ans.}$$

Problem 14.4. A steel rod is 2 m long and 50 mm in diameter. An axial pull of 100 kN is suddenly applied to the rod. Calculate the instantaneous stress induced and also the instantaneous elongation produced in the rod. Take $E = 200 \text{ GN/m}^2$.

Sol. Given :

Length, $L = 2 \text{ m} = 2 \times 1000 = 2000 \text{ mm}$

Diameter, $d = 50 \text{ mm}$

$$\therefore \text{Area, } A = \frac{\pi}{4} \times 50^2 = 625 \pi \text{ mm}^2$$

Suddenly applied load,

$$P = 100 \text{ kN} = 100 \times 1000 \text{ N}$$

$$\begin{aligned} \text{Value of } E &= 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2 \quad (\because G = \text{Giga} = 10^9) \\ &= \frac{200 \times 10^9}{10^6} \text{ N/mm}^2 \quad (\because 1 \text{ m} = 1000 \text{ mm} \therefore \text{m}^2 = 10^6 \text{ mm}^2) \\ &= 200 \times 10^3 \text{ N/mm}^2 \end{aligned}$$

Using equation (14.5) for suddenly applied load,

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{100 \times 1000}{625 \pi} \text{ N/mm}^2 = 101.86 \text{ N/mm}^2. \quad \text{Ans.}$$

Let dL = Elongation

$$\text{Then } dL = \frac{P}{E} \times L = \frac{101.86}{200 \times 10^3} \times 2000 = 1.0186 \text{ mm}. \quad \text{Ans.}$$

14.5. EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED WITH IMPACT

The load dropped from a certain height before the load commences to stretch the bar is a case of a load applied with impact. Consider a vertical rod fixed at the upper end and having a collar at the lower end as shown in Fig. 14.2. Let the load be dropped from a height on the collar. Due to this impact load, there will be some extension in the rod.

Let P = Load dropped (*i.e.*, load applied with impact)

L = Length of the rod,

A = Cross-sectional area of the rod,

V = Volume of rod $= A \times L$,

h = Height through which load is dropped,

δL = Extension of the rod due to load P ,

E = Modulus of elasticity of the material of rod,

σ = Stress induced in the rod due to impact load.

The strain in the bar is given by,

$$\text{Strain} = \frac{\text{Stress}}{E}$$

i.e.,

$$\frac{\delta L}{L} = \frac{\sigma}{E}$$

$$\therefore \delta L = \frac{\sigma}{E} \times L \quad \dots(14.6)$$

$$\begin{aligned} \text{Work done by the load} &= \text{Load} \times \text{Distance moved} \\ &= P(h + \delta L) \quad \dots(i) \end{aligned}$$

The strain energy stored by the rod,

$$U = \frac{\sigma^2}{2E} \times V = \frac{\sigma^2}{2E} \times AL$$

$\dots(ii)$

Equating the work done by the load to the strain energy stored, we get

$$P(h + \delta L) = \frac{\sigma^2}{2E} \cdot AL \quad \dots(14.7)$$

Important Conclusions

(i) If δL is very small in comparison with h .

The work done by load $= P \cdot h$

Equating the work done by the load to the strain energy stored in the rod, we get

$$P \cdot h = \frac{\sigma^2}{2E} \cdot AL$$

$$\therefore \sigma^2 = \frac{2E \cdot P \cdot h}{A \cdot L} \quad \text{and} \quad \sigma = \sqrt{\frac{2EP h}{A \cdot L}} \quad \dots(14.8)$$

(ii) In equation (14.7), if $h = 0$, this becomes the case of suddenly applied load.

$$\therefore \sigma = \frac{2P}{A}$$

Once the stress σ is known, the corresponding instantaneous extension (δL) and the strain energy (U) can be obtained.

Problem 14.5. The maximum instantaneous extension, produced by an unknown falling weight through a height of 4 cm in a vertical bar of length 3 m and of cross-sectional area 5 cm², is 2.1 mm.

Determine :

- (i) the instantaneous stress induced in the vertical bar, and
- (ii) the value of unknown weight. Take $E = 2 \times 10^5 \text{ N/mm}^2$.

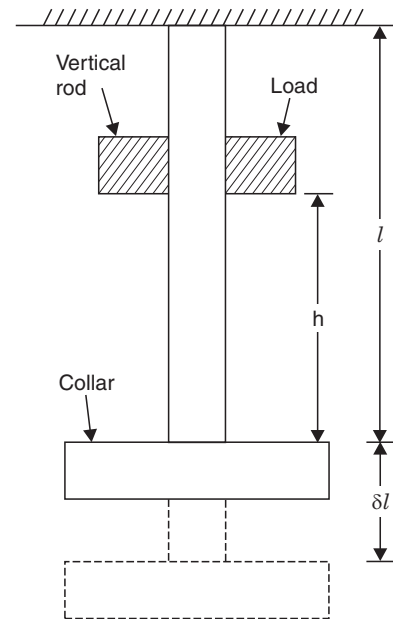


Fig. 14.2

Sol. Given :

Instantaneous extension, $\delta L = 2.1 \text{ mm}$

Length of bar, $L = 3 \text{ m} = 3000 \text{ mm}$

Area of bar, $A = 5 \text{ cm}^2 = 500 \text{ mm}^2$

$\therefore$ Volume of bar, $V = 500 \times 3000 = 1500000 \text{ mm}^3$

Height through which weight falls, $h = 4 \text{ cm} = 40 \text{ mm}$

Modulus of elasticity, $E = 2 \times 10^5 \text{ N/mm}^2$

Let $\sigma =$ Instantaneous stress produced, and
 $P =$ Unknown weight.

We know $E = \frac{\text{Stress}}{\text{Strain}}$ or $\text{Stress} = E \times \text{Strain}$

$$\begin{aligned} \therefore \text{Instantaneous stress} &= E \times \text{Instantaneous strain} = E \times \frac{\delta L}{L} \\ &= 2 \times 10^5 \times \frac{2.1}{3000} \text{ N/mm}^2 = \mathbf{140 \text{ N/mm}^2}. \quad \text{Ans.} \end{aligned}$$

Equating the work done by the falling weight to the strain energy stored, we get

$$P(h + \delta L) = \frac{\sigma^2}{2E} \times V$$

$$\text{or} \quad P(40 + 2.1) = \frac{140^2}{2 \times 2 \times 10^5} \times 1500000 = 73500$$

$$\text{or} \quad P = \frac{73500}{42.1} = \mathbf{1745.8 \text{ N}}. \quad \text{Ans.}$$

Problem 14.6. A crane-chain whose sectional area is 6.25 cm^2 carries a load of 10 kN . As it is being lowered at a uniform rate of 40 m per minute, the chain gets jammed suddenly, at which time the length of the chain unwound is 10 m . Estimate the stress induced in the chain due to the sudden stoppage. Neglect the weight of the chain. Take $E = 2.1 \times 10^5 \text{ N/mm}^2$.

(AMIE, Summer 1989)

Sol. Given :

Area, $A = 6.25 \text{ cm}^2 = 625 \text{ mm}^2$

Load, $W = 10 \text{ kN} = 10,000 \text{ N}$

Velocity, $V = 40 \text{ m/min} = \frac{40}{60} \text{ m/s} = \frac{2}{3} \text{ m/s}$

Length of chain unwound = $10 \text{ m} = 10 \times 1000 \text{ mm}$

$\therefore L = 10,000 \text{ mm}$

Value of $E = 2.1 \times 10^5 \text{ N/mm}^2$

Let $\sigma =$ Stress induced in the chain due sudden stoppage.

$$\begin{aligned} \text{K.E. of the crane} &= \frac{1}{2} m V^2 = \frac{1}{2} \left(\frac{W}{g} \right) \times V^2 \\ &= \frac{1}{2} \left(\frac{10000}{9.81} \right) \times \left(\frac{2}{3} \right)^2 \text{ N m} = 226.5 \text{ N m} \\ &= 226.5 \times 1000 \text{ N mm} = 226500 \text{ N mm} \end{aligned}$$

...(i)

When the chain gets jammed suddenly, the whole of the K.E. of the crane is absorbed in the chain. But the energy stored or absorbed in the chain

$$\begin{aligned}
 &= \frac{\sigma^2}{2E} \times A \times L \\
 &= \frac{\sigma^2}{2 \times 2.1 \times 10^5} \times 625 \times 10,000 \text{ Nmm} \quad \dots(ii)
 \end{aligned}$$

Now K.E. of crane = Energy stored in the chain

$$\text{or} \quad 226500 = \frac{\sigma^2}{2 \times 2.1 \times 10^5} \times 625 \times 10,000$$

$$\therefore \quad \sigma^2 = \frac{226500 \times 2 \times 2.1 \times 10^5}{625 \times 10,000}$$

$$\begin{aligned}
 \therefore \quad \sigma &= \sqrt{\frac{226500 \times 2 \times 2.1 \times 10^5}{625 \times 10,000}} \\
 &= 123.37 \text{ N/mm}^2. \quad \text{Ans.}
 \end{aligned}$$

Problem 14.7. A cage weighing 60 kN is attached to the end of a steel wire rope. It is lowered down a mine shaft with a constant velocity of 1 m/s. What is the maximum stress produced in the rope when its supporting drum is suddenly jammed? The free length of the rope at the moment of jamming is 15 m, its net cross-sectional area is 25 cm² and $E = 2 \times 10^5 \text{ N/mm}^2$. The self-weight of the wire rope may be neglected. (AMIE, Winter 1990)

Sol. Given :

Weight, $W = 60 \text{ kN} = 60,000 \text{ N}$

Velocity, $V = 1 \text{ m/s}$

Free length, $L = 15 \text{ m} = 15,000 \text{ mm}$

Area, $A = 25 \text{ cm}^2 = 25 \times 100 \text{ mm}^2$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$

$$\begin{aligned}
 \text{K.E. of the cage} &= \frac{1}{2} m V^2 = \frac{1}{2} \left(\frac{W}{g} \right) \times V^2 \\
 &= \frac{1}{2} \times \left(\frac{60,000}{9.81} \right) \times 1^2 \text{ Nm} = \frac{30000}{9.81} \text{ Nm} \\
 &= \frac{30000 \times 1000}{9.81} \text{ Nmm} \quad \dots(i)
 \end{aligned}$$

This energy is to be absorbed (or stored) by the rope.

Let σ = Maximum stress produced in the rope when its supporting drum is suddenly jammed.

But the maximum energy stored

$$= \frac{\sigma^2}{2E} \times A \times L = \frac{\sigma^2}{2 \times 2 \times 10^5} \times 2500 \times 15000 \text{ Nmm} \quad \dots(ii)$$

But K.E. of the cage = Energy stored in the rope

$$\frac{30000 \times 1000}{9.81} = \frac{\sigma^2}{2 \times 2 \times 10^5} \times 2500 \times 15000$$

$$\text{or } \sigma^2 = \frac{30000 \times 1000 \times 2 \times 2 \times 10^5}{9.81 \times 2500 \times 15000}$$

$$\text{or } \sigma = \sqrt{\frac{30000 \times 1000 \times 2 \times 2 \times 10^5}{9.81 \times 2500 \times 15000}} = 180.61 \text{ N/mm}^2. \quad \text{Ans.}$$

HIGHLIGHTS

1. The energy stored in a body due to straining effect is known as strain energy.
2. Resilience is the total strain energy stored in a body. Resilience is also defined as the capacity of a strained body for doing work on the removal of the straining force.
3. The maximum strain energy stored in a body is known as proof resilience. The proof resilience is given by,

$$\text{Proof resilience} = \frac{\sigma^2}{2E} \times \text{Volume}$$

where σ = Stress at the elastic limit.

4. The proof resilience of a body per unit volume is known as modulus of resilience.
5. The maximum stress induced in a body is given by

$$\sigma = \frac{P}{A} \quad \text{..... if the load } P \text{ is applied gradually}$$

$$= 2 \frac{P}{A} \quad \text{..... if the load } P \text{ is applied suddenly}$$

where A = Cross-sectional area of the body,

h = Height through which load falls,

E = Modulus of rigidity,

L = Length of the body.

6. The maximum stress induced in a body due to suddenly applied load is twice the stress induced when the same load is applied gradually.

EXERCISE 14

(A) Theoretical Questions

1. Define the following terms :
 (i) Resilience
 (ii) Strain energy
 (iii) Impact loading, and
 (iv) Spring.
2. Define resilience, proof resilience and modulus of resilience.
3. Explain the terms : Gradually applied load, suddenly applied load, and load applied with an impact.
4. If the extension produced in a rod due to impact load is very small in comparison with the height through which the load falls, prove that stress induced in the body will be given by

$$\sigma = \sqrt{\frac{2EP_h}{A \cdot L}}$$

5. Explain the following terms : (i) Proof stress, (ii) Proof resilience, and (iii) Modulus of resilience.

(Bhavnagar University, Feb. 1992)

(B) Numerical Problems

1. A tensile load of 50 kN is gradually applied to a circular bar of 5 cm diameter and 4 m long. If the value of $E = 2.0 \times 10^5 \text{ N/mm}^2$, determine : (i) stretch in the rod, (ii) stress in the rod, and (iii) strain energy absorbed by the rod. [Ans. (i) 0.0509 cm, (ii) 25.465 N/mm², (iii) 12.73 Nm]
2. If in question 1, the tensile load of 50 kN is applied suddenly, determine : (i) maximum instantaneous stress induced, (ii) instantaneous elongation in the rod, and (iii) strain energy absorbed in the rod. [Ans. (i) 50.93 N/mm², (ii) 0.1018 cm, (iii) 50.93 Nm]
3. Calculate instantaneous stress produced in a bar 10 cm² in area and 4 m long by the sudden application of a tensile load of unknown magnitude, if the extension of the bar due to suddenly applied load is 1.35 mm. Also determine the suddenly applied load. Take $E = 2 \times 10^5 \text{ N/mm}^2$.
[Ans. 67.5 N/mm², 33.75 kN]
4. The maximum instantaneous, produced by an unknown falling weight through a height of 4 cm in a vertical bar of length 5 m and of cross-sectional area 5 cm², is 1.80 mm. Determine : (i) the instantaneous stress induced in the vertical bar and (ii) the values of unknown weight.
Take $E = 2 \times 10^5 \text{ N/mm}^2$. [Ans. (i) 72 N/mm² and (ii) 775.1 N]
5. An unknown weight falls through a height of 20 mm on a collar rigidly attached to the lower end of a vertical bar 5 m long and 800 mm² in section. If the maximum extension of the rod is to be 2.5 mm, what is the corresponding stress and magnitude of the unknown weight ? Take $E = 2 \times 10^5 \text{ N/mm}^2$.
[Ans. 1000 kgf/cm², 444.44 kgf]

15

Bending of Beams

15.1. INTRODUCTION

When some external load acts on a beam, the shear force and bending moments are set up at all sections of the beam. Due to the shear force and bending moment, the beam undergoes certain deformation. The material of the beam will offer resistance or stresses against these deformations. These stresses with certain assumptions can be calculated. The stresses introduced by bending moment are known as *bending stresses*. In this chapter, the theory of pure bending, expression for bending stresses, bending stress in symmetrical sections will be discussed.

15.2. THEORY OF PURE BENDING

If a length of a beam is subjected to a constant bending moment and no shear force (*i.e.*, zero shear force), then the stresses will be set up in that length of the beam due to B.M. only and that length of the beam is said to be in *pure bending* or simple bending. The stresses set up in that length of beam are known as bending stresses.

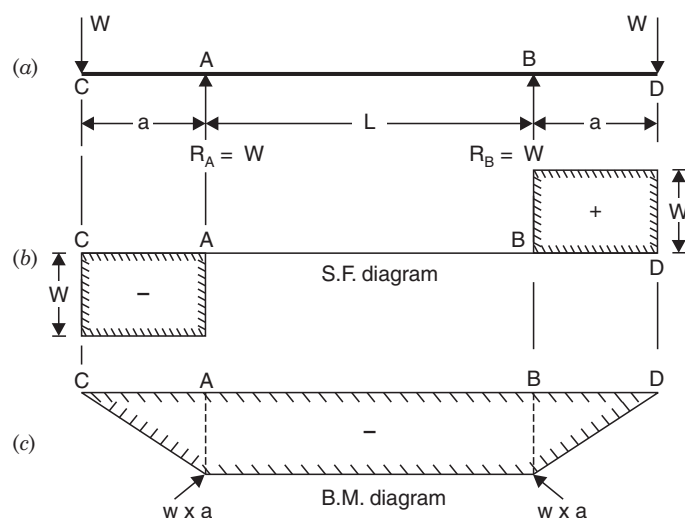


Fig. 15.1

A beam simply supported at A and B and overhanging by same length at each support is shown in Fig. 15.1. A point load W is applied at each end of the overhanging portion. The S.F. and B.M. for the beam are drawn as shown in Fig. 15.1 (b) and Fig. 15.1 (c)

respectively. From these diagrams, it is clear that there is no shear force between A and B but the B.M. between A and B is constant.

This means that between A and B , the beam is subjected to a constant bending moment only. This condition of the beam between A and B is known as pure bending or simple bending.

15.2.1. Theory of Simple Bending with Assumptions Made. Before discussing the theory of simple bending, let us see the assumptions made in the theory of simple bending. The following are the important assumptions :

1. The material of the beam is homogeneous* and isotropic**.
2. The value of Young's modulus of elasticity is the same in tension and compression.
3. The transverse sections which were plane before bending, remain plane after bending also.
4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.
5. The radius of curvature is large compared with the dimensions of the cross-section.
6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

Theory of Pure Bending

Fig. 15.2 (a) shows a part of a beam subjected to simple bending. Consider a small length δx of this part of beam. Consider two sections AB and CD which are normal to the axis of the beam $N-N$. Due to the action of the bending moment, the part of length δx will be deformed as shown in Fig. 15.2 (b). From this figure, it is clear that all the layers of the beam, which were originally of the same length, do not remain of the same length any more.

The top layer such as AC has deformed to the shape $A'C'$. This layer has been shortened in its length. The bottom layer BD has deformed to the shape $B'D'$. This layer has been elongated. From the Fig. 15.2 (b), it is clear that some of the layers have been shortened while some of them are elongated. At a level between the top and bottom of the beam, there will be a layer which is neither shortened nor elongated. This layer is known as **neutral layer** or **neutral surface**. This layer in Fig. 15.2 (b) is shown by $N'-N'$ and in Fig. 15.2 (a) by $N-N$. The line of intersection of the neutral layer on a cross-section of a beam is known as **neutral axis** (written as N.A.).

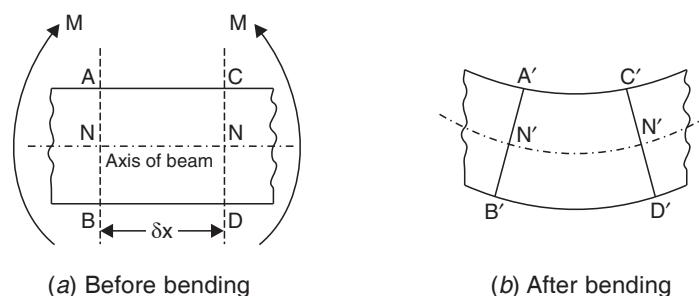


Fig. 15.2

*Homogeneous means the material is of the same kind throughout.

** Isotropic means that the elastic properties in all directions are equal.

The layers above $N-N$ (or $N'-N'$) have been shortened and those below, have been elongated. Due to the decrease in lengths of the layers above $N-N$, these layers will be subjected to compressive stresses. Due to the increase in the lengths of layers below $N-N$, these layers will be subjected to tensile stresses.

We also see that the top layer has been shortened maximum. As we proceed towards the layer $N-N$, the decrease in length of the layers decreases. At the layer $N-N$, there is no change in length. This means the compressive stress will be maximum at the top layer. Similarly the increase in length will be maximum at the bottom layer. As we proceed from bottom layer towards the layer $N-N$, the increase in length of layers decreases. Hence the amount by which a layer increases or decreases in length, depends upon the position of the layer with respect to $N-N$. This theory of bending is known as theory of simple bending.

15.2.2. Expression for Bending Stress. Fig. 15.3 (a) shows a small length δx of a beam subjected to a simple bending. Due to the action of bending, the part of length δx will be deformed as shown in Fig. 15.3 (b). Let $A'B'$ and $C'D'$ meet at O .

Let R = Radius of neutral layer $N'N'$

θ = Angle subtended at O by $A'B'$ and $C'D'$ produced.

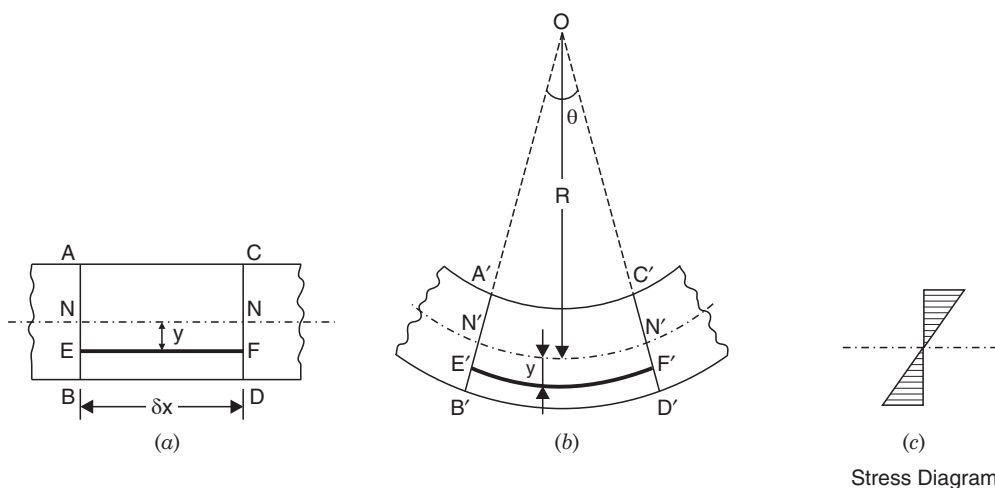


Fig. 15.3

1. Strain Variation Along the Depth of Beam. Consider a layer EF at a distance y below the neutral layer NN . After bending this layer will be elongated to $E'F'$.

Original length of layer $EF = \delta x$.

Also length of neutral layer $NN = \delta x$.

After bending, the length of neutral layer $N'N'$ will remain unchanged. But length of layer $E'F'$ will increase. Hence

$$N'N' = NN = \delta x.$$

Now from Fig. 15.3 (b),

$$N'N' = R \times \theta$$

and

$$E'F' = (R + y) \times \theta \quad (\because \text{Radius of } E'F' = R + y)$$

But $N'N' = NN = \delta x$.

Hence $\delta x = R \times \theta$

$\therefore$ Increase in the length of the layer EF

$$= E'F' - EF = (R + y) \theta - R \times \theta \quad (\because EF = \delta x = R \times \theta)$$

$$= y \times \theta$$

$\therefore$ Strain in the layer EF

$$= \frac{\text{Increase in length}}{\text{Original length}}$$

$$= \frac{y \times \theta}{EF} = \frac{y \times \theta}{R \times \theta} \quad (\because EF = \delta x = R \times \theta)$$

$$= \frac{y}{R}$$

As R is constant, hence the strain in a layer is proportional to its distance from the neutral axis. The above equation shows the variation of strain along the depth of the beam. The variation of strain is linear.

2. Stress Variation

Let σ = Stress in the layer EF

E = Young's modulus of the beam

Then $E = \frac{\text{Stress in the layer } EF}{\text{Strain in the layer } EF}$

$$= \frac{\sigma}{\left(\frac{y}{R}\right)} \quad \left(\because \text{Strain in } EF = \frac{y}{R}\right)$$

$$\therefore \sigma = E \times \frac{y}{R} = \frac{E}{R} \times y \quad \dots(15.1)$$

Since E and R are constant, therefore stress in any layer is directly proportional to the distance of the layer from the neutral layer. The equation (15.1) shows the variation of stress along the depth of the beam. The variation of stress is linear.

In the above case, all layers below the neutral layer are subjected to tensile stresses whereas the layers above neutral layer are subjected to compressive stresses. The Fig. 15.3 (c) shows the stress distribution.

The equation (15.1) can also be written as

$$\frac{\sigma}{y} = \frac{E}{R} \quad \dots(15.2)$$

15.3. NEUTRAL SURFACE AND NEUTRAL AXIS

The neutral axis of any transverse section of a beam is defined as the line of intersection of the neutral layer with the transverse section. It is written as N.A.

In Art. 15.2.2, we have seen that if a section of a beam is subjected to pure sagging moment, then the stresses will be compressive at any point above the neutral axis and tensile

below the neutral axis. There is no stress at the neutral axis. The stress at a distance y from the neutral axis is given by equation (15.1) as

$$\sigma = \frac{E}{R} \times y.$$

Fig. 15.4 shows the cross-section of a beam. Let N.A. be the neutral axis of the section. Consider a small layer at a distance y from the neutral axis. Let dA = Area of the layer.

Now the force on the layer

$$\begin{aligned} &= \text{Stress on layer} \times \text{Area of layer} \\ &= \sigma \times dA \\ &= \frac{E}{R} \times y \times dA \end{aligned}$$

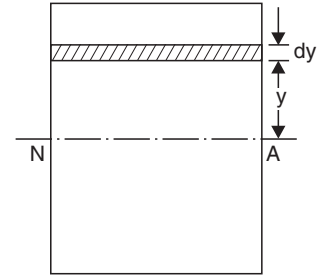


Fig. 15.4

$$\dots(i) \quad \left(\because \sigma = \frac{E}{R} \times y \right)$$

Total force on the beam section is obtained by integrating the above equation.

$\therefore$ Total force on the beam section

$$\begin{aligned} &= \int \frac{E}{R} \times y \times dA \\ &= \frac{E}{R} \int y \times dA \end{aligned} \quad (\because E \text{ and } R \text{ is constant})$$

But for pure bending, there is no force on the section of the beam (or force is zero).

$$\therefore \quad \frac{E}{R} \int y \times dA = 0$$

$$\text{or} \quad \int y \times dA = 0 \quad \left(\text{as } \frac{E}{R} \text{ cannot be zero} \right)$$

Now $y \times dA$ represents the moment of area dA about neutral axis. Hence $\int y \times dA$ represents the moment of entire area of the section about neutral axis. But we know that moment of any area about an axis passing through its centroid, is also equal to zero. Hence neutral axis coincides with the centroidal axis. Thus the centroidal axis of a section gives the position of neutral axis.

15.3.1. Moment of Resistance. Due to pure bending, the layers above the N.A. are subjected to compressive stresses whereas the layers below the N.A. are subjected to tensile stresses. Due to these stresses, the forces will be acting on the layers. These forces will have moment about the N.A. The total moment of these forces about the N.A. for a section is known as moment of resistance of that section.

The force on the layer at a distance y from neutral axis in Fig. 15.4 is given by equation (i), as

$$\text{Force on layer} = \frac{E}{R} \times y \times dA$$

Moment of this force about N.A.

$$\begin{aligned} &= \text{Force on layer} \times y \\ &= \frac{E}{R} \times y \times dA \times y \\ &= \frac{E}{R} \times y^2 \times dA \end{aligned}$$

Total moment of the forces on the section of the beam (or moment of resistance)

$$= \int \frac{E}{R} \times y^2 \times dA = \frac{E}{R} \int y^2 \times dA$$

Let M = External moment applied on the beam section. For equilibrium the moment of resistance offered by the section should be equal to the external bending moment.

$$\therefore M = \frac{E}{R} \int y^2 \times dA.$$

But the expression $\int y^2 \times dA$ represents the moment of inertia of the area of the section about the neutral axis. Let this moment of inertia be I .

$$\therefore M = \frac{E}{R} \times I \quad \text{or} \quad \frac{M}{I} = \frac{E}{R} \quad \dots(15.3)$$

But from equation (15.2), we have

$$\begin{aligned} \frac{\sigma}{y} &= \frac{E}{R} \\ \therefore \frac{M}{I} &= \frac{\sigma}{y} = \frac{E}{R} \quad \dots(15.4) \end{aligned}$$

The equation (15.4) is known as bending equation.

In equation (15.4), the different quantities are expressed in consistent units as given below :

M is expressed in N mm ; I in mm^4

σ is expressed in N/mm^2 ; y in mm

and E is expressed in N/mm^2 ; R in mm.

15.3.2. Condition of Pure Bending. The equation (15.4) is applicable to a member which is subjected to a constant bending moment and the member is absolutely free from shear force. But in actual practice, a member is subjected to such loading that the B.M. varies from section to section and also the shear force is not zero. But shear force is zero at a section where bending moment is maximum. *Hence the condition of pure bending* may be assumed to be satisfied at such a section. Hence the stresses produced due to maximum bending moment, are obtained from equation (15.4) as the shear forces at these sections are generally zero. Hence the theory and equations discussed in the above articles are quite sufficient and give results which enables the engineers to design beams and structures and calculate their stresses and strains with a reasonable degree of approximation where B.M. is maximum.

15.4. STRESSES IN BEAMS OF DIFFERENT CROSS-SECTIONS

The neutral axis (N.A.) of a symmetrical section (such as circular, rectangular or square) lies at a distance of $d/2$ from the outermost layer of the section where d is the diameter (for a circular section) or depth (for a rectangular or a square section). There is no stress at the neutral axis. But the stress at a point is directly proportional to its distance from the neutral axis. The maximum stress takes place at the outermost layer. For a simply supported beam, there is a compressive stress above the neutral axis and a tensile stress below it. If we plot these stresses, we will get a figure as shown in Fig. 15.5.

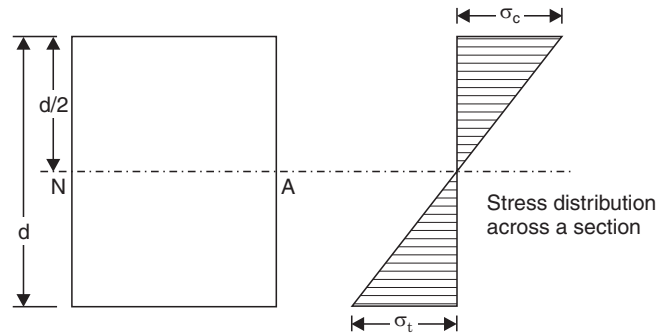


Fig. 15.5

Problem 15.1. A steel plate of width 120 mm and of thickness 20 mm is bent into a circular arc of radius 10 m. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take $E = 2 \times 10^5 \text{ N/mm}^2$.

Sol. Given :

Width of plate, $b = 120 \text{ mm}$

Thickness of plate, $t = 20 \text{ mm}$

∴ Moment of inertia, $I = \frac{bt^3}{12} = \frac{120 \times 20^3}{12} = 8 \times 10^4 \text{ mm}^4$

Radius of curvature, $R = 10 \text{ m} = 10 \times 10^3 \text{ mm}$

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Let σ_{max} = Maximum stress induced, and
 M = Bending moment.

Using equation (15.2), $\frac{\sigma}{y} = \frac{E}{R}$

∴ $\sigma = \frac{E}{R} \times y \quad \dots(i)$

Equation (i) gives the stress at a distance y from N.A.

Stress will be maximum, when y is maximum. But y will be maximum at the top layer or bottom layer.

∴ $y_{max} = \frac{t}{2} = \frac{20}{2} = 10 \text{ mm}.$

Now equation (i) can be written as

$$\begin{aligned} \sigma_{max} &= \frac{E}{R} \times y_{max} \\ &= \frac{2 \times 10^5}{10 \times 10^3} \times 10 = 200 \text{ N/mm}^2. \quad \text{Ans.} \end{aligned}$$

From equation (15.4), we have

$$\begin{aligned} \frac{M}{I} &= \frac{E}{R} \\ \therefore M &= \frac{E}{R} \times I = \frac{2 \times 10^5}{10 \times 10^3} \times 8 \times 10^4 \\ &= 16 \times 10^5 \text{ N mm} = 1.6 \text{ kNm}. \quad \text{Ans.} \end{aligned}$$

Problem 15.2. Calculate the maximum stress* induced in a cast iron pipe of external diameter 40 mm, of internal diameter 20 mm and of length 4 metre when the pipe is supported at its ends and carries a point load of 80 N at its centre.

Sol. Given :

External dia., $D = 40 \text{ mm}$

Internal dia., $d = 20 \text{ mm}$

Length, $L = 4 \text{ m} = 4 \times 1000 = 4000 \text{ mm}$

Point load, $W = 80 \text{ N}$

In case of simply supported beam carrying a point load at the centre, the maximum bending moment is at the centre of the beam.

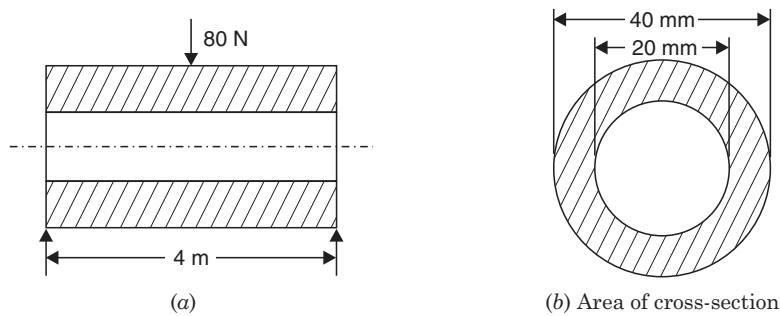


Fig. 15.6

$$\begin{aligned} \text{And maximum B.M.} &= \frac{W \times L}{4} \\ \therefore \text{Maximum B.M.} &= \frac{80 \times 4000}{4} = 8 \times 10^4 \text{ Nmm} \\ \therefore M &= 8 \times 10^4 \text{ Nmm} \end{aligned}$$

Fig. 15.6 (b) shows the cross-section of the pipe.

Moment of inertia of hollow pipe,

$$\begin{aligned} I &= \frac{\pi}{64} [D^4 - d^4] \\ &= \frac{\pi}{64} [40^4 - 20^4] = \frac{\pi}{64} [2560000 - 160000] \\ &= 117809.7 \text{ mm}^4 \end{aligned}$$

Now using equation (15.4),

$$\frac{M}{I} = \frac{\sigma}{y} \quad \dots(i)$$

when y is maximum, stress will be maximum. But y is maximum at the top layer from the N.A.

$$\therefore y_{\max} = \frac{D}{2} = \frac{40}{2} = 20 \text{ mm}$$

*The bending stress will be maximum at the section where B.M. is maximum. This is because

$$\frac{M}{I} = \frac{\sigma}{y} \quad \text{or} \quad \sigma = \frac{M}{I} \times y.$$

The above equation (i) can be written as

$$\begin{aligned}\frac{M}{I} &= \frac{\sigma_{max}}{y_{max}} \\ \therefore \sigma_{max} &= \frac{M}{I} \times y_{max} \\ &= \frac{8 \times 10^4 \times 20}{117809.7} = \mathbf{13.58 \text{ N/mm}^2}. \quad \text{Ans.}\end{aligned}$$

15.4.1. Section Modulus. Section modulus is defined as the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis. It is denoted by the symbol Z . Hence mathematically section modulus is given by,

$$Z = \frac{I}{y_{max}} \quad \dots(15.5)$$

where I = M.O.I. about neutral axis

and y_{max} = Distance of the outermost layer from the neutral axis.

From equation (15.4), we have

$$\frac{M}{I} = \frac{\sigma}{y}$$

The stress σ will be maximum, when y is maximum. Hence above equation can be written as

$$\begin{aligned}\frac{M}{I} &= \frac{\sigma_{max}}{y_{max}} \\ \therefore M &= \sigma_{max} \cdot \frac{I}{y_{max}} \\ \text{But } \frac{I}{y_{max}} &= Z \\ \therefore M &= \sigma_{max} \cdot Z \quad \dots(15.6)\end{aligned}$$

In the above equation, M is the maximum bending moment (or moment of resistance offered by the section). Hence moment of resistance offered the section is maximum when section modulus Z is maximum. Hence section modulus represent the strength of the section.

15.4.2. Section Modulus for Various Shapes of Beam Sections

1. Rectangular Section

Moment of inertia of a rectangular section about an axis through its C.G. (or through N.A.) is given by,

$$I = \frac{bd^3}{12}$$

Distance of outermost layer from N.A. is given by,

$$y_{max} = \frac{d}{2}$$

$\therefore$ Section modulus is given by,

$$Z = \frac{I}{y_{max}} = \frac{bd^3}{12 \times \left(\frac{d}{2}\right)} = \frac{bd^3}{12} \times \frac{2}{d} = \frac{bd^2}{6} \quad \dots(15.7)$$

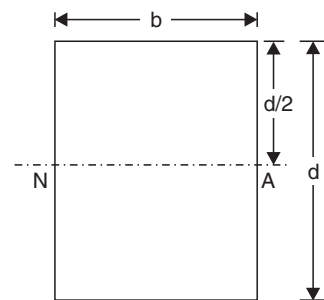


Fig. 15.7

2. Hollow Rectangular Section

Here

$$I = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$= \frac{1}{12} [BD^3 - bd^3]$$

$$y_{max} = \frac{D}{2}$$

$$\therefore Z = \frac{I}{y_{max}}$$

$$= \frac{\frac{1}{12} [BD^3 - bd^3]}{\left(\frac{D}{2}\right)}$$

$$= \frac{1}{6D} [BD^3 - bd^3] \quad \dots(15.8)$$

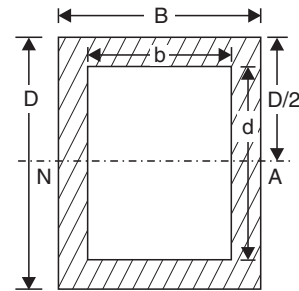


Fig. 15.8

3. Circular Section

For a circular section,

$$I = \frac{\pi}{64} d^4 \quad \text{and} \quad y_{max} = \frac{d}{2}$$

$$\therefore Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} d^4}{\left(\frac{d}{2}\right)} = \frac{\pi}{32} d^3 \quad \dots(15.9)$$

4. Hollow Circular Section

Here

$$I = \frac{\pi}{64} [D^4 - d^4]$$

and

$$y_{max} = \frac{D}{2}$$

$$\therefore Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} [D^4 - d^4]}{\left(\frac{D}{2}\right)}$$

$$= \frac{\pi}{32D} [D^4 - d^4] \quad \dots(15.10)$$

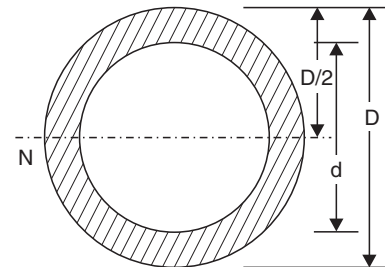


Fig. 15.9

Problem 15.3. A cantilever of length 2 metre fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm × 60 mm, find the stress at the failure.

Sol. Given :

Length, $L = 2 \text{ m} = 2 \times 10^3 \text{ mm}$

Load, $W = 2 \text{ kN} = 2000 \text{ N}$

Section of beam is 40 mm × 60 mm.

$\therefore$ Width of beam, $b = 40 \text{ mm}$

Depth of beam, $d = 60 \text{ mm}$

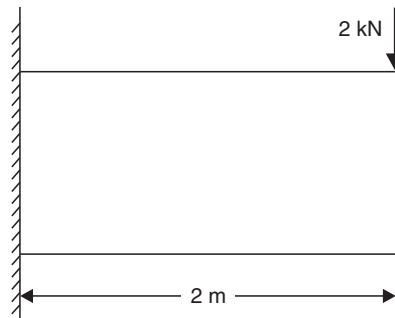


Fig. 15.10

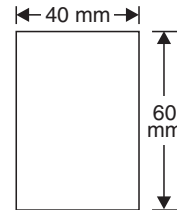


Fig. 15.10 (a)

Fig. 15.10 (a) shows the section of the beam.

Section modulus of a rectangular section is given by equation (15.7).

$$\therefore Z = \frac{bd^2}{6} = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

Maximum bending moment for a cantilever shown in Fig. 15.10 is at the fixed end.

$$\therefore M = W \times L = 2000 \times 2 \times 10^3 = 4 \times 10^6 \text{ Nmm}$$

Let σ_{max} = Stress at the failure

Using equation (15.6), we get

$$M = \sigma_{max} \cdot Z$$

$$\therefore \sigma_{max} = \frac{M}{Z} = \frac{4 \times 10^6}{24000} = 166.67 \text{ N/mm}^2. \text{ Ans.}$$

Problem 15.4. A rectangular beam 200 mm deep and 300 mm wide is simply supported over a span of 8 m. What uniformly distributed load per metre the beam may carry, if the bending stress is not to exceed 120 N/mm².

Sol. Given :

Depth of beam, $d = 200 \text{ mm}$

Width of beam, $b = 300 \text{ mm}$

Length of beam, $L = 8 \text{ m}$

Max. bending stress,

$$\sigma_{max} = 120 \text{ N/mm}^2$$

Let w = Uniformly distributed load per m length over the beam

(Fig. 15.11 (a) shows the section of the beam).

Section modulus for a rectangular section is given by equation (15.7).

$$\therefore Z = \frac{bd^2}{6} = \frac{300 \times 200^2}{6} = 2000000 \text{ mm}^3$$

Max. B.M. for a simply supported beam carrying uniformly distributed load as shown in Fig. 15.11 is at the centre of the beam. It is given by

$$M = \frac{w \times L^2}{8} = \frac{w \times 8^2}{8} \quad (\because L = 8 \text{ m})$$

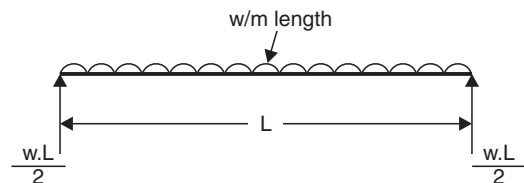


Fig. 15.11

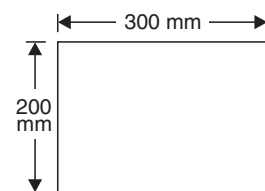


Fig. 15.11 (a)

$$\begin{aligned}
 &= 8w \text{ Nm} = 8w \times 1000 \text{ Nmm} \\
 &= 8000w \text{ Nmm} \quad (\because 1 \text{ m} = 1000 \text{ mm})
 \end{aligned}$$

Now using equation (15.6), we get

$$\begin{aligned}
 M &= \sigma_{\max} \cdot Z \\
 \text{or } 8000w &= 120 \times 2000000
 \end{aligned}$$

$$\therefore w = \frac{120 \times 2000000}{8000} = 30 \times 1000 \text{ N/m} = \mathbf{30 \text{ kN/m. Ans.}}$$

Problem 15.5. A rectangular beam 300 mm deep is simply supported over a span of 4 metres. Determine the uniformly distributed load per metre which the beam may carry, if the bending stress should not exceed 120 N/mm². Take $I = 8 \times 10^6 \text{ mm}^4$.

(Annamalai University, 1991)

Sol. Given :

Depth, $d = 300 \text{ mm}$

Span, $L = 4 \text{ m}$

Max. bending stress, $\sigma_{\max} = 120 \text{ N/mm}^2$

Moment of inertia, $I = 8 \times 10^6 \text{ mm}^4$

Let, $w = \text{U.D.L. per metre length over the beam in N/m.}$

The bending stress will be maximum, where bending moment is maximum. For a simply supported beam carrying U.D.L., the bending moment is maximum at the centre of the beam [i.e., at point C of Fig. 15.11 (b)]

$$\begin{aligned}
 \therefore \text{Max. B.M.} &= 2w \times 2 - 2w \times 1 \\
 &= 4w - 2w
 \end{aligned}$$

$$= 2w \text{ Nm}$$

$$= 2w \times 1000 \text{ Nmm}$$

$$\text{or } M = 2000w \text{ Nmm}$$

Now using equation (15.6), we get

$$M = \sigma_{\max} \times Z \quad \dots(i)$$

$$\begin{aligned}
 \text{where } Z &= \frac{I}{y_{\max}} = \frac{8 \times 10^6}{150} \quad \left(\because y_{\max} = \frac{d}{2} = \frac{300}{2} = 150 \text{ mm} \right)
 \end{aligned}$$

Hence above equation (i) becomes as

$$2000w = 120 \times \frac{8 \times 10^6}{150}$$

$$\text{or } w = \frac{120 \times 8 \times 10^6}{2000 \times 150} = \mathbf{3200 \text{ N/m. Ans.}}$$

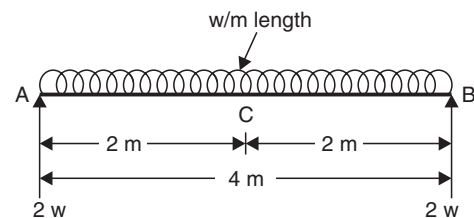


Fig. 15.11 (b)

Problem 15.6. A square beam $20 \text{ mm} \times 20 \text{ mm}$ in section and 2 m long is supported at the ends. The beam fails when a point load of 400 N is applied at the centre of the beam. What uniformly distributed load per metre length will break a cantilever of the same material 40 mm wide, 60 mm deep and 3 m long ?

Sol. Given :

Depth of beam, $d = 20 \text{ mm}$

Width of beam, $b = 20 \text{ mm}$

Length of beam, $L = 2 \text{ m}$

Point load, $W = 400 \text{ N}$

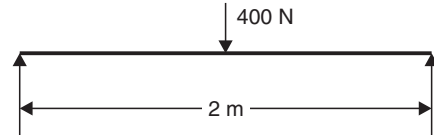


Fig. 15.12

In this problem, the maximum stress for the simply supported beam is to be calculated first. As the material of the cantilever is same as that of simply supported beam, hence maximum stress for the cantilever will also be same as that of simply supported beam.

Fig. 15.12 (a) shows the section of beam.

The section modulus for the rectangular section of simply supported beam is given by equation (15.7).

$$\therefore Z = \frac{bd^2}{6} = \frac{20 \times 20^2}{6} = \frac{4000}{3} \text{ mm}^3$$

Max. B.M. for a simply supported beam carrying a point load at the centre (as shown in Fig. 15.12) is given by,

$$M = \frac{w \times L}{4} = \frac{400 \times 2}{4} = 200 \text{ Nm}$$

$$= 200 \times 1000 = 200000 \text{ Nmm}$$

Let σ_{max} = Max. stress induced

Now using equation (15.6), we get

$$M = \sigma_{max} \cdot Z$$

$$\text{or } 200000 = \sigma_{max} \times \frac{4000}{3}$$

$$\therefore \sigma_{max} = \frac{200000 \times 3}{4000} = 150 \text{ N/mm}^2$$

Now let us consider the cantilever as shown in Fig. 15.13.

Let w = Uniformly distributed load per m run.

Maximum stress will be same as in case of simply supported beam.

$$\therefore \sigma_{max} = 150 \text{ N/mm}^2$$

Width of cantilever, $b = 40 \text{ mm}$

Depth of cantilever, $d = 60 \text{ mm}$

Length of cantilever, $L = 3 \text{ m}$

Fig. 15.13 (a) shows the section of cantilever beam.

Section modulus of rectangular section of cantilever = $\frac{bd^2}{6}$

$$\therefore Z = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

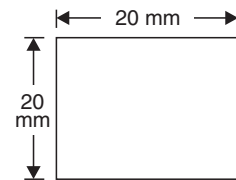


Fig. 15.12 (a)

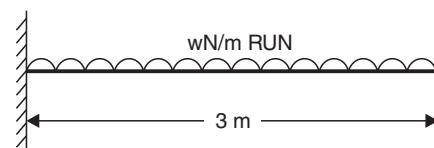


Fig. 15.13

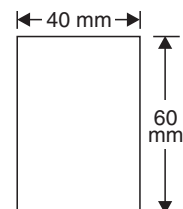


Fig. 15.13 (a)

Maximum B.M. for a cantilever

$$= \frac{wL^2}{2} = \frac{w \times 3^2}{2} = 4.5w \text{ Nm} = 4.5 \times 1000w \text{ Nmm}$$

$$\therefore M = 4.5 \times 1000w \text{ Nmm}$$

Now using equation (15.6), we get

$$M = \sigma_{\max} \cdot Z$$

or $4.5 \times 1000w = 150 \times 24000$

$$\therefore w = \frac{150 \times 24000}{4.5 \times 1000} = 800 \text{ N/m. Ans.}$$

Problem 15.7. A steel beam of hollow square section of 60 mm outer side and 50 mm inner side is simply supported on a span of 4 metres. Find the maximum concentrated load the beam at the middle of the span if the bending stress is not to exceed 120 N/mm².

(U.P. Tech. University, May 2009)

Sol. Given:

$$L = 4 \text{ m}, \sigma_b = 120 \text{ N/mm}^2$$

Let $W = \text{Max. concentrated load.}$

The bending moment is maximum at the centre and it is

equal to $\frac{W \times L}{4}$.

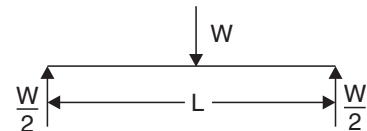


Fig. 15.14

$$\therefore \text{B.M.} = \frac{W \times L}{4} = \frac{W \times 4}{4} = W \text{ Nm}$$

or $M = W \times 1000 = 1000 W \text{ Nmm}$

M.O.I. for hollow section,

$$\begin{aligned} I &= \frac{60 \times 60^3}{12} - \frac{50 \times 50^3}{12} \text{ mm}^4 \\ &= \frac{10^4}{12} \times [1296 - 625] \text{ mm}^4 \\ &= \frac{10^4}{12} [671] \text{ mm}^4 = 55.91 \times 10^4 \\ y_{\max} &= \frac{60}{2} = 30 \text{ mm} \end{aligned}$$

Using $\frac{M}{I} = \frac{\sigma_b}{y}$

$$\therefore \frac{1000 W}{55.91 \times 10^4} = \frac{120}{30}$$

Here,

$M = \text{B.M. in N mm}$

$\sigma_b = \text{bending stress in N/mm}^2$

$I = \text{M.O.I. in mm}^4$

$$\therefore W = \frac{10^4 \times 671}{12 \times 1000} \times \frac{120}{30}$$

$$= \frac{10 \times 671}{3} \text{ N} = 2233.3 \text{ N. Ans.}$$

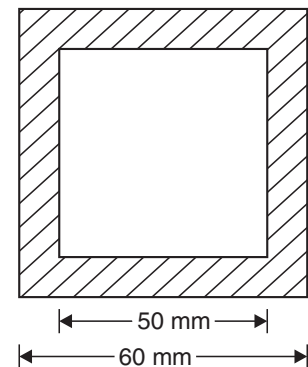


Fig. 15.14 (a)

Problem 15.8. A rolled steel joist of I section has the dimensions : as shown in Fig. 15.15. This beam of I section carries a u.d.l. of 40 kN/m run on a span of 10 m, calculate the maximum stress produced due to bending.

Sol. Given :

u.d.l., $w = 40 \text{ kN/m} = 40000 \text{ N/m}$

Span, $L = 10 \text{ m}$

Moment of inertia about the neutral axis

$$= \frac{200 \times 400^3}{12} - \frac{(200 - 10) \times 360^3}{12}$$

$$= 1066666666 - 738720000$$

$$= 327946666 \text{ mm}^4$$

Maximum B.M. is given by,

$$M = \frac{w \times L^2}{8} = \frac{40000 \times 10^2}{8}$$

$$= 500000 \text{ Nm}$$

$$= 500000 \times 1000 \text{ Nmm}$$

$$= 5 \times 10^8 \text{ Nmm}$$

Now using the relation,

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\therefore \sigma = \frac{M}{I} \times y$$

or
$$\sigma_{\max} = \frac{M}{I} \times y_{\max} = \frac{5 \times 10^8}{327946666} \times 200 \quad (\because y_{\max} = 200 \text{ mm})$$

$$= 304.92 \text{ N/mm}^2. \text{ Ans.}$$

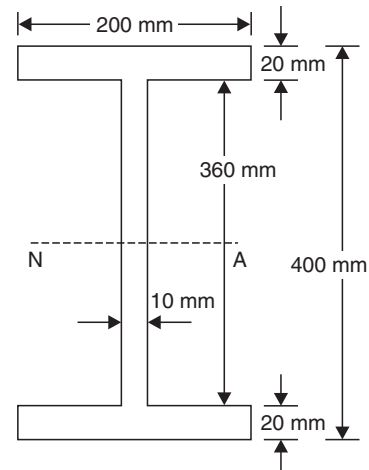


Fig. 15.15

HIGHLIGHTS

1. The stresses produced due to constant bending moment (with zero shear force) are known as bending stresses.
2. The bending equation is given by,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

where M = Bending moment

σ = Bending stress

I = Moment of inertia about N.A.

y = Distance of the fibre from N.A.

R = Radius of curvature

E = Young's modulus of beam.

3. The bending stress in any layer is directly proportional to the distance of the layer from the neutral axis (N.A.).
4. The bending stress on the neutral axis is zero.
5. The neutral axis of a symmetrical section (such as circular, rectangular or square) lies at a distance of $\frac{d}{2}$ from the outermost layer of the section where d is the depth of the section.

6. If the top layer of the section is subjected to compressive stress then the bottom layer of the section will be subjected to tensile stress.
7. The ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis is known as section modulus. It is denoted by Z .

$$\therefore Z = \frac{I}{y_{max}}$$

8. Section modulus for various sections are given as :

$$\begin{aligned} Z &= \frac{bd^2}{6} && \text{...For rectangular section} \\ &= \frac{1}{6D} (BD^3 - bd^3) && \text{...For a hollow rectangular section} \\ &= \frac{\pi d^3}{32} && \text{...For a circular section} \\ &= \frac{\pi}{32D} [D^4 - d^4] && \text{...For a hollow circular section.} \end{aligned}$$

EXERCISE 15

(A) Theoretical Questions

1. Define the terms : bending stress in a beam, neutral axis and section modulus.
2. What do you mean by 'simple bending' or 'pure bending' ? What are the assumptions made in the theory of simple bending ?
3. Derive an expression for bending stress at a layer in a beam.
4. What do you understand by the term neutral axis and neutral surface ?
(U.P. Tech. University, May 2009)
5. Prove that relation,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

where M = Bending moment, I = M.O.I.
 σ = Bending stress, y = Distance from N.A.
 E = Young's modulus, and R = Radius of curvature.

(Bangalore University, Jan. 1990)

6. What do you mean by section modulus ? Find an expression for section modulus for a rectangular, circular and hollow circular sections.
7. Explain the terms : Neutral axis, section modulus, and moment of resistance.
(Bangalore University, July 1988)
8. Show that for a beam subjected to pure bending, neutral axis coincides with the centroid of the cross-section.
(Bangalore University, March 1989)
9. Prove that the bending stress in any fibre is proportional to the distance of that fibre from neutral layer in a beam.
(Bhavnagar University, 1992)

(B) Numerical Problems

1. A steel plate of width 60 mm and of thickness 10 mm is bent into a circular arc of radius 10 m. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take $E = 2 \times 10^5 \text{ N/mm}^2$.
[Ans. 100 N/mm² ; 100 Nm]

2. A cast iron pipe of external diameter 60 mm, internal diameter of 40 mm, and of length 5 m is supported at its ends. Calculate the maximum bending stress induced in the pipe if it carries a point load of 100 N at its centre.
[Ans. 7.34 N/mm²]
3. A rectangular beam 300 mm deep is simply supported over a span of 4 m. What uniformly distributed load per metre, the beam may carry if the bending stress is not to exceed 120 N/mm²? Take $I = 8 \times 10^6 \text{ mm}^4$.
[Ans. 3.2 kN/m]
4. A cast iron cantilever of length 1.5 metre fails when a point load W is applied at the free end. If the section of the beam is 40 mm $\times$ 60 mm and the stress at the failure is 120 N/mm², find the point load applied.
[Ans. 1.92 kN]
5. A cast iron beam 20 mm $\times$ 20 mm in section and 100 cm long is simply supported at the ends. It carries a point load W at the centre. The maximum stress induced is 120 N/mm². What uniformly distributed load will break a cantilever of the same material 50 mm wide, 100 mm deep and 2 m long?
[Ans. 5 kN per m run]
6. A timber beam is 120 mm wide and 200 mm deep and is used on a span of 4 metres. The beam carries a uniformly distributed load of 2.8 kN/m run over the entire length. Find the maximum bending stress induced.
[Ans. 7 N/mm²]
7. A timber cantilever 200 mm wide and 300 mm deep is 3 m long. It is loaded with a U.D.L. of 3 kN/m over the entire length. A point load of 2.7 kN is placed at the free end of the cantilever. Find the maximum bending stress produced.
[Ans. 7.2 N/mm²]
8. A timber beam is freely supported on supports 6 m apart. It carries a uniformly distributed load of 12 kN/m run and a point load of 9 kN at 3.5 m from the right support. Design a suitable section of the beam making depth twice the width, if the stress in timber is not to exceed 8 N/mm².
[Ans. 230 mm $\times$ 460 mm]

16

Theory of Torsion

16.1. TORQUE AND TWIST

A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft. The torque is equal to the product of the force applied (tangentially to the ends of a shaft) and radius of the shaft. Due to the application of the torques at the two ends, the shaft is subjected to a twisting moment. This causes the shear stresses and shear strains in the material of the shaft. Twisting moment is due to twist.

16.2. SHEAR STRESS PRODUCED IN A CIRCULAR SHAFT SUBJECTED TO TORSION

When a circular shaft is subjected to torsion, shear stresses are set up in the material of the shaft. To determine the magnitude of shear stress at any point on the shaft, consider a shaft fixed at one end AA and free at the end BB as shown in Fig. 16.1. Let CD is any line on the outer surface of the shaft. Now let the shaft is subjected to a torque T at the end BB as shown in Fig. 16.2. As a result of this torque T , the shaft at the end BB will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses. The point D will shift to D' and hence line CD will be deflected to CD' as shown in Fig. 16.2 (a). The line OD will be shifted to OD' as shown in Fig. 16.2 (b).

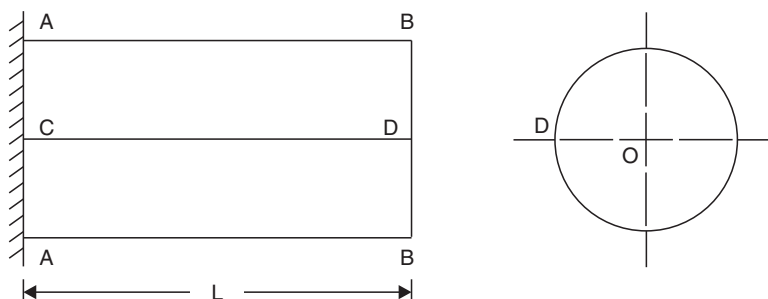


Fig. 16.1. Shaft fixed at one end AA before torque T is applied.

Let R = Radius of shaft

L = Length of shaft

T = Torque applied at the end BB

τ = Shear stress induced at the surface of the shaft due to torque T

C = Modulus of rigidity of the material of the shaft

$\phi = \angle DCD'$ also equal to shear strain
 $\theta = \angle DOD'$ and is also called angle of twist.

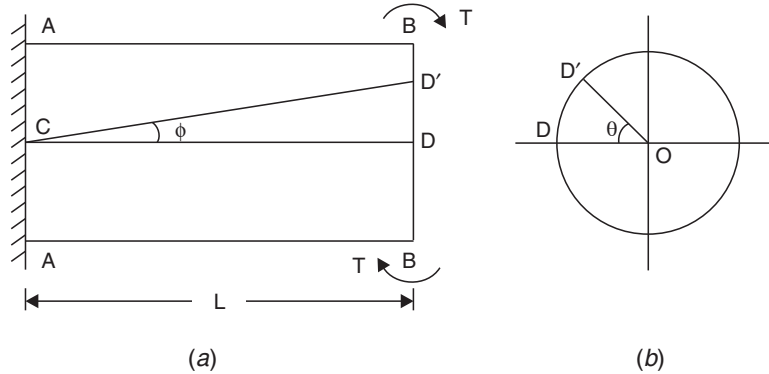


Fig. 16.2. Shaft fixed at AA and subjected to torque T at BB.

Now distortion at the outer surface due to torque T

$$= DD'$$

$\therefore$ Shear strain at outer surface

= Distortion per unit length

$$= \frac{\text{Distortion at the outer surface}}{\text{Length of shaft}} = \frac{DD'}{L}$$

$$= \frac{DD'}{CD} = \tan \phi$$

$$= \phi$$

(if ϕ is very small then $\tan \phi \simeq \phi$)

$\therefore$ Shear strain at outer surface,

$$\phi = \frac{DD'}{L} \quad \dots(i)$$

Now from Fig. 16.2 (b).

$$\text{Arc } DD' = OD \times \theta = R\theta$$

($\because OD = R = \text{Radius of shaft}$)

Substituting the value of DD' in equation (i), we get

Shear strain at outer surface

$$\phi = \frac{R \times \theta}{L} \quad \dots(ii)$$

Now the modulus of rigidity (C) of the material of the shaft is given as

$$C = \frac{\text{Shear stress induced}}{\text{Shear strain produced}} = \frac{\text{Shear stress at the outer surface}}{\text{Shear strain at outer surface}}$$

$$= \frac{\tau}{\left(\frac{R\theta}{L}\right)} \quad \left(\because \text{From equation (ii), shear strain} = \frac{R\theta}{L} \right)$$

$$= \frac{\tau \times L}{R\theta}$$

$\therefore$

$$\frac{C\theta}{L} = \frac{\tau}{R} \quad \dots(16.1)$$

$$\therefore \tau = \frac{R \times C \times \theta}{L}$$

Now for a given shaft subjected to a given torque (T), the values of C , θ and L are constant. Hence shear stress produced is proportional to the radius R .

$$\therefore \tau \propto R \quad \text{or} \quad \frac{\tau}{R} = \text{constant} \quad \dots(iii)$$

If q is the shear stress induced at a radius ' r ' from the centre of the shaft then

$$\frac{\tau}{R} = \frac{q}{r} \quad \dots(16.2)$$

But

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

From equation (16.1)

$$\therefore \frac{\tau}{R} = \frac{C\theta}{L} = \frac{q}{r} \quad \dots(16.3)$$

From equation (iii), it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

16.2.1. Assumptions Made in the Derivation of Shear Stress Produced in a Circular Shaft Subjected to Torsion. The derivation of shear stress produced in a circular shaft subjected to torsion, is based on the following assumptions :

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. The shaft is of uniform circular section throughout.
4. Cross-sections of the shaft, which are plane before twist remain plain after twist.
5. All radii which are straight before twist remain straight after twist.

16.3. MAXIMUM TORQUE TRANSMITTED BY A CIRCULAR SOLID SHAFT

The maximum torque transmitted by a circular solid shaft, is obtained from the maximum shear stress induced at the outer surface of the solid shaft. Consider a shaft subjected to a torque T as shown in Fig. 16.3.

Let τ = Maximum shear stress induced at the outer surface

R = Radius of the shaft

q = Shear stress at a radius ' r ' from the centre.

Consider an elementary circular ring of thickness ' dr ' at a distance ' r ' from the centre as shown in Fig. 16.3. Then the area of the ring,

$$dA = 2\pi r dr$$

From equation (16.2), we have

$$\frac{\tau}{R} = \frac{q}{r}$$

$\therefore$ Shear stress at the radius r ,

$$q = \frac{\tau}{R} r = \tau \frac{r}{R}$$

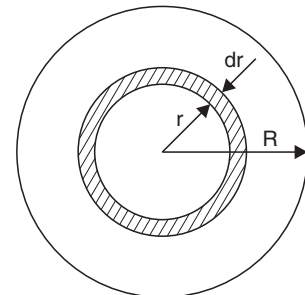


Fig. 16.3

∴ Turning force on the elementary circular ring

= Shear stress acting on the ring × Area of ring

$$= q \times dA$$

$$= \tau \times \frac{r}{R} \times 2\pi r dr \quad \left(\because q = \tau \times \frac{r}{R} \right)$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr$$

Now turning moment due to the turning force on the elementary ring,

dT = Turning force on the ring × Distance of the ring from the axis

$$= \frac{\tau}{R} \times 2\pi r^2 dr \times r$$

$$= \frac{\tau}{R} \times 2\pi r^3 dr \quad \dots [16.3 (A)]$$

∴ The total turning moment (or total torque) is obtained by integrating the above equation between the limits O and R

$$\begin{aligned} \therefore T &= \int_0^R dT = \int_0^R \frac{\tau}{R} \times 2\pi r^3 dr \\ &= \frac{\tau}{R} \times 2\pi \int_0^R r^3 dr = \frac{\tau}{R} \times 2\pi \left[\frac{r^4}{4} \right]_0^R \\ &= \frac{\tau}{R} \times 2\pi \times \frac{R^4}{4} = \tau \times \frac{\pi}{2} \times R^3 \\ &= \tau \times \frac{\pi}{2} \times \left(\frac{D}{2} \right)^3 \quad \left(\because R = \frac{D}{2} \right) \\ &= \tau \times \frac{\pi}{2} \times \frac{D^3}{8} = \tau \times \frac{\pi D^3}{16} = \frac{\pi}{16} \tau D^3 \quad \dots (16.4) \end{aligned}$$

Problem 16.1. A solid shaft of 150 mm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the maximum shear stress induced to the shaft is 45 N/mm².

Sol. Given :

Diameter of the shaft, $D = 150$ mm

Maximum shear stress, $\tau = 45$ N/mm²

Let T = Maximum torque transmitted by the shaft.

$$\begin{aligned} \text{Using equation (16.4), } T &= \frac{\pi}{16} \tau D^3 = \frac{\pi}{16} \times 45 \times 150^3 \\ &= 29820586 \text{ Nmm} = \mathbf{29820.586 \text{ Nm.}} \quad \mathbf{Ans.} \end{aligned}$$

Problem 16.2. The shearing stress in a solid shaft is not to exceed 40 N/mm² when the torque transmitted is 20000 Nm. Determine the minimum diameter of the shaft.

Sol. Given :

Maximum shear stress, $\tau = 40$ N/mm²

Torque transmitted, $T = 20000$ Nm = 20000 × 10³ Nmm

Let D = Minimum diameter of the shaft in mm.

Using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$D = \left(\frac{16T}{\pi \tau} \right)^{1/3} = \left(\frac{16 \times 20000 \times 10^3}{\pi \times 40} \right)^{1/3} = 136.2 \text{ mm. Ans.}$$

16.4. TORQUE TRANSMITTED BY A HOLLOW CIRCULAR SHAFTS

Torque transmitted by a hollow circular shaft is obtained in the same way as for a solid shaft. Consider a hollow shaft. Let it is subjected to a torque T as shown in Fig. 16.4. Take an elementary circular ring of thickness ' dr ' at a distance r from the centre as shown in Fig. 16.4.

Let R_0 = Outer radius of the shaft

R_i = Inner radius of the shaft

r = Radius of elementary circular ring

dr = Thickness of the ring

τ = Maximum shear stress induced at outer surface of the shaft

q = Shear stress induced on the elementary ring

dA = Area of the elementary circular ring
 $= 2\pi r \times dr$

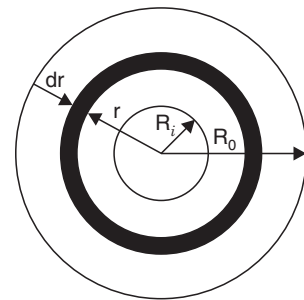


Fig. 16.4. Hollow shaft.

Shear stress at the elementary ring is obtained from equation (16.2) as

$$\frac{\tau}{R_0} = \frac{q}{r} \quad (\because \text{Here outer radius } R = R_0)$$

$$\therefore q = \frac{\tau}{R_0} \times r$$

$$\therefore \text{Turning force on the ring} = \text{Stress} \times \text{Area} = q \times dA$$

$$\begin{aligned} &= \frac{\tau}{R_0} r \times 2\pi r dr \\ &= 2\pi \frac{\tau}{R_0} r^2 dr \end{aligned} \quad \left(\because q = \frac{\tau}{R_0} r \right)$$

Turning moment (dT) on the ring,

$$dT = \text{Turning force} \times \text{Distance of the ring from centre}$$

$$= 2\pi \frac{\tau}{R_0} r^2 dr \times r = 2\pi \frac{\tau}{R_0} r^3 dr$$

The total turning moment (or total torque T) is obtained by integrating the above equation between the limits R_i and R_0 .

$$\begin{aligned} \therefore T &= \int_{R_i}^{R_0} dT = \int_{R_i}^{R_0} 2\pi \frac{\tau}{R_0} r^3 dr \\ &= 2\pi \frac{\tau}{R_0} \int_{R_i}^{R_0} r^3 dr \end{aligned}$$

($\because \tau$ and R_0 are constant and can be taken outside the integral)

$$\begin{aligned}
&= 2\pi \frac{\tau}{R_0} \left[\frac{r^4}{4} \right]_{R_i}^{R_0} = 2\pi \frac{\tau}{R_0} \left[\frac{R_0^4 - R_i^4}{4} \right] \\
&= \frac{\pi}{2} \tau \left[\frac{R_0^4 - R_i^4}{R_0} \right] \quad \dots(16.5)
\end{aligned}$$

Let D_0 = Outer diameter of the shaft
 D_i = Inner diameter of the shaft.

Then $R_0 = \frac{D_0}{2}$ and $R_i = \frac{D_i}{2}$.

Substituting the values of R_0 and R_i in equation (16.5),

$$\begin{aligned}
T &= \frac{\pi}{2} \tau \left[\frac{\left(\frac{D_0}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4}{\left(\frac{D_0}{2}\right)} \right] = \frac{\pi}{2} \tau \left[\frac{\frac{D_0^4}{16} - \frac{D_i^4}{16}}{\frac{D_0}{2}} \right] \\
&= \frac{\pi}{2} \tau \left[\frac{D_0^4 - D_i^4}{16} \times \frac{2}{D_0} \right] \\
&= \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0} \right] \quad \dots(16.6)
\end{aligned}$$

16.5. POWER TRANSMITTED BY SHAFTS

Once the expression for torque (T) for a solid or a hollow shaft is obtained, power transmitted by the shafts can be determined.

Let N = r.p.m. of the shaft
 T = Mean torque transmitted in Nm
 ω = Angular speed of shaft.

Then Power = $\frac{2\pi NT}{60}$ watts ...(16.7)

$$= \omega \times T \quad \left(\because \frac{2\pi N}{60} = \omega \right)$$

$$= T \times \omega \quad \dots[16.7 (A)]$$

Problem 16.3. In a hollow circular shaft of outer and inner diameters of 20 cm and 10 cm respectively, the shear stress is not to exceed 40 N/mm². Find the maximum torque which the shaft can safely transmit.

Sol. Given :

Outer diameter, $D_0 = 20 \text{ cm} = 200 \text{ mm}$

Inner diameter, $D_i = 10 \text{ cm} = 100 \text{ mm}$

Maximum shear stress, $\tau = 40 \text{ N/mm}^2$

Let T = Maximum torque transmitted by the shaft.

*The torque T obtained by this formula is the average (or mean) torque.

Using equation (16.6),

$$\begin{aligned} T &= \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0} \right] = \frac{\pi}{16} \times 40 \left[\frac{200^4 - 100^4}{200} \right] \\ &= \frac{\pi}{16} \times 40 \left[\frac{16 \times 10^8 - 1 \times 10^8}{200} \right] = 58904860 \text{ Nmm} \\ &= \mathbf{58904.86 \text{ Nm.}} \quad \mathbf{Ans.} \end{aligned}$$

Problem 16.4. Two shafts of the same material and of same lengths are subjected to the same torque, if the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal diameter is $2/3$ of the outside diameter and the maximum shear stress developed in each shaft is the same, compare the weights of the shafts.

Sol. Given :

Two shafts of the same material and same lengths (one is solid and other is hollow) transmit the same torque and develops the same maximum stress.

Let T = Torque transmitted by each shaft

τ = Max. shear stress developed in each shaft

D = Outer diameter of the solid shaft

D_0 = Outer diameter of the hollow shaft

D_i = Inner diameter of the hollow shaft $= \frac{2}{3} D_0$

W_s = Weight of the solid shaft

W_h = Weight of the hollow shaft

L = Length of each shaft

w = Weight density of the material of each shaft.

Torque transmitted by the solid shaft is given by equation (16.4)

$$T = \frac{\pi}{16} \tau D^3 \quad \dots(i)$$

Torque transmitted by the hollow shaft is given by equation (16.6),

$$\begin{aligned} T &= \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0} \right] = \frac{\pi}{16} \tau \left[\frac{D_0^4 - (2/3 D_0)^4}{D_0} \right] \\ &= \frac{\pi}{16} \tau \left[\frac{D_0^4 - \frac{16}{81} D_0^4}{D_0} \right] = \frac{\pi}{16} \tau \times \frac{65 D_0^4}{81 \times D_0} \\ &= \frac{\pi}{16} \tau \times \frac{65 D_0^3}{81} \quad \dots(ii) \end{aligned}$$

As torque transmitted by solid and hollow shafts are equal, hence equating equations (i) and (ii),

$$\frac{\pi}{16} \tau D^3 = \frac{\pi}{16} \tau \times \frac{65}{81} D_0^3$$

Cancelling $\frac{\pi}{16} \tau$ to both sides

or
$$D^3 = \frac{65}{81} D_0^3$$

$$\therefore D = \left[\frac{65}{81} D_0^3 \right]^{1/3} = \left(\frac{65}{81} \right)^{1/3} D_0 = 0.929 D_0 \quad \dots(iii)$$

Now weight of solid shaft, $W_s = \rho \times g \times \text{Volume of solid shaft}$
 $= w \times \text{Area of cross-section} \times \text{Length}$
 $= w \times \frac{\pi}{4} D^2 \times L \quad \dots(iv)$

Weight of hollow shaft,

$$\begin{aligned} W_h &= \rho \times g \times \text{Area of cross-section of hollow shaft} \times \text{Length} \\ &= w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L = w \times \frac{\pi}{4} [D_0^2 - (2/3 D_0)^2] \times L \\ &= w \times \frac{\pi}{4} \left[D_0^2 - \frac{4}{9} D_0^2 \right] \times L = w \times \frac{\pi}{4} \times \frac{5}{9} D_0^2 \times L \quad \dots(v) \end{aligned}$$

Dividing equation (iv) by equation (v),

$$\begin{aligned} \frac{W_s}{W_h} &= \frac{w \times \frac{\pi}{4} D^2 \times L}{w \times \frac{\pi}{4} \times \frac{5}{9} D_0^2 \times L} = \frac{9D^2}{5D_0^2} \\ &= \frac{9}{5} \times \frac{(0.929 D_0)^2}{D_0^2} \quad [\because D = 0.929 D_0 \text{ from equation (iii)}] \\ &= \frac{9}{5} \times 0.929^2 \times \frac{D_0^2}{D_0^2} = \frac{1.55}{1} \end{aligned}$$

$$\therefore \frac{\text{Weight of solid shaft}}{\text{Weight of hollow shaft}} = \frac{1.55}{1} \quad \text{Ans.}$$

Problem 16.5. A hollow shaft of external diameter 120 mm transmits 300 kW power at 200 r.p.m. Determine the maximum internal diameter if the maximum stress in the shaft is not to exceed 60 N/mm².

Sol. Given :

External dia., $D_0 = 120$ mm
 Power, $P = 300$ kW = 300,000 W
 Speed, $N = 200$ r.p.m.
 Max. shear stress, $\tau = 60$ N/mm²

Let D_i = Internal dia. of shaft

Using equation (16.7),

$$\begin{aligned} P &= \frac{2\pi NT}{60} \quad \text{or} \quad 300,000 = \frac{2\pi \times 200 \times T}{60} \\ \therefore T &= \frac{300,000 \times 60}{2\pi \times 200} = 14323.9 \text{ Nm} \\ &= 14323.9 \times 1000 \text{ Nmm} = 14323900 \text{ Nmm} \end{aligned}$$

Now using equation (16.6),

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$\text{or} \quad 14323900 = \frac{\pi}{16} \times 60 \times \frac{(120^4 - D_i^4)}{120}$$

$$\text{or} \quad \frac{14323900 \times 16 \times 120}{\pi \times 60} = 120^4 - D_i^4$$

$$145902000 = 207360000 - D_i^4$$

$$\text{or} \quad D_i^4 = 207360000 - 145902000 = 61458000$$

$$\therefore D_i = (61458000)^{1/4} = \mathbf{88.5 \text{ mm. Ans.}}$$

Problem 16.6. Find the maximum shear stress induced in a solid circular shaft of diameter 15 cm when the shaft transmits 150 kW power at 180 r.p.m.

Sol. Given :

Diameter of shaft, $D = 15 \text{ cm} = 150 \text{ mm}$

Power transmitted, $P = 150 \text{ kW} = 150 \times 10^3 \text{ W}$

Speed of shaft, $N = 180 \text{ r.p.m.}$

Let τ = Maximum shear stress induced in the shaft

Power transmitted is given by equation (16.7) as

$$P = \frac{2\pi NT}{60}$$

$$150 \times 10^3 = \frac{2\pi \times 180 \times T}{60}$$

$$\therefore T = \frac{150 \times 10^3 \times 60}{2\pi \times 180} = 7957.7 \text{ Nm} = 7957700 \text{ Nmm}$$

Now using equation (16.4) as,

$$T = \frac{\pi}{16} \tau D^3$$

$$7957700 = \frac{\pi}{16} \times \tau \times 150^3$$

$$\therefore \tau = \frac{16 \times 7957700}{\pi \times 150^3} = \mathbf{12 \text{ N/mm}^2. \text{ Ans.}}$$

Problem 16.7. A solid steel shaft has to transmit 75 kW at 200 r.p.m. Taking allowable shear stress as 70 N/mm^2 , find suitable diameter for the shaft, if the maximum torque transmitted at each revolution exceeds the mean by 30%.

Sol. Given :

Power transmitted, $P = 75 \text{ kW} = 75 \times 10^3 \text{ W}$

R.P.M. of the shaft, $N = 200$

Shear stress, $\tau = 70 \text{ N/mm}^2$

Let T = Mean torque transmitted

$T_{\max}$ = Maximum torque transmitted = $1.3 T$

D = Suitable diameter of the shaft

Power is given by the relation,

$$= \frac{2\pi NT}{60}$$

$$\text{or} \quad 75 \times 10^3 = \frac{2\pi \times 200 \times T}{60}$$

$$\therefore T = \frac{75 \times 10^3 \times 60}{2\pi \times 200} = 3580.98 \text{ Nm} = 3580980 \text{ Nmm}$$

$$\therefore T_{max} = 1.3 T = 1.3 \times 3580980 = 4655274 \text{ Nmm.}$$

Maximum torque transmitted by a solid shaft is given by equation (16.4) as,

$$T_{max} = \frac{\pi}{16} \times \tau \times D^3$$

or
$$4655274 = \frac{\pi}{16} \times 70 \times D^3$$

$$\therefore D = \left(\frac{16 \times 4655274}{\pi \times 70} \right)^{1/3} = 69.57 \text{ mm} \simeq 70 \text{ mm. Ans.}$$

16.5.1. Expression for Torque in terms of Polar Moment of Inertia. Polar moment of inertia of a plane area is defined as the moment of inertia of the area about an axis perpendicular to the plane of the figure and passing through the C.G. of the area. It is denoted by symbol J .

The torque in terms of polar moment of inertia (J) is obtained from equation [16.3 (A)] of Art. 16.3.

The moment (dT) on the circular ring is given by equation [16.3 (A)] as

$$\begin{aligned} dT &= \frac{\tau}{R} 2\pi r^3 dr = \frac{\tau}{R} 2\pi r \times r^2 dr = \frac{\tau}{R} r^2 \times 2\pi r \times dr \\ &= \frac{\tau}{R} r^2 dA \quad (\because dA = 2\pi r dr \text{ see Fig. 16.3}) \end{aligned}$$

$$\therefore \text{Total torque, } T = \int_0^R dT = \int_0^R \frac{\tau}{R} r^2 dA = \frac{\tau}{R} \int_0^R r^2 dA \quad \dots(i)$$

But $r^2 dA =$ Moment of inertia of the elementary ring about an axis perpendicular to the plane of Fig. 16.3 and passing through the centre of the circle.

$\therefore \int_0^R r^2 dA =$ Moment of inertia of the circle about an axis perpendicular to the plane of the circle and passing through the centre of the circle

$$= \text{Polar moment of inertia } (J) = \frac{\pi}{32} D^4.$$

Hence equation (i) becomes as

$$\begin{aligned} T &= \frac{\tau}{R} \times J & \left(\because J = \int_0^R r^2 dA \right) \\ \therefore \frac{T}{J} &= \frac{\tau}{R} & \dots(16.8) \end{aligned}$$

But from equation (16.1), we have

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$\therefore \frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L} \quad \dots(16.9)$$

where C = Modulus of rigidity

θ = Angle of twist in radiation

L = Length of the shaft.

16.5.2. Polar Modulus. Polar modulus is defined as the ratio of the polar moment of inertia to the radius of the shaft. It is also called torsional section modulus. It is denoted by Z_p . Mathematically,

$$Z_p = \frac{J}{R}$$

$$(a) \text{ For a solid shaft, } J = \frac{\pi}{32} D^4$$

$$\therefore Z_p = \frac{\frac{\pi}{32} D^4}{R} = \frac{\frac{\pi}{32} D^4}{D/2} = \frac{\pi}{16} D^3 \quad \dots(16.10)$$

$$(b) \text{ For a hollow shaft, } J = \frac{\pi}{32} (D_0^4 - D_i^4) \quad \dots(16.11)$$

$$\therefore Z_p = \frac{\frac{\pi}{32} [D_0^4 - D_i^4]}{R} \quad \text{(Here } R \text{ is the outer radius)}$$

$$\left(\because R = \frac{D_0}{2} \right)$$

$$= \frac{\frac{\pi}{32} [D_0^4 - D_i^4]}{D_0/2} = \frac{\pi}{16D_0} \times [D_0^4 - D_i^4] \quad \dots(16.12)$$

Problem 16.8. Determine the diameter of a solid shaft which will transmit 300 kW at 250 r.p.m. The maximum shear stress should not exceed 30 N/mm² and twist should not be more than 1° in a shaft length of 2 m. Take modulus of rigidity = 1 × 10⁵ N/mm².

Sol. Given :

Power transmitted, $P = 300 \text{ kW} = 300 \times 10^3 \text{ W}$

Speed of the shaft, $N = 250 \text{ r.p.m.}$

Maximum shear stress, $\tau = 30 \text{ N/mm}^2$

Twist in shaft, $\theta = 1^\circ = \frac{\pi}{180} = 0.01745 \text{ radian}$

Length of shaft, $L = 2 \text{ m} = 2000 \text{ mm}$

Modulus of rigidity, $C = 1 \times 10^5 \text{ N/mm}^2$

Let D = Diameter of the shaft.

Power is given by the relation,

$$P = \frac{2\pi NT}{60}$$

$$\text{or } 300 \times 10^3 = \frac{2\pi \times 250 \times T}{60}$$

$$\therefore T = \frac{300 \times 10^3 \times 60}{2\pi \times 250} = 11459.1 \text{ Nm} = 11459.1 \times 10^3 \text{ Nmm}$$

(i) Diameter of the shaft when maximum shear stress,

$$\tau = 30 \text{ N/mm}^2$$

Maximum torque transmitted by a solid shaft is given by equation (16.4) as

$$T = \frac{\pi}{16} \times \tau \times D^3$$

$$\therefore 11459100 = \frac{\pi}{16} \times 30 \times D^3$$

$$\therefore D = \left(\frac{16 \times 11459100}{\pi \times 30} \right)^{1/3} = 124.5 \text{ mm} \quad \dots(i)$$

(ii) Diameter of shaft when twist should not be more than 1° .

Using equation (16.9),

$$\frac{T}{J} = \frac{C\theta}{L}$$

where J = Polar moment of inertia of solid shaft

$$= \frac{\pi}{32} D^4$$

$$\therefore \frac{11459100}{\frac{\pi}{32} D^4} = \frac{10^5 \times 0.01745}{2000}$$

$$\therefore D^4 = \frac{32 \times 2000 \times 11459100}{10^5 \times \pi \times 0.01745} = 13377.81 \times 10^4$$

$$\therefore D = (13377.81 \times 10^4)^{1/4} = 107.5 \text{ mm} \quad \dots(ii)$$

The suitable diameter of the shaft is the greater* of the two values given by equations (i) and (ii).

$\therefore$ Diameter of the shaft = **124.5 mm say 125 mm. Ans.**

*(If diameter is taken smaller of the two values say 107.5 mm, then from equation

$T = \frac{\pi}{16} \tau D^3$, the value of shear stress will be

$$11459100 = \frac{\pi}{16} \tau \times (107.5)^3$$

$$\text{or} \quad 11459100 = 243920 \tau$$

$$\text{or} \quad \tau = \frac{11459100}{243920} = 46.978 \text{ N/mm}^2$$

which is more than the given value of 30 N/mm²).

Problem 16.9. In a tensile test, a tensile piece 25 mm in diameter, 200 mm gauge length stretched 0.0975 mm under a pull of 50,000 N. In a torsion test, the same rod twisted 0.025 radians over a length of 200 mm, when a torque of 400 Nm was applied. Evaluate the Poisson's ratio and the three elastic moduli for the material.

(U.P. Tech. University, May 2009)

Sol. Given:

Tensile Test

$$\begin{aligned}d &= 25 \text{ mm} \\L &= 200 \text{ mm} \\dL &= 0.0975 \text{ mm} \\P &= 50,000 \text{ N}\end{aligned}$$

Torsion Test

$$\begin{aligned}d &= 25 \text{ mm} \\ \theta &= 0.025 \text{ radian} \\L &= 200 \text{ mm} \\T &= 400 \text{ Nm} \\&= 400 \times 1000 \text{ Nmm} \\&= 400,000 \text{ Nmm}\end{aligned}$$

Find : μ , E , C and K

Tensile Test

$$\begin{aligned}E &= \frac{\text{Stress}}{\text{Strain}} = \frac{\left(\frac{P}{\frac{\pi}{4}d^2}\right)}{\left(\frac{\delta L}{L}\right)} = \frac{\frac{50,000}{\frac{\pi}{4} \times 25^2}}{\left(\frac{0.0975}{200}\right)} \text{ N/mm}^2 \\&= \frac{50,000 \times 4}{\pi \times 625} \times \frac{200}{0.0975} \text{ N/mm}^2 \\ \text{Stress} &= \frac{\text{Load}}{\text{Area}} = \frac{50,000}{\frac{\pi}{4}d^2} = \frac{50,000 \times 4}{\pi \times 25^2} \text{ N/mm}^2 \\&= 101.86 \text{ N/mm}^2 \\ \text{Strain} &= \frac{\delta L}{L} = \frac{0.0975}{200} = 0.0004875 \\ \therefore E &= \frac{\text{Stress}}{\text{Strain}} = \frac{101.86}{0.0004875} = \mathbf{208943.6 \text{ N/mm}^2. \text{ Ans.}}\end{aligned}$$

Torsion Test

Using $\frac{T}{J} = \frac{C \times \theta}{L}$

where $T = 400,000 \text{ Nmm}$
 $\theta = 0.025 \text{ radian}$
 $L = 200 \text{ mm}$

$$\begin{aligned}J &= \frac{\pi}{32}d^4 = \frac{\pi}{32} \times 25^4 = 38349.5 \text{ mm}^4 \\ \therefore \frac{400,000}{38349.5} &= \frac{C \times 0.025}{200} \\ \therefore C &= \frac{400,000 \times 200}{38349.5 \times 0.025} = \mathbf{83443 \text{ N/mm}^2. \text{ Ans.}}\end{aligned}$$

Value of μ

The relation between E and C is given by

$$\begin{aligned}E &= 2C(1 + \mu) \\ \therefore 208943.6 &= 2 \times 83443(1 + \mu)\end{aligned}$$

$$\text{or} \quad \frac{208943.6}{2 \times 83443} = 1 + \mu$$

$$\text{or} \quad 1.252 = 1 + \mu$$

$$\therefore \mu = 1.252 - 1 = \mathbf{0.252. \quad Ans.}$$

Value of K (bulk modulus)

The relation between E and K is given by

$$E = 3K(1 - 2\mu)$$

$$\therefore 208943.6 = 3 \times K(1 - 2 \times 0.252)$$

$$= 3K \times 0.496$$

$$\therefore K = \frac{208943.6}{3 \times 0.496} = \mathbf{140419 \text{ N/mm}^2. \quad Ans.}$$

HIGHLIGHTS

1. A shaft is in torsion, when equal and opposite torques are applied at the two ends of a shaft.
2. The relation of maximum shear stress induced in a shaft subjected to twisting moment is given by

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

where τ = Maximum shear stress,

R = Radius of shaft,

C = Modulus of rigidity,

θ = Angle of twist in radian, and

L = Length of the shaft.

3. When a circular shaft is subjected to torsion, the shear stress at any point varies linearly from the axis to the surface *i.e.*,

$$\frac{\tau}{R} = \frac{q}{r}$$

where τ = Maximum shear stress on the surface of the shaft

R = Radius of surface

q = Shear stress at a point which is at a radius ' r '.

4. The shear stress is maximum on the surface of the shaft and is zero at the axis of the shaft.
5. The torque transmitted by a solid shaft is given by

$$T = \frac{\pi}{16} \tau D^3$$

where D = Dia. of solid shaft and

τ = Max. shear stress.

6. The torque transmitted by a hollow circular shaft is given by

$$T = \frac{\pi}{16} \tau \left(\frac{D_0^4 - D_i^4}{D_0} \right)$$

where D_0 = External diameter, and

D_i = Internal diameter.

7. Relation between torque, polar moment of inertia and shear stress is given as

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L}$$

where $I =$ Polar moment of inertia

$$= \frac{\pi}{32} D^4 \quad \dots \text{For a solid shaft}$$

$$= \frac{\pi}{32} [D_0^4 - D_1^4] \quad \dots \text{For a hollow shaft.}$$

8. Strength of a shaft means the maximum torque or maximum power the shaft can transmit.
 9. The product of modulus of rigidity and polar moment of inertia of a shaft is known as torsional rigidity or stiffness of the shaft. Mathematically,

Torsional rigidity $= C \times J$

$$= \frac{T \times L}{\theta} \quad \left(\because \frac{T}{J} = \frac{C\theta}{L} \therefore C \times J = \frac{T \times L}{\theta} \right)$$

$= T$ if $L = 1$ metre and $\theta =$ one radian.

10. The power transmitted by a shaft is given by

$$P = \frac{2\pi NT}{60}$$

11. Polar moment of inertia (J) is given by

$$J = \frac{\pi}{32} D^4 \quad \dots \text{For a solid shaft}$$

$$= \frac{\pi}{32} (D^4 - d^4) \quad \dots \text{For hollow shaft.}$$

EXERCISE 16

(A) Theoretical Questions

1. State the assumptions made in the theory of pure torsion. (*U.P. Tech. University, May 2009*)
2. Derive an expression for the shear stress produced in a circular shaft which is subject to torsion. What are the assumptions made in the derivation?
3. Prove that the torque transmitted by a solid shaft when subjected to torsion is given by

$$T = \frac{\pi}{16} \tau D^3$$

where $D =$ Dia. of solid shaft and $\tau =$ Max. shear stress.

4. When a circular shaft is subjected to torsion show that the shear stress varies linearly from the axis to the surface.
5. Derive the relation for a circular shaft when subjected to torsion as given below

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L}$$

where $T =$ Torque transmitted,

$J =$ Polar moment of inertia,

$\tau =$ Max. shear stress,

$R =$ Radius of the shaft,

$C =$ Modulus rigidity,

$\theta =$ Angle of twist, and

$L =$ Length of the shaft.

6. Define the term 'Polar modulus'. Find the expressions for polar modulus for a solid shaft and for a hollow shaft.
7. What do you mean by 'strength of a shaft' ?
8. Define torsional rigidity of a shaft. Prove that the torsional rigidity is the torque required to produce a twist of one radian in a unit length of the shaft.

(B) Numerical Problems

1. A solid shaft of 20 cm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the maximum shear stress induced in the shaft is 50 N/mm^2 . [Ans. 78539.8 Nm]
2. The shearing stress in a solid shaft is not to exceed 45 N/mm^2 when the torque transmitted is 40000 Nm. Determine the minimum diameter of the shaft. [Ans. 16.49 mm]
3. Find the maximum torque transmitted by a hollow circular shaft of external diameter 30 cm and internal diameter 15 cm if the shear stress is not to exceed 40 N/mm^2 . [Ans. 198.8 kN]
4. Two shafts of the same material and of same lengths are subjected to the same torque, if the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal diameter is 0.7 times the outside diameter and the maximum shear stress developed in each shaft is the same, compare the weights of the shafts. [Ans. $\frac{1.633}{1}$]
5. Find the maximum shear stress induced in a solid circular shaft of diameter 20 cm when the shaft transmit 187.5 kW at 200 r.p.m. [Ans. 5.7 N/mm^2]
6. A solid circular shaft is to transmit 375 kW at 150 r.p.m.
 - (i) Find the diameter of the shaft if the shear stress is not to exceed 65 N/mm^2 .
 - (ii) What percent saving in weight would be obtained if this shaft is replaced by a hollow shaft whose internal diameter equal to $\frac{2}{3}$ rd of its external diameter, the length, the material and maximum shear stress being the same ? [Ans. (i) 12.29 cm, (ii) 35.71%]
7. Determine the diameter of a solid shaft which will transmit 337.5 kW at 300 r.p.m. The maximum shear stress should not exceed 35 N/mm^2 and twist should not be more than 1° in a shaft length of 2.5 m. Take modulus of rigidity = $9 \times 10^4 \text{ N/mm}^2$. [Ans. 11.57 cm]

OBJECTIVE TYPE QUESTIONS

Objective Type Questions

UNIT 1. FORCE SYSTEMS AND FRICTION

(A) Force Systems

1. Triangle law of forces states that if two forces acting at a point are represented in magnitude and direction by the two sides of the triangle taken in order, then their resultant is given by the
 - (a) third side of the triangle taken in the same order
 - (b) third side of the triangle taken in the opposite order
 - (c) sum of the two forces acting
 - (d) none of the above.
2. Law of polygon of forces states that
 - (a) if a number of forces acting at a point are represented by the sides of a polygon taken in order, then their resultant is represented in magnitude and direction by the closing side of the polygon, taken in the same order.
 - (b) if a number of forces acting at a point are represented in magnitude and direction by the sides of a polygon taken in order, then their resultant is represented in magnitude and direction by the closing side of the polygon, taken in the opposite order.
 - (c) the resultant of a number of forces acting on a point is the sum of all forces.
 - (d) none of the above.
3. Two forces P and Q are acting at an angle θ , their resultant (R) is given by
 - (a) $R = \sqrt{P^2 + Q^2 + 2AB \sin 2\theta}$
 - (b) $R = \sqrt{P^2 + Q^2 + 2AB \cos \theta}$
 - (c) $R = \sqrt{P^2 + Q^2 - 2AB \cos \theta}$
 - (d) $R = \sqrt{P^2 + Q^2 + 2AB \cos 2\theta}$.
4. Two forces A and B are acting at an angle θ and their resultant R makes an angle α with the force A , then
 - (a) $\tan \alpha = \frac{B \sin \theta}{B + A \cos \theta}$
 - (b) $\tan \alpha = \frac{A \sin \theta}{A + B \cos \theta}$
 - (c) $\tan \alpha = \frac{B \sin \theta}{A + B \cos \theta}$
 - (d) $\tan \alpha = \frac{A \cos \theta}{B + A \sin \theta}$.
5. Two forces A and B are acting at an angle θ and their resultant R makes an angle α with the force A , then
 - (a) $\cos \alpha = \frac{A + B \sin \theta}{\sqrt{A^2 + B^2 - 2AB \cos \theta}}$
 - (b) $\cos \alpha = \frac{A + B \sin \theta}{\sqrt{A^2 + B^2 + 2AB \cos \theta}}$
 - (c) $\cos \alpha = \frac{B \sin \theta}{\sqrt{A^2 + B^2 + 2AB \cos \theta}}$
 - (d) none of the above.

6. Lami's theorem states that if
- (a) three forces acting at a point are in equilibrium, they can be represented by the three sides of a triangle.
 - (b) the three forces acting at a point can be represented in magnitude and direction by the sides of a triangle, the forces are in equilibrium.
 - (c) three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.
 - (d) none of the above.
7. The forces which do not meet at a point are called
- (a) non-coplanar forces
 - (b) coplanar forces
 - (c) non-concurrent forces
 - (d) concurrent forces.
8. The forces whose lines of action do not lie in the same plane, are called
- (a) non-coplanar forces
 - (b) coplanar forces
 - (c) non-concurrent forces
 - (d) none of the above.
9. The forces, whose lines of action lie on the same line, are known as
- (a) coplanar forces
 - (b) concurrent forces
 - (c) collinear forces
 - (d) none of the above.
10. The forces, whose lines of action does not lie in the same plane but are meeting at one point, are known as
- (a) coplanar concurrent forces
 - (b) non-coplanar concurrent forces
 - (c) non-coplanar non-concurrent forces
 - (d) none of the above.
11. The forces, whose lines of action lie in the same plane and are meeting at one point, are known as
- (a) coplanar concurrent forces
 - (b) coplanar non-concurrent forces
 - (c) non-coplanar concurrent forces
 - (d) none of the above.
12. Coplanar concurrent forces means the lines of action of forces
- (a) lie in the same plane
 - (b) lie in the same plane but the forces are not meeting at one point
 - (c) lie in the same plane and forces are meeting at one point
 - (d) none of the above.
13. The forces, which meet at a point, are known as
- (a) collinear forces
 - (b) coplanar forces
 - (c) concurrent forces
 - (d) none of the above.
14. The forces, which lies in the same plane, are called
- (a) collinear forces
 - (b) coplanar forces
 - (c) concurrent forces
 - (d) none of the above.
15. Tick mark the correct statement.
- (a) The algebraic sum of the resolved parts of a number of force in a given direction is equal to their resultant.
 - (b) The algebraic sum of the resolved parts of a number of force in a given direction is equal to two times their resultant.
 - (c) The algebraic sum of the resolved parts of a number of forces in a given direction is equal to the resolved component of the resultant in that direction.
 - (d) None of the above.

16. The statement—if three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, the forces are in equilibrium—is known as
(a) Lami's theorem (b) Law of polygon of forces
(c) Law of triangle of forces (d) Newton's law of forces.
17. The statement—the algebraic sum of the moments taken about any point in the plane of forces is zero—is known as
(a) Law of polygon of forces (b) Lami's theorem
(c) Newton's law of forces (d) Law of moments.
18. Two couples will balance one another when they are in the same plane and
(a) have equal moments and their direction of rotation is same
(b) have unequal moments and their direction of rotation is opposite
(c) have equal moments and their direction of rotation is opposite
(d) none of the above.

(B) Friction

19. Limiting force of friction is defined as the frictional force which exists when a body
(a) is moving with maximum velocity (b) is stationary
(c) just begins to slide over the surface (d) none of the above.
20. Co-efficient of friction is the ratio of
(a) force of friction to reaction between two bodies
(b) force of friction to normal reaction between two bodies
(c) force of limiting friction to reaction between two bodies
(d) force of limiting friction to normal reaction between two bodies.
21. Angle of friction (ϕ) is the angle between the
(a) limiting friction and normal reaction
(b) limiting friction and the resultant of limiting friction and normal reaction
(c) normal reaction and the resultant of limiting friction and normal reaction
(d) none of the above.
22. The co-efficient of friction (μ) in terms of angle of friction (ϕ) is given by
(a) $\phi = \tan \mu$ (b) $\mu = \sin \phi$
(c) $\mu = \tan \phi$ (d) $\mu = \frac{1}{\tan \phi}$.
23. The force of friction which exists when the body is in motion is called
(a) static friction (b) limiting friction
(c) dynamic friction (d) none of the above.
24. Dynamic friction is always
(a) more than static friction (b) more than limiting friction
(c) less than limiting friction (d) none of the above.
25. A body of weight W is resting on a horizontal plane. A force P is applied parallel to the plane to move the body. The value of P , necessary to move the body against the resistance of friction is

- (a) $W/\tan \phi$ (b) $W \sin \phi$ (c) $W \tan \phi$ (d) $W \cos \phi$

where ϕ = Angle of friction.

- 26.** The force P is applied at an angle θ with the horizontal plane on which a body of weight W is placed as shown in Fig. 1. The value of P , necessary to move the body is equal to

- (a) $\frac{W \cos \phi}{\cos (\theta - \phi)}$ (b) $W \frac{\sin \phi}{\cos (\theta - \phi)}$
 (c) $\frac{W \tan \phi}{\sin (\theta - \phi)}$ (d) none of the above

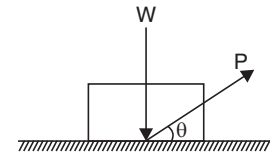


Fig. 1

where ϕ = Angle of friction.

- 27.** Refer to Fig. 1, the necessary force P to move the body will be minimum, when

- (a) $\theta = 2\phi$ (b) $\theta = \phi$
 (c) $\theta = \phi/2$ (d) none of the above

where ϕ = Angle of friction.

- 28.** Refer to Fig. 1, the minimum force P to move the body is

- (a) $W \tan \phi$ (b) $W \cos \phi$
 (c) $W \sin \phi$ (d) $W/\sin \phi$

where ϕ = Angle of friction.

- 29.** The necessary force P , applied at an angle θ with the horizontal plane, on which a body of weight W is placed, to move the body is (see Fig. 2)

- (a) $\frac{W \cos \phi}{\cos (\theta - \phi)}$ (b) $\frac{W \sin \phi}{\cos (\theta - \phi)}$
 (c) $\frac{W \tan \phi}{\cos (\theta + \phi)}$ (d) $\frac{W \tan \phi}{\cos (\theta - \phi)}$

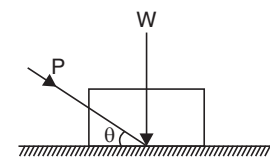


Fig. 2

where ϕ = Angle of friction.

- 30.** Refer to Fig. 2, the minimum force P to move the body is equal to

- (a) $W \tan \phi$ (b) $W \sin \phi$
 (c) $W \cos \phi$ (d) $W/\tan \phi$

where ϕ = Angle of friction.

- 31.** Refer to Fig. 2, the force P will be minimum when

- (a) $\theta = 2\phi$ (b) $\theta = \phi$
 (c) $\theta = (90 - \phi)$ (d) $\theta = (90 - 2\phi)$.

- 32.** The necessary force P , applied parallel to an inclined plane having inclination α with horizontal to move a body of weight W , up the plane is equal to (Refer to Fig. 3)

- (a) $\frac{W \sin \alpha}{\cos \phi}$ (b) $\frac{W \sin (\alpha + \phi)}{\cos \phi}$
 (c) $\frac{W \cos (\alpha + \phi)}{\cos \phi}$ (d) $\frac{W \tan (\alpha + \phi)}{\cos \phi}$

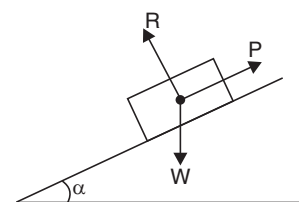


Fig. 3

where ϕ = Angle of friction.

33. Refer to Fig. 3, if the body is on the point of moving down the plane, then necessary force P will be equal to

(a) $\frac{W \sin (\alpha + \phi)}{\cos \phi}$	(b) $\frac{W \sin (\alpha + \phi)}{\tan \phi}$
(c) $\frac{W \sin (\alpha - \phi)}{\cos \phi}$	(d) $\frac{W \sin (\alpha - \phi)}{\tan \phi}$

34. Fig. 4 shows a body of weight W placed on an inclined plane having inclination α with the horizontal. Force P is applied horizontally. When the body is on the point of moving up the plane, the necessary value of P is

(a) $W \sin (\alpha + \phi)$	(b) $W \tan (\alpha + \phi)$
(c) $W \cos (\alpha + \phi)$	(d) $W \sin (\alpha - \phi)$

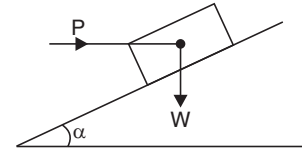


Fig. 4

35. Refer to Fig. 4, if the body is on the point of moving down the plane, the necessary force P would be

(a) $W \sin (\alpha - \phi)$	(b) $W \tan (\alpha + \phi)$
(c) $W \tan (\alpha - \phi)$	(d) $W \sin (\alpha + \phi)$

36. A ladder of weight 250 N is placed against a smooth vertical wall and a rough horizontal floor ($\mu = 0.3$) as shown in Fig. 5. If the ladder is on the point of sliding, the reaction at A will be

(a) 250 N	(b) 261 N
(c) 125 N	(d) 500 N.

37. Refer to Fig. 5, the reaction at B will be

(a) 250 N	(b) 125 N
(c) 75 N	(d) 500 N.

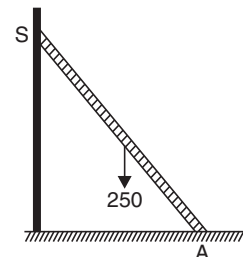


Fig. 5

38. If in question 36, a man of weight 500 N stands on the ladder at its middle point. The reaction at A will be (when ladder is on the point of sliding)

(a) 750 N	(b) 783 N
(c) 250 N	(d) 1000 N.

39. For question 38, the reaction at B will be

(a) 750 N	(b) 783 N
(c) 225 N	(d) 1000 N.

40. A horizontal force of 400 N is applied on a body of weight 1200 N, placed on a horizontal plane. If the body is just on the point of motion, the angle of friction would be

(a) 20°	(b) $18^\circ 26'$
(c) 10°	(d) 25° .

41. A body of weight W is placed on an inclined plane. The angle made by the inclined plane with horizontal, when the body is on the point of moving down is called

(a) angle of inclination	(b) angle of repose
(c) angle of friction	(d) angle of limiting friction.

UNIT 2. BASIC STRUCTURAL ANALYSIS

(A) Plane Trusses

1. The number of members (n) and number of joints (j) in a perfect frame is given by
 (a) $n = (3j - 2)$ (b) $n = (2j - 3)$
 (c) $j = (2n - 3)$ (d) $j = (3n - 2)$.
2. A frame, which has got less number of members than given by the formula $n = (2j - 3)$, is called a
 (a) perfect frame (b) deficient frame
 (c) redundant frame (d) none of the above.
3. A frame, which has got more number of members than given by the formula $n = (2j - 3)$, is called a
 (a) perfect frame (b) deficient frame
 (c) redundant frame (d) none of the above.
4. A frame, which has got the number of members equal to the number of members given by $n = (2j - 3)$, is called a
 (a) perfect frame (b) deficient frame
 (c) redundant frame (d) none of the above.
5. The resultant of two forces each equal to $\frac{P}{4}$ and acting at right angles is
 (a) $\frac{P}{2}$ (b) $\frac{P}{2\sqrt{2}}$
 (c) $\sqrt{2P}$ (d) $\frac{P}{\sqrt{2}}$.
6. Two forces of magnitudes 4 and 5 N act at an angle of 60° , the resultant force is equal to
 (a) 6 N (b) $\sqrt{61}$
 (c) 7 N (d) 9 N.
7. A body will be in equilibrium when
 (a) the algebraic sum of vertical components of all forces is zero
 (b) the algebraic sum of horizontal components of all forces is zero
 (c) the algebraic sum of moments of all forces about a point is zero
 (d) all the above.

(B) Beams (Shear Force and Bending Moment Diagrams)

8. A simply supported beam of span (l) carries a point load (W) at the centre of the beam. The bending moment diagram will be a
 (a) parabola with maximum ordinate at the centre of the beam
 (b) parabola with maximum ordinate at one end of the beam
 (c) triangle with maximum ordinate at the centre of the beam
 (d) triangle with maximum ordinate at one end of the beam.

9. A simply supported beam of span (l) carries a uniformly distributed load (w N per unit length) over the whole span. The bending moment diagram will be a
- (a) parabola with maximum ordinate at the centre of the beam
 - (b) parabola with maximum ordinate at one end of the beam
 - (c) triangle with maximum ordinate at the centre of the beam
 - (d) triangle with maximum ordinate at one end of the beam.
10. A cantilever of length (l) carries a point load (W) at the free end. The bending moment diagram will be a
- (a) parabola with maximum ordinate at the centre of the beam
 - (b) parabola with maximum ordinate at the cantilever end
 - (c) triangle with maximum ordinate at the free end
 - (d) triangle with maximum ordinate at the cantilever end.
11. A cantilever of length (l) carries a uniformly distributed load over the whole length. The bending moment diagram will be
- (a) parabola with maximum ordinate at the centre
 - (b) parabola with maximum ordinate at the cantilever end
 - (c) triangle with maximum ordinate at the free end
 - (d) triangle with maximum ordinate at the cantilever end.
12. A simply supported beam of span (l) carries a point load (W) at the centre of the beam. The shear force diagram will be
- (a) a rectangle
 - (b) a triangle
 - (c) two equal and opposite rectangles
 - (d) two equal and opposite triangles.
13. A simply supported beam of span (l) carries a uniformly distributed load over the whole span. The shear force diagram will be
- (a) a rectangle
 - (b) a triangle
 - (c) two equal and opposite rectangles
 - (d) two equal and opposite triangles.
14. A cantilever of length (l) carries a point load (W) at the free end. The shear force diagram will be
- (a) two equal and opposite rectangles
 - (b) a rectangle
 - (c) two equal and opposite triangles
 - (d) a triangle.
15. A cantilever of length (l) carries a uniformly distributed load over the whole length. The shear force diagram will be
- (a) two equal and opposite rectangles
 - (b) a rectangle
 - (c) two equal and opposite triangles
 - (d) a triangle.
16. The bending moment on a section is maximum where shearing force is
- (a) minimum
 - (b) maximum
 - (c) zero
 - (d) equal
 - (e) changing sine
 - (f) none of the above.
17. The point of zero bending moment, where the continuous curve of bending moment changes sign, is called
- (a) the point of contra-flexure
 - (b) the point of inflation
 - (c) the point of virtual hinge
 - (d) all of the above.

18. The point of contra-flexure occurs only in
(a) continuous beams (b) cantilever beams
(c) overhanging beams (d) simply supported beams
(e) all of the above (f) none of the above.
19. The bending moment at a section, where shear force is zero, will be
(a) zero (b) maximum
(c) minimum (d) either minimum or maximum
(e) none of the above.
20. A beam of uniform strength is one which has same
(a) bending stress at every section (b) deflection throughout the beam
(c) bending moment throughout the beam (d) shear force throughout the beam.
21. The shear force and bending moment are zero at the free end of a cantilever, if it carries a
(a) point load at the free end
(b) uniformly distributed load over the whole length
(c) point load in the middle of its length
(d) none of the above.
22. A simply supported beam carries a uniformly distributed load of w kgf per unit length over the whole span (l). The shear force at the centre is
(a) $\frac{wl}{2}$ (b) $\frac{wl^2}{8}$
(c) $\frac{wl}{4}$ (d) zero.
23. For question 22, the shear force at the supported ends will be
(a) $\pm \frac{wl}{2}$ (b) $\frac{wl^2}{8}$
(c) $\frac{wl}{8}$ (d) zero.
24. For question 22, the bending moment at the supported ends will be
(a) $\pm \frac{wl}{2}$ (b) $\frac{wl^2}{8}$
(c) $\frac{wl}{4}$ (d) zero.
25. For question 22, the bending moment at the centre will be
(a) $\pm \frac{wl}{2}$ (b) $\frac{wl^2}{8}$
(c) $\frac{wl}{4}$ (d) zero.
26. For question 22, the point of contra-flexure is at
(a) the supported end (b) the middle of the beam
(c) distance $\frac{l}{4}$ from the supported ends (d) none of the above.

- 27.** A cantilever of length (l) carries a uniformly distributed load w kgf per unit length for the whole length. The shear force at the free end will be
- (a) wl (b) $\frac{wl^2}{2}$
 (c) $\frac{wl}{2}$ (d) zero.
- 28.** In question 27, the shear force at the fixed end will be
- (a) wl (b) $\frac{wl^2}{2}$
 (c) $\frac{wl}{2}$ (d) zero.
- 29.** In question 27, the bending moment at the fixed end will be
- (a) wl (b) $\frac{wl^2}{2}$
 (c) $\frac{wl}{2}$ (d) zero.
- 30.** In question 27, the bending moment at the free end will be
- (a) wl (b) $\frac{wl^2}{2}$
 (c) $\frac{wl}{2}$ (d) zero.
- 31.** In question 27, point of contra-flexure is at
- (a) the free end (b) the fixed end
 (c) the middle of the beam (d) no where.
- 32.** Fig. 6 shows a simply supported beam of span 10 m. It carries a point load of 200 N at a distance 7 m from the end A. The reaction at A will be
- (a) 140 N (b) 60 N
 (c) 420 N (d) 100 N.
- 33.** For the beam shown in Fig. 6, the reaction at B will be
- (a) 140 N (b) 60 N
 (c) 420 N (d) 100 N.
- 34.** For the beam shown in Fig. 6, the maximum bending moment is at
- (a) point A (b) point B
 (c) point C (d) point between A and C.
- 35.** For the beam shown in Fig. 6, the maximum bending moment
- (a) 140 kgf-m (b) 60 kgf-m
 (c) 420 kgf-m (d) 100 kgf-m.
- 36.** For the beam shown in Fig. 6, the shear force diagram will consist of
- (a) a rectangle (b) a triangle
 (c) two rectangles (d) two triangles.

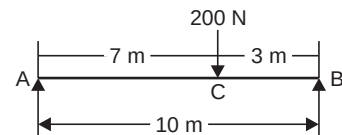


Fig. 6

UNIT 3. CENTROID AND MOMENT OF INERTIA

(A) Centre of Gravity

1. The C.G. of a triangle lies at the point of concurrence of
 - (a) the right bisectors of the angle of the triangle
 - (b) the medians of the triangle
 - (c) the altitudes from the vertices on the opposite side
 - (d) none of the above.
2. The C.G. of solid hemisphere lies on the central radius at a distance

(a) $\frac{3r}{4}$ from the plane base	(b) $\frac{3r}{8}$ from the plane base
(c) $\frac{8r}{3}$ from the plane base	(d) none of the above.
3. The C.G. of a semi-circular lamina lies on the central radius at a distance of

(a) $\frac{4r}{3\pi}$ from base diameter	(b) $\frac{3r}{8}$ from base diameter
(c) $\frac{8r}{3}$ from base diameter	(d) none of the above.
4. The C.G. of a solid right circular cone lies on the axis at a height
 - (a) half of the total height above the base
 - (b) one-third of the total height above the base
 - (c) one-fourth of the total height above the base
 - (d) none of the above.
5. The C.G. of a thin hollow right circular cone lies on the axis at a height

(a) half of the total height	(b) one-third of the total height
(c) one-fourth of the total height	(d) none of the above.
6. The C.G. of a semi-circular arc is at the central radius at a distance of

(a) $\frac{3r}{4}$ from base diameter	(b) $\frac{3r}{8}$ from base diameter
(c) $\frac{2r}{\pi}$ above base diameter	(d) $\frac{r}{2\pi}$ above base diameter.
7. The C.G. of a quadrant of a circle is at a distance of

(a) $\frac{3r}{4\pi}$ from the axis	(b) $\frac{4r}{3\pi}$ from the axis
(c) $\frac{3r}{8}$ from the axis	(d) $\frac{8r}{3}$ from the axis.
8. The C.G. of a thin hollow hemisphere is at a distance of

(a) $\frac{r}{3}$ from base	(b) $\frac{r}{2}$ from base
(c) $\frac{r}{4}$ from base	(d) none of the above.

9. The C.G. of a solid cone lies on the central axis at a distance of
 (a) $\frac{1}{3}$ rd of the total height above base (b) half the total height above base
 (c) $\frac{1}{4}$ th of the total height above base (d) $\frac{2}{5}$ th of total height above base.
10. The C.G. of a hollow cone lies on the central axis above base at a distance of
 (a) $\frac{1}{3}$ rd of the total height (b) half of the total height
 (c) $\frac{1}{4}$ th of the total height (d) $\frac{2}{5}$ th of the total height.
11. The C.G. of a hollow pyramid lies on the central axis above base at a distance of
 (a) $\frac{1}{4}$ th of the total height (b) $\frac{1}{3}$ rd of the total height
 (c) half of the total height (d) $\frac{2}{5}$ th of the total height.
12. The C.G. of a hollow cylinder lies on the vertical axis above base at a distance of
 (a) $\frac{1}{4}$ th of the total height (b) $\frac{1}{3}$ rd of the total height
 (c) half of the total height (d) $\frac{2}{5}$ th of the total height.
13. The C.G. of a parabola as shown in Fig. 7 lies on the axis OB at a distance of
 (a) $\frac{1}{5}$ th of OB
 (b) $\frac{1}{4}$ th of OB
 (c) $\frac{2}{5}$ th of OB
 (d) $\frac{1}{3}$ rd of OB .

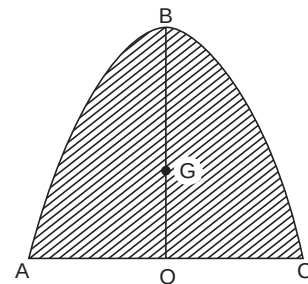


Fig. 7

(B) Moment of Inertia

14. If A = area of the body, I_G = moment of inertia of body about an axis passing through its C.G. and I_0 = moment of inertia of the body about an axis parallel to the axis passing through C.G. and at a distance x , then according to the theorem of parallel axis
 (a) $I_G = I_0 + Ax^2$ (b) $I_0 = I_G + Ax^2$
 (c) $I_0 = I_G - Ax^2$ (d) none of the above.
15. If I_x = M.O.I. about x -axis and I_y = M.O.I. about y -axis, then moment of inertia about z -axis is given by
 (a) $I_z = I_x - I_y$ (b) $I_z = I_y - I_x$
 (c) $I_z = I_x \times I_y$ (d) none of the above.
16. A thin rod of length L and mass M will have moment of inertia about an axis passing through one of its edge and perpendicular to the rod
 (a) $\frac{ML^2}{12}$ (b) $\frac{ML^2}{6}$
 (c) $\frac{ML^3}{3}$ (d) $\frac{ML^2}{3}$.

17. Moment of inertia of a rectangular section having b = Width and d = Depth about x -axis is given by

$$(a) I_x = \frac{bd^3}{12}$$

$$(b) I_x = \frac{b^3d}{12}$$

$$(c) I_x = \frac{b^2d^2}{6}$$

(d) none of the above.

18. M.O.I. of a circular section of diameter d about an axis passing through its C.G. lying in the plane of the section is given by

$$(a) I_x = \frac{\pi d^4}{32}$$

$$(b) I_x = \frac{\pi d^4}{64}$$

$$(c) I_x = \frac{\pi d^4}{16}$$

(d) none of the above.

19. M.O.I. of a triangular section about an axis passing through its base is given by

$$(a) I = \frac{bh^3}{12}$$

$$(b) I = \frac{bh^3}{32}$$

$$(c) I = \frac{bh^2}{36}$$

(d) none of the above.

where b = Width at a base and h = Height of triangle.

20. M.O.I. of a triangular section, about an axis passing through its C.G. is

$$(a) I = \frac{bh^3}{12}$$

$$(b) I = \frac{bh^3}{32}$$

$$(c) I = \frac{bh^3}{36}$$

(d) none of the above.

21. M.O.I. of a solid sphere of mass M and radius R is given by

$$(a) I = \frac{MR^2}{12}$$

$$(b) I = \frac{2}{5} MR^2$$

$$(c) I = \frac{MR^2}{36}$$

$$(d) I = \frac{3}{5} MR^2.$$

22. M.O.I. of a thin spherical shell of mass M and radius r is given as

$$(a) \frac{2}{5} Mr^2$$

$$(b) \frac{2}{3} Mr^2$$

$$(c) \frac{3}{5} Mr^2$$

$$(d) \frac{4}{3} Mr^2.$$

23. M.O.I. of a solid cone about its vertical axis is

$$(a) \frac{10}{3} Mr^2$$

$$(b) \frac{5}{3} Mr^2$$

$$(c) \frac{3}{10} Mr^2$$

$$(d) \frac{3}{5} Mr^2.$$

24. The units of moment of inertia of mass are

$$(a) \text{kg-m}^3$$

$$(b) \text{kg-m}^2$$

$$(c) \text{kg-m}$$

$$(d) \text{kg-m}^4.$$

- 25.** The units of moment of inertia of area are
(a) kg-m^3 (b) kg-m^2
(c) m^4 (d) none of the above.
- 26.** M.O.I. of a solid cone about an axis passing through its C.G. and parallel to base is
(a) $\frac{3}{10} Mr^2$ (b) $\frac{3}{5} Mr^2$
(c) $\frac{3}{20} M \left(r^2 + \frac{h^2}{4} \right)$ (d) $\frac{10}{3} Mr^2$.
- 27.** M.O.I. of a solid circular cylinder about vertical axis is
(a) $\frac{3}{10} Mr^2$ (b) $\frac{3}{5} Mr^2$
(c) $\frac{3}{2} Mr^2$ (d) $\frac{Mr^2}{2}$.
- 28.** M.O.I. of elliptical section, of major axis = D and minor axis = B , is
(a) $\frac{\pi}{64} D^4$ (b) $\frac{\pi}{64} d^4$
(c) $\frac{\pi}{64} BD^3$ (d) $\frac{\pi}{64} B^2 D^2$.
- 29.** Polar moment of inertia of a circle (I_0) is given by
(a) $\frac{\pi}{64} d^4$ (b) $\frac{\pi}{32} d^4$
(c) $\frac{\pi}{16} d^4$ (d) $\frac{\pi}{32} d^3$.
- 30.** Polar moment of inertia of a rectangle ($b \times h$) is given by
(a) $\frac{bh^3}{12}$ (b) $\frac{hb^3}{12}$
(c) $\frac{bh(b^2 + h^2)}{12}$ (d) $\frac{b^2 h^2 (b + h)}{12}$.
- 31.** Polar moment of inertia of an equilateral triangle of sides x is given by
(a) $\frac{x^4}{16}$ (b) $\frac{x^4}{16\sqrt{3}}$
(c) $\frac{x^4}{32}$ (d) $\frac{x^4}{64}$.
- 32.** Polar moment of inertia is
(a) the moment of inertia of an area about an axis parallel to centroidal axis
(b) equal to moment of inertia
(c) the moment of an area about an axis which is not lying in the plane of the area
(d) the moment of inertia of an area about a line or axis perpendicular to the plane of the area.

33. The moment of inertia of a triangle (having base = b and height = h) with respect to an axis through the apex and parallel to the base is
- (a) $\frac{bh^3}{12}$ (b) $\frac{bh^3}{36}$
 (c) $\frac{bh^3}{4}$ (d) $\frac{bh^3}{10}$
34. Newton's second law for rotary motion states that
- (a) rate of change of rotation of a body about a fixed axis is directly proportional to the impressed external force and takes place in the direction of force.
 (b) rate of change of momentum is directly proportional to the impressed force and takes place in the direction of force.
 (c) rate of change of rotation (angular momentum) is directly proportional to the impressed external torque and takes place in the direction of force.
 (d) none of the above.
35. The relation between external torque (T) acting on a body and the angular acceleration (α) is given by
- (a) $T = \frac{I}{\alpha}$ (b) $T = \frac{\alpha}{I}$
 (c) $T = I\alpha$ (d) none of the above
- where T is in newton-metres.

UNIT 4. KINEMATICS AND KINETICS OF RIGID BODY

(A) Kinematics of Rigid Body

- Rate of change of displacement of a body is called
 (a) acceleration (b) velocity
 (c) momentum (d) none of the above.
- Rate of change of velocity of body is called
 (a) acceleration (b) velocity
 (c) momentum (d) none of the above.
- The product of mass and velocity of a body is called
 (a) acceleration (b) velocity
 (c) momentum (d) none of the above.
- If a body is moving with a uniform acceleration (a), then final velocity (V) of the body after time ' t ' is equal to
 (a) $ut + \frac{1}{2}at^2$ (b) $u + at$
 (c) $u^2 + 2aS$ (d) none of the above.
 where u = Initial velocity, S = Distance travelled in t seconds.
- If a body is moving with a uniform acceleration (a), then the distance travelled by a body after time ' t ' is equal to
 (a) $ut + \frac{1}{2}at^2$ (b) $u + at$
 (c) $u^2 + 2aV$ (d) none of the above.

6. If a body is moving with a uniform acceleration (a), then the distance travelled by the body in n th second is given by
- (a) $\frac{u+a}{2}(1-2n)$ (b) $\frac{u+a}{2}(n-2)$
 (c) $u + \frac{a}{2}(2n-1)$ (d) none of the above.
7. If a body is moving in a curved path, the motion of the body is called
- (a) rectilinear (b) rotational
 (c) curvilinear (d) none of the above.
8. If a body is moving in a straight line, the motion of the body is called
- (a) rectilinear (b) rotational
 (c) curvilinear (d) none of the above.
9. If a body is moving in a circular path, the motion of the body is called
- (a) rectilinear (b) rotational
 (c) curvilinear (d) none of the above.
10. Rate of change of angular velocity is called
- (a) acceleration (b) angular acceleration
 (c) kinetic energy (d) none of the above.
11. The relation between linear acceleration (f) and angular acceleration (α) is given by
- (a) $\alpha = a \times r$ (b) $\alpha = \frac{a}{r}$
 (c) $\alpha = \frac{1}{a \times r}$ (d) $\alpha = \frac{r}{a}$.
12. The angular displacement by a rotating body in the n th second is equal to
- (a) $\omega_0 + \left(\frac{n-2}{2}\right)\alpha$ (b) $\left(\frac{\omega_0 \times n}{2}\right) \times \alpha$
 (c) $\omega_0 + \left(\frac{2n-1}{2}\right)\alpha$ (d) $\omega_0 + \left(\frac{1-2n}{2}\right)\alpha$
 where ω_0 = Initial angular velocity of the body moving in a circle
 α = Uniform angular acceleration.
13. The linear velocity (V) of a rotating body is given by
- (a) $V = \frac{\omega}{r}$ (b) $V = \omega \times r$
 (c) $V = \frac{1}{\omega r}$ (d) none of the above.
14. The expression ($\frac{1}{2}mV^2$) denotes
- (a) centrifugal force (b) kinetic energy
 (c) potential energy (d) none of the above.

(B) Kinetics of Rigid Body

15. The expression $\left(\frac{mV^2}{r}\right)$ represents
 (a) centrifugal force (b) kinetic energy
 (c) potential energy (d) none of the above.
16. The expression $\left(\frac{1}{2}I\omega^2\right)$ represents
 (a) centrifugal force (b) kinetic energy
 (c) kinetic energy of rotation (d) potential energy.
17. A force P of high magnitude acts on a body for a small interval of time (Δt) . The product of P and Δt is called
 (a) impulsive force (b) kinetic energy of the body
 (c) impulse (d) none of the above.
18. The force P in question 17 is called
 (a) impulse (b) impulsive force
 (c) propelling force (d) none of the above.
19. Energy lost by a body (of mass m and moving with a velocity V) when it strikes another body (of mass M at rest) due to impact is equal to
 (a) $\frac{mV^2}{2g}\left(1 + \frac{m}{m+M}\right)$ (b) $\frac{mV^2}{2g}(m+M-1)$
 (c) $\frac{mV^2}{2g}\left(1 - \frac{m}{m+M}\right)$ (d) none of the above.
20. Tension in a cable supporting a lift, when lift is going up is equal to
 (a) $W\left(1 - \frac{a}{g}\right)$ (b) $W\left(1 + \frac{a}{g}\right)$
 (c) $W\left(W - \frac{W}{g}\right)$ (d) $W\left(g + \frac{a}{g}\right)$
 where f = Uniform acceleration of lift ; and W = Weight carried by lift.
21. Tension a cable supporting a lift, when lift is going down is equal to
 (a) $W\left(1 - \frac{a}{g}\right)$ (b) $W\left(1 + \frac{a}{g}\right)$
 (c) $f\left(W - \frac{W}{g}\right)$ (d) $W\left(g + \frac{a}{g}\right)$
 where f = Uniform acceleration of lift ; and W = Weight carried by lift.
22. When two bodies of mass (m and $2m$) are connected by a light inextensible string and pass over a smooth pulley, then acceleration of one body is
 (a) equal to the acceleration of the other body
 (b) two time the acceleration of the other body
 (c) half the acceleration of the other body
 (d) none of the above.

- 23.** When two bodies of mass (M and $2M$) are connected by a light inextensible string and pass over a smooth pulley, then
- tension in both sides of the string will be equal
 - tension in one side of the string is two times the tension in the other side of the string
 - tension in one side of the string is half the tension in the other side of the string
 - none of the above.

- 24.** Two bodies of masses m_1 and m_2 are connected by a light in extensible string and pass over a smooth pulley. If the mass m_1 is coming down, then the acceleration of both the bodies is equal to

$$\begin{array}{ll} (a) \frac{g(m_1 + m_2)}{(m_1 - m_2)} & (b) \frac{g(m_1 - m_2)}{(m_1 + m_2)} \\ (c) \frac{g(m_1 m_2)}{m_1 - m_2} & (d) \frac{g(m_1 m_2)}{m_1 + m_2} \end{array}$$

- 25.** For question 24, the tension in the string will be equal to

$$\begin{array}{ll} (a) \frac{2(m_1 - m_2)}{m_1 + m_2} & (b) \frac{2m_1 m_2}{m_1 - m_2} \\ (c) \frac{2m_1 m_2}{m_1 + m_2} & (d) \frac{2(m_1 + m_2)}{m_1 - m_2} \end{array}$$

- 26.** Fig. 8 shows the two bodies of masses m_1 and m_2 connected by a light inextensible string and a passing over a smooth pulley. Mass m_2 lies on a smooth horizontal plane. When mass m_1 is moving downward the acceleration of the two bodies is equal to

$$\begin{array}{ll} (a) \frac{m_1 g}{m_1 - m_2} \text{ m/s}^2 & (b) \frac{m_1 g}{m_1 + m_2} \text{ m/s}^2 \\ (c) \frac{m_2 g}{m_1 + m_2} & (d) \frac{m_2 g}{m_1 - m_2} \end{array}$$

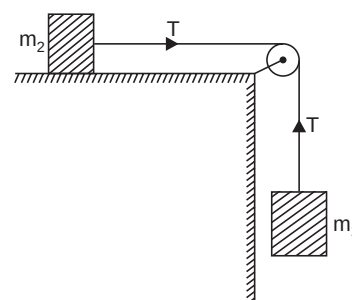


Fig. 8

- 27.** Refer to Fig. 8, the tension (T) will be equal to

$$\begin{array}{ll} (a) \frac{m_1 m_2}{m_1 - m_2} \text{ N} & (b) \frac{m_1 m_2}{m_1 + m_2} \\ (c) \frac{m_1 - m_2}{m_1 m_2} & (d) \frac{m_1 + m_2}{m_1 m_2} \end{array}$$

- 28.** If the weight W_2 in Fig. 8 is resting on a rough horizontal plane (having co-efficient of friction as μ), then the acceleration is equal to

$$\begin{array}{ll} (a) \frac{(W_1 + \mu W_2)g}{(W_1 + W_2)} & (b) \frac{(\mu W_1 + W_2)g}{(W_1 + W_2)} \\ (c) \frac{(W_1 - \mu W_2)g}{(W_1 + W_2)} & (d) \frac{(\mu W_1 + \mu W_2)g}{W_1 - W_2} \end{array}$$

29. If the weight W_2 in Fig. 8 is resting on a rough horizontal plane (having co-efficient of friction as μ), then tension in the string is equal to

(a) $\frac{(1+\mu)W_1W_2}{W_1+W_2}$ (b) $\frac{(1-\mu)W_1W_2}{W_1-W_2}$
 (c) $\frac{\mu W_1W_2}{W_1+W_2}$ (d) $\frac{\mu W_1W_2}{W_1-W_2}$

30. Two weights W_1 and W_2 are connected by a light inextensible string. Weight W_2 is placed on a smooth inclined plane of inclination α and W_1 hangs freely as shown in Fig. 9. If W_1 moves downwards then acceleration is equal to

(a) $\frac{W_1 - W_2 \sin \alpha}{(W_1 + W_2)}$ (b) $\frac{(W_1 - W_2 \sin \alpha)g}{W_1 + W_2}$
 (c) $\frac{W_1 + W_2}{W_1 - W_2 \sin \alpha}$ (d) $\frac{(W_1 - W_2)g}{W_1 - W_2 \sin \alpha}$

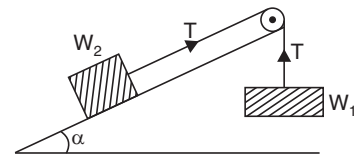


Fig. 9

31. If in Fig. 9, weight W_2 is placed on a rough inclined plane of inclination α and co-efficient of friction μ , then the acceleration is equal to

(a) $\frac{(W_1 - W_2 \sin \alpha - \mu W_2 \cos \alpha)}{(W_1 + W_2)}$ (b) $\frac{(W_1 + W_2 \sin \alpha - \mu W_2 \cos \alpha)}{(W_1 + W_2)}$
 (c) $\frac{(W_1 - W_2 \sin \alpha - \mu W_2 \cos \alpha)g}{(W_1 + W_2)}$ (d) $\frac{(W_1 - W_2 \sin \alpha)g}{(W_1 + W_2)}$

32. The time taken by a ball (of weight 500 N) to return back to earth, if it is thrown vertically upwards with a velocity 4.9 m/s is equal to

(a) $\frac{1}{2}$ s (b) 1 s
 (c) 2 s (d) 3 s.

33. The maximum height attained by a ball (to weight 500 N) which is thrown vertically upwards with a velocity 4.9 m/s is equal to

(a) 100 cm (b) 245 cm (c) 122.5 cm (d) 980 cm.

34. A tower is of height 100 m. A stone is thrown up from the foot of water with a velocity of 20 m/s and at the same time another stone is dropped from the top of the tower. The two stones will meet after

(a) 10 s (b) 5 s (c) 2 s (d) 7.5 s.

35. The maximum height reached by a stone (of weight 50 N) which is thrown vertically upward with an initial velocity 19.6 m/s would be

(a) 20 m (b) 19.6 m (c) 30 m (d) 25 m.

36. A body is moving with a velocity of 2 m/s. If the velocity of body becomes 5 m/s after 4 seconds, the acceleration of the body would be

(a) 1 m/s² (b) 0.75 m/s² (c) 1.5 m/s² (d) 0.375 m/s².

37. A body is rotating with an angular velocity of 5 radian/s. After 4 seconds, the angular velocity of the body becomes 1.3 radians per sec. The angular acceleration of the body would be

(a) 3 rad/s² (b) 2 rad/s² (c) 1 rad/s² (d) 1.5 rad/s².

38. A flywheel starting from rest and accelerating uniformly performs 20 revolution in 4 seconds. The angular velocity of flywheel after 8 seconds would be
 (a) 30 rad/s (b) 35 rad/s (c) 40 rad/s (d) 55 rad/s.
39. A body is moving in a straight line with an initial velocity of 4 m/s. After 5 seconds the velocity of the body becomes 9 m/s. The distance travelled by the body in third second would be
 (a) 6 m (b) 5.5 m (c) 6.5 m (d) 4 m.
40. A body is rotating with an angular velocity of 5 radians/s. After 4 seconds, the angular acceleration of the body becomes 13 radians/s. If the body is rotating with uniform acceleration, the angle covered by the body in the third second would be
 (a) 20 radians (b) 25 radians (c) 15 radians (d) 10 radians.
41. A body is moving with a velocity of 10 m/s. The time required, to stop the body within a distance of 5 m, is equal to
 (a) 3 second (b) 5 second (c) 1 second (d) 0.5 second.
42. A stone dropped into a well is heard to strike the water after 4 seconds. If the velocity of sound is 350 m/s, the depth of well would be
 (a) 150 m (b) 70.75 m (c) 100 m (d) 35.375 m.
43. A light string passes over a smooth, weightless pulley and has weights 40 N and 60 N attached to its end as shown in Fig. 10. The tension in string will be
 (a) 60 N (b) 50 N
 (c) 48 N (d) 20 N.
44. Refer to Fig. 10, the acceleration, with which the weight 60 N descends, is
 (a) $\frac{g}{5}$ (b) $\frac{g}{4}$
 (c) $2g$ (d) $5g$.
45. Refer to Fig. 10, the pressure on pulley would be
 (a) 100 N (b) 96 N
 (c) 20 N (d) 50 N.
46. Fig. 11 shows two weights 40 N and 60 N connected by a light inextensible string and passes over a smooth weightless pulley. The weight 40 N is resting on a rough horizontal plane with $\mu = 0.3$ and weight 60 N hangs freely and is moving downward. The tension in the string would be
 (a) 20 N (b) 100 N
 (c) 31.2 N (d) 50 N.
47. Refer to Fig. 11, the acceleration with which weight 60 N descends would be
 (a) 0.08 g (b) 20 g
 (c) 100 g (d) 50 g.

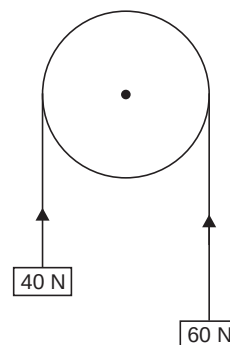


Fig. 10

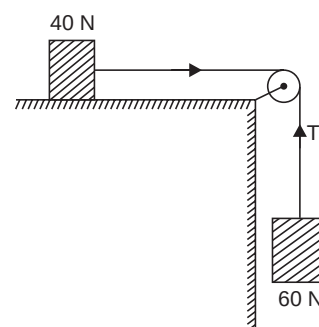


Fig. 11

48. Two weights of 50 N and 150 N (of two blocks *A* and *B* respectively) are connected by a string and frictionless and weightless pulleys as shown in Fig. 12. The tension in the string would be

(a) 100 N (b) 200 N
(c) 64.3 N (d) 50 N.

49. Refer to Fig. 12, the acceleration to block *A* would be

(a) $\frac{g}{7}$ (b) $\frac{g}{5}$
(c) $\frac{2g}{5}$ (d) $\frac{2g}{7}$.

50. Refer to Fig. 12, the acceleration of block *B* would be

(a) $\frac{g}{7}$ (b) $\frac{g}{5}$
(c) $\frac{2g}{5}$ (d) $\frac{2g}{7}$.

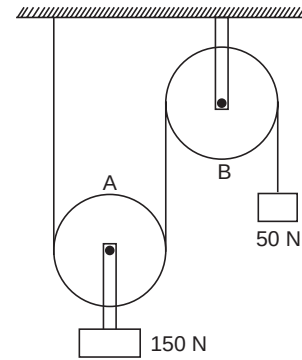


Fig. 12

UNIT 5. MECHANICS OF DEFORMABLE SOLIDS

(A) Simple Stress and Strain

- With elastic limit in a loaded material, stress is
 - inversely proportional to strain
 - directly proportional to strain
 - equal to strain
 - none of the above.
- The ratio of linear stress to linear strain is known as
 - Poisson's ratio
 - bulk modulus
 - modulus of rigidity
 - modulus of elasticity.
- The ratio of lateral strain to longitudinal strain is called
 - Poisson's ratio
 - bulk modulus
 - modulus of rigidity
 - modulus of elasticity.
- The ratio of shear stress to shear strain is called
 - Poisson's ratio
 - bulk modulus
 - modulus of rigidity
 - modulus of elasticity.
- The ratio of normal stress of each face of a solid cube to volumetric strain is called
 - Poisson's ratio
 - bulk modulus
 - modulus of rigidity
 - modulus of elasticity.
- Hooke's law holds good upto
 - proportional limit
 - yield point
 - elastic limit
 - plastic limit.
- The property of a material by virtue of which a body returns to its original shape after removal of the load is known as
 - ductility
 - plasticity
 - elasticity
 - resilience.

8. A tensile force (P) is acting on a body of length (l) and area of cross-section (A). The change in length would be

(a) $\frac{P}{lAE}$ (b) $\frac{PE}{Al}$
 (c) $\frac{Pl}{AE}$ (d) $\frac{Al}{PE}$

9. The modulus of elasticity (E) and modulus of rigidity (C) are related by

(a) $C = \frac{mE}{3(m-2)}$ (b) $C = \frac{mE}{2(m+1)}$
 (c) $C = \frac{3(m-2)}{mE}$ (d) $C = \frac{2(m+1)}{mE}$

where $\frac{1}{m}$ = Poisson's ratio.

10. The modulus of elasticity (E) and bulk modulus (K) are related by

(a) $K = \frac{mE}{3(m-2)}$ (b) $K = \frac{mE}{2(m+1)}$
 (c) $K = \frac{3(m-2)}{mE}$ (d) $K = \frac{2(m+1)}{mE}$

where $\frac{1}{m}$ = Poisson's ratio.

11. Fig. 13 shows the stress-strain curve for mild steel. Hook's law holds good upto

- (a) point A (b) point B
 (c) point C (d) point D.

12. The point A in Fig. 13 represents

- (a) proportional limit
 (b) upper yield point
 (c) lower yield point
 (d) breaking point.

13. The point C in Fig. 13 represents

- (a) elastic limit
 (c) lower yield point

- (b) upper yield point
 (d) breaking point.

14. The point D in Fig. 13 represents

- (a) elastic limit
 (c) lower yield point

- (b) upper yield point
 (d) breaking point.

15. The point F in Fig. 13 represents

- (a) elastic limit
 (c) lower yield point

- (b) upper yield point
 (d) breaking point.

16. The stress corresponding to point E in Fig. 13 is called

- (a) breaking stress
 (c) yield point stress

- (b) ultimate stress
 (d) nominal stress.

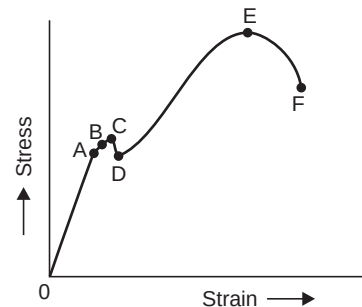


Fig. 13

17. The stress corresponding to point *F* in Fig. 13 is called
 (a) breaking stress (b) ultimate stress
 (c) yield point stress (d) nominal stress.
18. A rod of steel of 1 cm^2 in cross-sectional area and 100 cm long is subjected to an axial pull of 20000 N. If $E = 2 \times 10^5 \text{ N/cm}^2$, the elongation will be
 (a) 1 cm (b) 0.2 cm (c) 0.1 cm (d) 0.15 cm.
19. The modulus of elasticity in S.I. units is expressed as
 (a) N/m (b) N/m^2 (c) Nm/s (d) no unit.
20. The ratio of modulus of rigidity to modulus of elasticity for a Poisson's ratio of 0.25 would be
 (a) 0.5 (b) 0.4 (c) 0.3 (d) 1.0.
21. The ratio of bulk modulus to modulus of elasticity for a Poisson's ratio of 0.25 would be
 (a) $2/3$ (b) $1/3$ (c) $4/3$ (d) 1.0.
22. The relation between modulus of elasticity (E), modulus of rigidity (C) and bulk modulus (K) is given by
 (a) $E = \frac{3KC}{C + 9K}$ (b) $E = \frac{9KC}{C + 3K}$
 (c) $E = \frac{C + 9K}{3KC}$ (d) $E = \frac{C + 3K}{9KC}$.
23. The ratio of modulus of rigidity of bulk modulus for a Poisson's ratio of 0.25 would be
 (a) $2/3$ (b) $2/5$ (c) $3/5$ (d) $4/5$.
24. The work done in producing strain on a material per unit volume is called
 (a) resilience (b) ductility (c) elasticity (d) plasticity.
25. The property by virtue of which a metal can be beaten into plates is called
 (a) ductility (b) malleability (c) resilience (d) plasticity.
26. The extension per unit length of the rod due to suddenly applied load as compared to the same load gradually applied to the same rod is
 (a) same (b) double (c) three times (d) half.
27. The stress due to suddenly applied load as compared to the stress due to the same load gradually applied to the same rod is
 (a) half (b) same (c) double (d) three times.
28. The stress and strain curves for four different materials, i.e., mild steel, cast iron, brass and cast aluminium are shown in Fig. 14. The curve A is for
 (a) mild steel (b) cast iron
 (c) brass (d) cast aluminium.
29. Curve B in Fig. 14 is for
 (a) mild steel (b) cast iron
 (c) brass (d) cast aluminium.
30. Curve C in Fig. 14 is for
 (a) mild steel (b) cast iron
 (c) brass (d) cast aluminium.

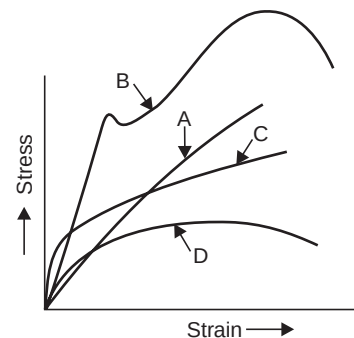


Fig. 14

31. Curve D in Fig. 14 is for
 (a) mild steel (b) cast iron (c) brass (d) cast aluminium.
32. A tensile test is performed on a specimen of 4 cm^2 cross-section. The maximum load on the specimen was 4000 N. The area of cross-section at neck was 2 cm^2 . The specimen will have the ultimate tensile stress as
 (a) 1000 N/cm^2 (b) 2000 N/cm^2 (c) 3000 N/cm^2 (d) 500 N/cm^2 .

(B) Pure Bending of Beams

33. The ratio of moment of inertia about the neutral axis to the distance of the most distant point of the section from the neutral axis is called
 (a) moment of inertia (b) section modulus
 (c) polar moment of inertia (d) modulus of rigidity.
34. The relation between maximum stress (σ) offered by a section, moment of resistance (M) and section modulus (Z) is given by
 (a) $M = \frac{\sigma}{Z}$ (b) $M = \frac{Z}{\sigma}$
 (c) $M = \sigma \times Z$ (d) $M = \frac{1}{\sigma \times Z}$.
35. Every cross-section of a shaft, which is subjected to a twisting moment, is under
 (a) compressive stress (b) shear stress
 (c) tensile stress (d) bending stress.
36. The shear stress at any point of a shaft, subjected to twisting moment, is
 (a) proportional to its distance from the central axis of the shaft
 (b) inversely proportional to its distance from the central axis of the shaft
 (c) proportional to the square of its distance from the central axis of the shaft
 (d) none of the above.

(C) Torsion

37. When a shaft is subjected to tension, the relation between maximum shear stress (τ_0), modulus of rigidity of the shaft (C) and angle of twist (θ) is given by
 (a) $\frac{C\theta}{\tau_0} = \frac{R}{L}$ (b) $\frac{C\theta}{L} = \frac{\tau_0}{R}$
 (c) $\frac{C\theta}{R} = \frac{\tau_0}{L}$ (d) $\frac{C}{L\theta} = \frac{\tau_0}{R}$
 where L = Length of shaft and R = Radius of shaft.
38. A solid shaft of diameter D transmits the torque equal to
 (a) $\frac{\pi}{32} \tau D^3$ (b) $\frac{\pi}{64} \tau D^3$
 (c) $\frac{\pi}{16} \tau D^3$ (d) $\frac{\pi}{8} \tau D^3$
 where τ = Maximum allowable shear stress.

39. The torque transmitted by a hollow shaft to external diameter (D) and internal diameter (d) is equal to

$$\begin{array}{ll} (a) \frac{\pi}{32} \tau [D^3 - d^3] & (b) \frac{\pi}{16} \tau [D^3 - d^3] \\ (c) \frac{\pi}{16} \tau \left[\frac{D^4 - d^4}{D} \right] & (d) \frac{\pi}{32} \tau \left[\frac{D^4 - d^4}{D} \right]. \end{array}$$

40. Polar moment of inertia of a solid circular shaft of diameter D is equal to

$$\begin{array}{ll} (a) \frac{\pi D^3}{32} & (b) \frac{\pi D^4}{32} \\ (c) \frac{\pi D^3}{64} & (d) \frac{\pi D^4}{64}. \end{array}$$

41. Polar moment of inertia of a hollow circular shaft is equal to

$$\begin{array}{ll} (a) \frac{\pi}{32} [D^3 - d^3] & (b) \frac{\pi}{32} [D^4 - d^4] \\ (c) \frac{\pi}{64} [D^3 - d^3] & (d) \frac{\pi}{64} [D^4 - d^4]. \end{array}$$

42. Polar modulus of a shaft section is equal to

- product of polar moment of inertia and maximum radius of the shaft
- ratio of polar moment of inertia of maximum radius of the shaft
- sum of polar moment of inertia and maximum radius of the shaft
- difference of polar moment of inertia and maximum radius of the shaft.

43. The torsional rigidity of a shaft is defined as the torque required to produce

- maximum twist in the shaft
- maximum shear stress in the shaft
- minimum twist in the shaft
- a twist of one radian per unit length of the shaft.

44. The flexural rigidity of the deflection of beams is expressed as

$$\begin{array}{ll} (a) \frac{I}{E} & (b) \frac{E}{I} \\ (c) EI & (d) \frac{I}{EI} \end{array}$$

where E = Modulus of elasticity and I = Moment of inertia.

45. The greatest twisting moment which a given shaft section can resist is equal to

- polar modulus $\times \tau$
- polar modulus/ τ
- τ /polar modulus
- none of the above.

46. Choose the correct statement.

- Shafts of the same material and length having the same polar modulus have the same strength.
- For a shaft of a given material, the magnitude of polar modulus is a measure of its strength in resisting torsion.

- (c) From a number of shafts of the same length and material the shaft with greatest polar modulus will resist the maximum twisting moment.
- (d) all of the above.
- (e) none of the above.
47. Two shafts, one solid and the other hollow, are made of the same materials and are having same length and weight. The hollow shaft as compared to solid shaft is
- (a) more strong (b) less strong
- (c) having same strength (d) none of the above.
48. Two shafts, one solid and the other hollow, are of same length and material. They are subjected to the same torque and attain the same permissible maximum shear stress at the same time. The weight of hollow shaft will be
- (a) more (b) less
- (c) same as of solid shaft (d) none of the above.
49. The torque transmitted, by a solid shaft of diameter 40 mm if the shear stress is not to exceed 400 kg/cm^2 , would be
- (a) $1.6 \times \pi \text{ kg-m}$ (b) $16 \pi \text{ kg-m}$ (c) $0.8 \times \pi \text{ kg-m}$ (d) $0.4 \times \pi \text{ kg-m}$.
50. If in question 49, the diameter of the solid shaft is doubled, then torque transmitted would be
- (a) same (b) double (c) four times (d) eight times.
51. If in question 49, the diameter of the solid shaft is made 20 mm, then torque transmitted would be
- (a) same (b) one-half (c) one-eighth (d) one-fourth.
52. The torsion equation is given by
- (a) $\frac{T}{J} = \frac{\tau}{R} = \frac{L}{C\theta}$ (b) $\frac{T}{R} = \frac{\tau}{J} = \frac{C\theta}{L}$
- (c) $\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L}$ (d) $\frac{T}{\tau} = \frac{R}{J} = \frac{C\theta}{L}$.
53. The assumption made, while determining the shear stress in a circular shaft subjected to torsion, is that
- (a) the material of the shaft is uniform
- (b) the twist along the shaft is uniform
- (c) cross-sections of the shaft is plane and circular before and after the twist
- (d) all of the above
- (e) none of the above.
54. Choose the wrong statement
- (a) Shafts of the same material and length having the same polar modulus have the same strength.
- (b) The measure of the strength of the shaft in resisting torsion is given by the magnitude of the polar modulus.
- (c) The polar modulus of a solid shaft $= \frac{\pi d^3}{16}$.
- (d) Polar modulus is the same as polar moment of inertia.

(D) Additional Questions

55. A cantilever of length (l) carries a load whose intensity varies uniformly from zero at the free end to w per unit length at the fixed end, the bending moment diagram will be a
 (a) straight line curve (b) parabolic curve
 (c) cubic curve (d) combination of (a) and (b).
56. A simply supported beam is overhanging equally on both sides and carries a uniformly distributed load of w per unit length over the whole length. The length between the supports is (l) and length of overhang to one side is ' a '. If $l > 2a$, then the number of points of contraflexure will
 (a) zero (b) one (c) two (d) three.
57. If in question 56, $l = 2a$, the number of points of contraflexure will be
 (a) zero (b) one (c) two (d) three.
58. If in question 56, $l < 2a$, the number of points of contraflexure will be
 (a) zero (b) one (c) two (d) three.
59. In question 56, the shear force diagram will consist of
 (a) two triangles (b) two rectangles (c) four triangles (d) four rectangles.

(E) Tick Mark the True or False Statement

60. There will be an increase in length and decrease in diameter when a thin cylinder is subjected to an internal fluid pressure (p).
 (a) True (b) False.
61. The term $(T \times \omega / 1000)$ represents the power transmitted by a shaft in kW where T = torque in Nm and ω is in rad/s.
 (a) True (b) False.
62. The torque required to produce a twist of one radian in unit length of a shaft, is known as torsional rigidity of the shaft.
 (a) True (b) False.
63. The polar modulus of a shaft is expressed by J/R where J = polar moment of inertia and R is the radius of shaft.
 (a) True (b) False.
64. The term $\pi d^3 / 16$ represents the polar moment of inertia of a solid shaft.
 (a) True (b) False.
65. The polar modulus of a hollow shaft of outer diameter (D) and inner diameter (d) is given by $\pi(D^3 - d^3) / 16$.
 (a) True (b) False.
66. The strength of a shaft will be 27 times than that of another shaft if both the shafts are made of the same material but the dia. of the first shaft should be three times the dia. of the second shaft.
 (a) True (b) False.
67. The polar moment of inertia of a hollow shaft is $\pi(D^4 - d^4) / 64$ where D = outer diameter and d = inner diameter.
 (a) True (b) False.

68. The stress is having the unit of N/m^2 .
(a) True (b) False.
69. The ratio of tensile stress to tensile strain upto elastic limit is known as Young's modulus.
(a) True (b) False.
70. The ratio of compressive stress to compressive strain upto elastic limit is known as bulk modulus.
(a) True (b) False.
71. The ratio of shear stress to shear strain upto elastic limit is known as modulus of rigidity.
(a) True (b) False.
72. The stress is directly proportional to the corresponding strain within elastic limit. This is known as Hook's law.
(a) True (b) False.
73. The ratio of longitudinal strain of lateral strain is known as Poisson's ratio.
(a) True (b) False.
74. If P is tensile force acting on a body of area of cross-section A , length L and of Young's modulus E , then change in length is equal to $P \times L/AE$.
(a) True (b) False.
75. The strain energy stored in a body, when load is applied gradually is equal to $f^2 \times V/2E$ where f = stress in the body, V = volume of the body and E = Young's modulus.
(a) True (b) False.
76. The stress produced in a body due to suddenly applied load is one-half of the stress produced when the same load is applied gradually.
(a) True (b) False.
77. The B.M. is maximum when shear force along a section is zero.
(a) True (b) False.
78. The intensity of stress which causes unit strain is called modulus of elasticity.
(a) True (b) False.
79. Elasticity is the property of a material by virtue of which a body does not return to its original shape after removal of the load.
(a) True (b) False.
80. Ductility is the property of a material which allows it to be drawn into smaller section.
(a) True (b) False.
81. The unit of stress and Young's modulus are same.
(a) True (b) False.
82. The strain at right angles to longitudinal strain is known as lateral strain.
(a) True (b) False.
83. The Poisson's ratio for most of materials varies from 3 to 4.
(a) True (b) False.
84. The length of a bar will decrease but width and thickness will increase if the bar is subjected to an axial push.
(a) True (b) False.

85. The strength of a shaft is judged by the torque transmitted by it.
(a) True (b) False.
86. Fatigue testing machine is used to determine the impact test.
(a) True (b) False.
87. The ratio of ultimate stress to working stress is known as factor of safety.
(a) True (b) False.
88. In case of cast iron, the percentage reduction in area under tensile test, is approximately 12%.
(a) True (b) False.
89. The tensile test is performed generally on ductile materials.
(a) True (b) False.
90. The compression test is performed generally on brittle materials.
(a) True (b) False.
91. Cantilever is a beam which is fixed at one end and free at the other.
(a) True (b) False.
92. In case of simply supported beams, the bending moment is maximum at the supports.
(a) True (b) False.
93. A simply supported beam of span L carries a point load W at the centre of the beam. The maximum B.M. will be equal to $W \times L/2$.
(a) True (b) False.
94. In question 93, the B.M. diagram will be rectangle.
(a) True (b) False.
95. In question 93, the shear force diagram will be an isosceles triangle.
(a) True (b) False.
96. In question 93, the maximum bending moment will be at the centre.
(a) True (b) False.
97. In question 93, the shear force will change sign at the centre.
(a) True (b) False.
98. The point where shear force changes sign, is known as point of contraflexure.
(a) True (b) False.
99. The point of contraflexure lies at the centre of a simply supported beam carrying a uniformly distributed load.
(a) True (b) False.

(F) Fill in the Blanks

100. In beams, the point of contraflexure occurs.
(a) fixed (b) overhanging.
101. The shear force is at the free end of a cantilever beam.
(a) maximum (b) zero.
102. A beam of uniform strength has same at every section.
(a) bending moment (b) bending stress.

103. The bending stress in the beam will if the section modulus of the beam is increased.
(a) decrease (b) increase.
104. A neutral axis of a beam is subjected to stress.
(a) maximum (b) zero.
105. For eccentrically loaded struts members are preferred.
(a) solid (b) hollow.
106. A square beam and a circular beam of the same areas of cross-sections are subjected to equal bending moments, then will be more economical.
(a) circular beam (b) square beam.
107. A beam is simply supported at both ends. The maximum deflection is equal to $PL^3/48EI$. This deflection is due to
(a) U.D.L. (b) central load P at the middle.
108. The section modulus is for a rectangular section.
(a) $\frac{bd^3}{12}$ (b) $\frac{bd^2}{6}$
where b = Width and d = Depth of the section.
109. The section modulus is for a hollow rectangular section.
(a) $\frac{BD^3 - bd^3}{6D}$ (b) $\frac{BD^2 - bd^2}{6}$
where B, D = Outer width and depth, b, d = Inner width and depth.
110. The section modulus is for a circular section.
(a) $\frac{\pi d^3}{16}$ (b) $\frac{\pi d^3}{32}$
where d = Dia. of circular section.
111. The bending equation gives the relation between and moment of inertia.
(a) bending moment (b) shear force.
112. The natural axis of a symmetrical circular section lies at a distance of from the outermost layer.
(a) $\frac{2d}{3}$ (b) $\frac{d}{2}$
where d = Dia. of the section.
113. The shear stress distribution is across a rectangular section.
(a) linear (b) parabolic.
114. The maximum shear stress is for a rectangular section.
(a) at outermost layer (b) at the N.A.
115. The maximum shear stress for a rectangular section is given by, $q_{\max} = \dots q_{\text{av}}$.
(a) 2.0 (b) 1.5
where q_{av} = Average shear stress.
116. The shear stress is at the N.A. for a circular section.
(a) minimum (b) maximum.

117. The relation between maximum shear stress and average shear stress for a circular section is given by, $q_{\max} = \dots q_{\text{av}}$.
- (a) $\frac{3}{4}$ (b) $\frac{4}{3}$.
118. The shear stress is maximum at for a triangular section.
- (a) N.A. (b) a height of $h/2$.

ANSWERS

Unit 1

1. (b)	2. (b)	3. (b)	4. (c)	5. (b)	6. (c)
7. (c)	8. (a)	9. (c)	10. (b)	11. (a)	12. (c)
13. (c)	14. (b)	15. (c)	16. (c)	17. (d)	18. (c)
19. (c)	20. (d)	21. (c)	22. (c)	23. (c)	24. (c)
25. (c)	26. (b)	27. (b)	28. (c)	29. (c)	30. (b)
31. (c)	32. (b)	33. (c)	34. (b)	35. (c)	36. (b)
37. (c)	38. (b)	39. (c)	40. (b)	41. (b)	

Unit 2

1. (b)	2. (b)	3. (c)	4. (a)	5. (b)	6. (b)
7. (d)	8. (c)	9. (a)	10. (d)	11. (b)	12. (c)
13. (d)	14. (b)	15. (d)	16. (c)	17. (d)	18. (c)
19. (d)	20. (a)	21. (b)	22. (d)	23. (a)	24. (d)
25. (b)	26. (d)	27. (d)	28. (a)	29. (b)	30. (d)
31. (d)	32. (b)	33. (a)	34. (c)	35. (c)	36. (c)

Unit 3

1. (b)	2. (b)	3. (a)	4. (c)	5. (b)	6. (c)
7. (b)	8. (b)	9. (c)	10. (a)	11. (b)	12. (c)
13. (c)	14. (b)	15. (c)	16. (d)	17. (a)	18. (b)
19. (c)	20. (a)	21. (b)	22. (b)	23. (c)	24. (b)
25. (c)	26. (c)	27. (d)	28. (c)	29. (b)	30. (c)
31. (b)	32. (d)	33. (c)	34. (c)	35. (c)	

Unit 4

1. (b)	2. (a)	3. (c)	4. (b)	5. (a)	6. (c)
7. (c)	8. (a)	9. (b)	10. (b)	11. (b)	12. (c)
13. (b)	14. (b)	15. (a)	16. (c)	17. (c)	18. (b)
19. (c)	20. (b)	21. (a)	22. (a)	23. (a)	24. (b)
25. (c)	26. (b)	27. (b)	28. (c)	29. (a)	30. (b)
31. (c)	32. (b)	33. (c)	34. (b)	35. (b)	36. (b)
37. (b)	38. (c)	39. (c)	40. (d)	41. (c)	42. (b)

43. (c)	44. (a)	45. (b)	46. (c)	47. (a)	48. (c)
49. (a)	50. (d)				

Unit 5

1. (b)	2. (d)	3. (a)	4. (c)	5. (b)	6. (a)
7. (c)	8. (c)	9. (b)	10. (a)	11. (a)	12. (a)
13. (b)	14. (c)	15. (d)	16. (b)	17. (a)	18. (c)
19. (b)	20. (b)	21. (a)	22. (c)	23. (c)	24. (a)
25. (b)	26. (b)	27. (c)	28. (b)	29. (a)	30. (c)
31. (d)	32. (a)	33. (b)	34. (c)	35. (b)	36. (a)
37. (b)	38. (c)	39. (c)	40. (b)	41. (b)	42. (b)
43. (d)	44. (c)	45. (a)	46. (d)	47. (a)	48. (b)
49. (a)	50. (d)	51. (c)	52. (c)	53. (d)	54. (d)
55. (c)	56. (c)	57. (b)	58. (a)	59. (c)	60. (a)
61. (a)	62. (a)	63. (a)	64. (b)	65. (a)	66. (a)
67. (b)	68. (a)	69. (a)	70. (b)	71. (a)	72. (a)
73. (b)	74. (a)	75. (a)	76. (b)	77. (a)	78. (a)
79. (b)	80. (a)	81. (a)	82. (b)	83. (b)	84. (a)
85. (a)	86. (b)	87. (a)	88. (b)	89. (a)	90. (a)
91. (a)	92. (b)	93. (b)	94. (b)	95. (b)	96. (a)
97. (a)	98. (b)	99. (b)	100. (b)	101. (b)	102. (b)
103. (a)	104. (b)	105. (b)	106. (b)	107. (b)	108. (b)
109. (a)	110. (b)	111. (a)	112. (b)	113. (b)	114. (b)
115. (b)	116. (b)	117. (b)	118. (b)		

EXAMINATION PAPER WITH SOLUTIONS

EXAMINATION PAPER

B. TECH.

(SEM. II) EXAMINATION, 2008–09

ENGG. MECHANICS

Time : 3 Hours

Total Marks : 100

Note : (1) This paper is in **three** sections. Section A carries 20 marks, Section B carries 30 marks and Section C carries **50** marks.

(2) Attempt **all** questions. Marks are indicated against each question/part.

(3) Assume missing data suitably, if any.

SECTION-A

1. You are required to answer all the parts:

2 × 10 = 20

Choose correct answer for the following parts:

(a) In order to determine the effects of a force acting on a body, we must know:

- (i) its magnitude
- (ii) its direction
- (iii) position or line of action
- (iv) all of these

(b) D'Alembert's principle is used for:

- (i) determining the stresses in truss
- (ii) reducing kinetics problem to equivalent statics problem
- (iii) stability of floating bodies
- (iv) solving kinematics problem

Fill in the blanks for the following three parts:

You will be awarded full marks, if all the entries in a part are correct otherwise will be awarded zero.

(c) A truss is said to be rigid in nature when there is no on application of any external

(d) The centre of gravity of a semicircle lies at a distance of whereas centre of gravity of a hemisphere lie at a distance of , both from its base measured along vertical axis.

(e) In a tensile test on mild steel specimen, the yield stress is than the ultimate stress and the breaking stress is than the ultimate stress if all the stresses are calculated on the basis of initial cross-sectional area of the specimen.

Match the columns for the following three parts:

You will be awarded full marks, if all the matches in a part are correct otherwise will be awarded zero.

(f) Match the following columns:

Column-I**Column-II**

(i) Coplanar forces	(P) Line of action of all forces lie in the same plane and pass through a common point
(ii) Concurrent forces	(Q) Line of action of all forces lie in the same plane
(iii) Concurrent coplanar forces	(R) Line of action of all forces lie along the same line
(iv) Collinear forces	(S) Line of action of all forces pass through a common point

(g) Column-II gives the mass moment of inertia of various solids about the central axis. Match the following columns:

Column-I**Column-II**

(i) Cylinder	(P) $3mr^2/2$
(ii) Sphere	(Q) $3mr^2/10$
(iii) Cone	(R) $2mr^2/5$
(iv) Thin circular disk	(S) $mr^2/2$

(h) Column-II gives maximum bending moments for the following loads. Match the following columns:

Column-I**Column-II**

(i) Cantilever with point load at free end	(P) $Wl/4$
(ii) Cantilever with uniformly distributed load on the entire cantilever	(Q) $Wl^2/8$
(iii) Simple supported beam with point load at mid span	(R) Wl
(iv) Simple supported beam with uniformly distributed load on the entire beam	(S) $Wl^2/2$

Choose the correct answer for the following **two** parts:

(i) Consider the following statements:

- (I) The magnitude of the moment does not change if the point of application of the force is transmitted along its line of action.
- (II) The magnitude of the moment does not change if the moment centre is moved along a line parallel to the line of action of the force:
- I alone is correct
 - I and II are correct
 - II alone is correct
 - Neither I nor II correct.

(j) Consider the following statements:

- (I) In truss analysis the bars are connected at their ends by frictional hinges.
- (II) In truss analysis the bars are assumed to be weightless.

- (i) *I* alone is correct
- (ii) *I* and *II* are correct
- (iii) *II* alone is correct
- (iv) Neither *I* nor *II* is correct

SECTION-B

2. Answer any **three** parts of the following:

10 × 3 = 30

- (a) A roller shown in Figure 1 is of mass 150 kg. What force P is necessary to start the roller over the block A ?

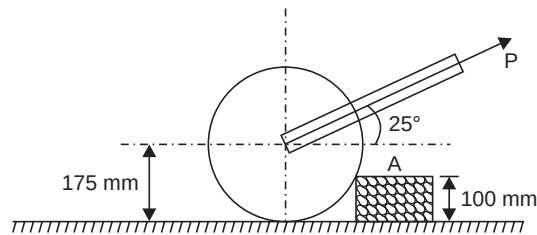


Figure 1

- (b) Calculate the values of shear force and bending moments for the cantilever beam shown in Figure 2. Also draw the shear force and bending moment diagrams.

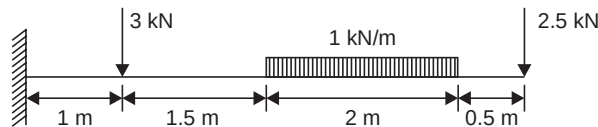


Figure 2

- (c) Find the moment of inertia of the area shown shaded in Figure 3 about edge AB.

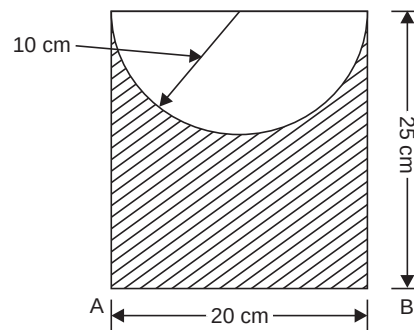


Figure 3

- (d) Two bodies A and B are connected by a thread and move along a rough horizontal plane ($\mu = 0.3$) under the action of a force 400 N applied to the body B as shown in Figure 4.

Determine the acceleration of the two bodies and the tension in the thread, using D'Alembert's principle.

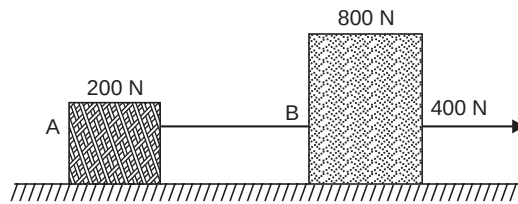


Figure 4

- (e) A member is subjected to point loads P_1 , P_2 , P_3 and P_4 as shown in Figure 5. Calculate the force P_3 necessary for equilibrium if $P_1 = 120 \text{ kN}$, $P_2 = 220 \text{ kN}$ and $P_4 = 160 \text{ kN}$. Determine also the change in length of the member. Take $E = 2 \times 10^5 \text{ N/mm}^2$.

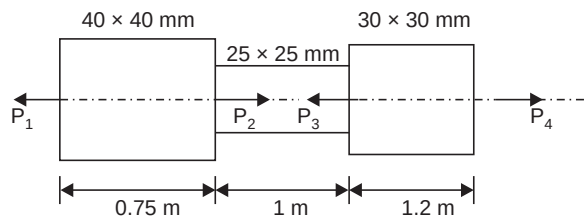


Figure 5

SECTION-C

3. Answer any **two** parts of the following:

5 × 2 = 10

- (a) Explain the following:
 (i) Laws of static friction
 (ii) Limiting angle of friction.
- (b) What is a free body diagram? Explain with suitable example.
- (c) Two rollers C and D produce vertical forces P and Q on the horizontal beam AB , as shown in Figure 6. Determine the distance x of the load P from the support A if the reaction at A is twice as great as the reaction at B . The weight of the beam is to be neglected. Given : $P = 18 \text{ kN}$, $Q = 9 \text{ kN}$, $l = 3.6 \text{ m}$, $c = 0.9 \text{ m}$.

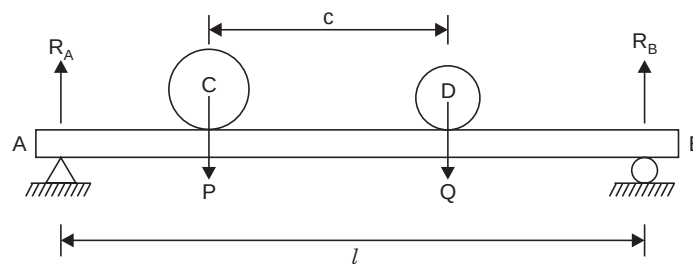


Figure 6

4. Answer any **one** of the following:

10

(a) Find the axial forces in all members of a truss as shown in Figure 7.

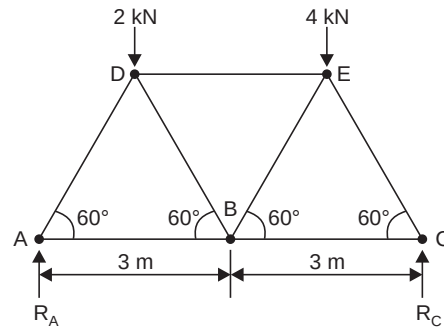


Figure 7

(b) Draw the shear force and bending moment diagram for the beam loaded as shown in Figure 8.

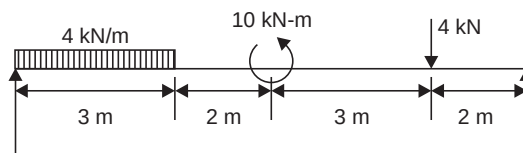


Figure 8

5. Answer any **two** parts of the following:

$5 \times 2 = 10$

(a) Explain any **two** of the following:

- (i) Parallel axis theorem
- (ii) Product of inertia of an area about its axis of symmetry
- (iii) Centre of gravity.

(b) Find the centroid of Figure 9.

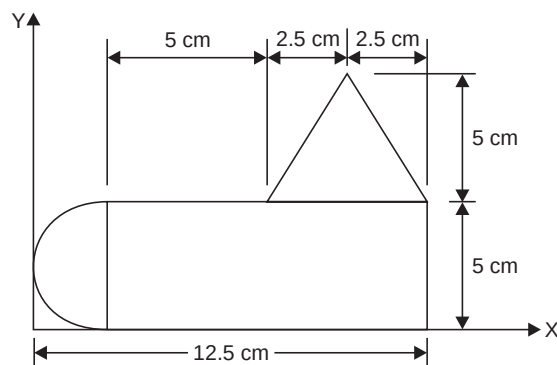


Figure 9

(c) Derive an expression of mass moment of inertia of a cylinder about the longitudinal axis.

- 6.** Answer any **two** parts of the following: **5 × 2 = 10**
- (a) What do you understand by the term kinematics? Explain different types of plane motion of rigid bodies with suitable example.
- (b) A wheel rotates for 5 seconds with constant angular acceleration and describes 100 radians during this time. It then rotates with a constant angular velocity and during the next 5 seconds describes 80 radians. Find the initial angular velocity and the angular acceleration.
- (c) What is energy? Explain the various forms of mechanical energies.
- 7.** Answer any **one** of the following: **10**
- (a) What do you understand by the term neutral axis and neutral surface? A steel beam of hollow square section of 60 mm outer side and 50 mm inner side is simply supported on a span of 4 metres. Find the maximum concentrated load the beam can carry at the middle of the span if the bending stress is not to exceed 120 N/mm².
- (b) State the assumptions made in the theory of pure torsion.
- In a tensile test, a test piece 25 mm in diameter, 200 mm gauge length stretched 0.0975 mm under a pull of 50,000 N. In a torsion test, the same rod twisted 0.025 radian over a length of 200 mm, when a torque of 400 Nm was applied. Evaluate the Poisson's ratio and the three elastic moduli for the material.

SOLUTIONS TO EXAMINATION PAPER

SECTION-A

- 1.** (a) (iv) (b) (ii) (c) Displacement, load
- (d) $\frac{3r}{4\pi}$ (e) more, less
- (f) (i) Q (ii) S (iii) P (iv) R
- (g) (i) S (ii) R (iii) Q (iv) P
- (h) (i) R (ii) S (iii) P (iv) Q
- (i) (ii) (j) (ii)

SECTION-B

- 2.** (a) Refer to solved problem 4.18 on page 93.
- (b) Solved problem 8.6 on page 231.
- (c) Solved problem 9.16 on page 308.
- (d) Solved problem 11.5 on page 403.
- (e) Exercise, Q. No. 8 on page 496. It is solved there.

SECTION-C

- 3.** (a) (i) Page 130 Art. 6.5.1 (ii) Page 129 Art. 6.3
- (b) Page 79 Art. 4.3
- (c) Solved problem 5.11 on page 121

4. (a) Taking moments about A, ($\Sigma M_A = 0$)

$$R_C \times 6 = 2 \times 1.5 + 4 \times 4.5 = 3 + 18 = 21 \text{ kN}$$

$$\therefore R_C = \frac{21}{6} = 3.5 \text{ kN and } R_A = (2 + 4) - R_C = 6 - 3.5 = 2.5 \text{ kN.}$$

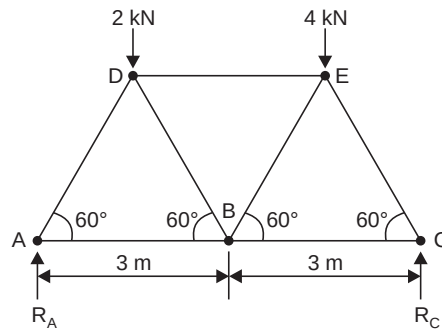


Figure 10

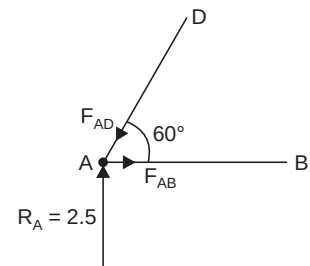
Joint A

$$\Sigma F_y = 0, F_{AD} \sin 60^\circ = 2.5$$

$$\therefore F_{AD} = \frac{2.5}{\sin 60^\circ} = 2.887 \text{ kN (comp.) Ans.}$$

$$\Sigma F_x = 0, F_{AD} \cos 60^\circ = F_{AB}$$

$$\therefore F_{AB} = 2.887 \times 0.5 = 1.443 \text{ kN (tensile) Ans.}$$



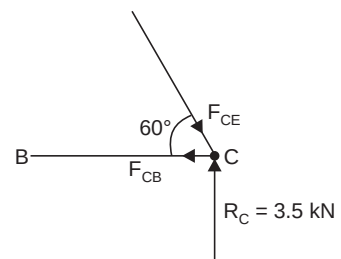
Joint C

$$\Sigma F_y = 0, F_{CE} \sin 60^\circ = 3.5,$$

$$\therefore F_{CE} = \frac{3.5}{\sin 60^\circ} = 4.041 \text{ kN (comp.) Ans.}$$

$$\Sigma F_x = 0, F_{CB} = F_{CE} \cos 60^\circ$$

$$= 4.041 \times 0.5 = 2.0205 \text{ kN (tensile) Ans.}$$



Joint D

$$F_{DA} = F_{AD} = 2.887 \text{ kN (already calculated)}$$

$$\Sigma F_x = 0, F_{DB} \cos 60^\circ + F_{DA} \cos 60^\circ = F_{DE} \quad \dots(i)$$

$$\Sigma F_y = 0, F_{DA} \sin 60^\circ - F_{DB} \sin 60^\circ - 2 = 0 \quad \dots(ii)$$

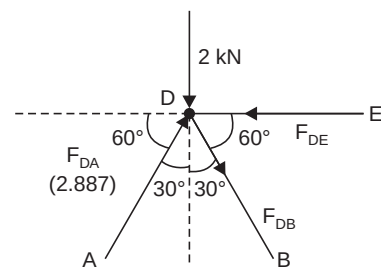
$$\therefore F_{DB} = \frac{F_{DA} \sin 60^\circ - 2}{\sin 60^\circ} = \frac{2.887 \times 0.866 - 2}{0.866}$$

$$= 5.77 \text{ kN (tensile) Ans.}$$

From equation (i),

$$5.77 \cos 60^\circ + 2.887 \cos 60^\circ = F_{DE}$$

$$\text{or } F_{DE} = 5.77 \times 0.5 + 2.887 \times 0.5 = 1.732 \text{ kN (comp.) Ans.}$$



Joint E

$$F_{ED} = F_{DE} = 1.732 \text{ kN (Already known)}$$

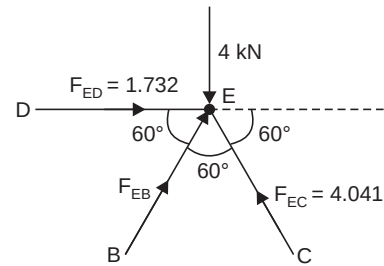
$$F_{EC} = F_{CE} = 4.041 \text{ kN (Already calculated)}$$

$$F_x = 0, F_{ED} + F_{EB} \cos 60^\circ - F_{EC} \cos 60^\circ = 0$$

$$\text{or } 1.732 + F_{EB} \times 0.5 - 4.041 \times 0.5 = 0$$

$$\text{or } F_{EB} = \frac{4.041 \times 0.5 - 1.732}{0.5}$$

$$= 5.77 \text{ kN (comp.) Ans.}$$



The forces in all the members have been calculated. There is no need to consider joint B.

$$4. (b) \quad \Sigma M_A = 0, R_B \times 10 = (4 \times 3) \times 1.5 - 10 + 4 \times (3 + 3 + 2)$$

$$= 18 - 10 + 32 = 40 \text{ kN}$$

$$\therefore R_B = \frac{40}{10} = 4 \text{ kN and } R_A = (4 \times 3) + 4 - 4 = 12 \text{ kN}$$

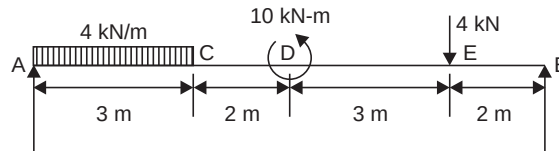


Figure 11

S.F. Diagram [Fig. 12 (A)]

(i) At A, the S.F. is equal to $R_A = 12 \text{ kN}$ vertically upwards.

(ii) At a distance x from A, between AC, S.F. is given as,

$$\text{S.F.} = R_A - x \times 4$$

$$= 12 - 4x \text{ (Linear variation)}$$

(iii) At C, where $x = 3 \text{ m}$,

$$\text{S.F.} = 12 - 4 \times 3 = 12 - 12 = 0$$

(iv) S.F. is constant between C and E and is equal to zero.

(v) At E, S.F. is -4 kN (i.e., vertically downwards)

(vi) At B, S.F. is equal to $R_B = 4 \text{ kN}$.

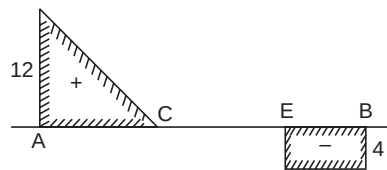


Figure 12(A) S.F. diagram.

B.M. Diagram [Fig. 12 (B)]

B.M. is zero at A and B. Hence $M_A = 0$ and $M_B = 0$

Portion AC (x varies from 0 to 3)

B.M. at a distance x from A,

$$M_x = R_A \times x - (x \times 4) \times \frac{x}{2}$$

$$= 12x - 2x^2 \text{ (Parabolic variation)}$$

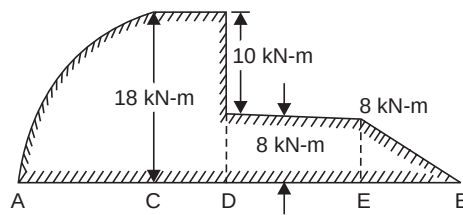


Figure 12(B) B.M. diagram.

B.M. at C where $x = 3$, is given as

$$M_C = 12 \times 3 - 2 \times 3^2 = 36 - 18 = 18 \text{ kN-m}$$

Portion CD (x varies from 3 to 5)

B.M. at a distance x from A ,

$$\begin{aligned} M_x &= R_A \times x - (4 \times 3)(x - 1.5) \\ &= 12x - 12(x - 1.5) \end{aligned}$$

Just left to D , B.M. is given as ($x = 5$)

$$\begin{aligned} M_D &= 12 \times 5 - 12(5 - 1.5) \\ &= 60 - 12 \times 3.5 = 60 - 42 = 18 \text{ kN-m} \end{aligned}$$

B.M. is constant between CD and equal to 18 kN-m (clockwise)

Just right to D , the B.M. drops by 10 kN-m and hence it is equal to $18 - 10 = 8$ kN-m.

Portion BE

B.M. at B is zero. B.M. at E , $M_E = R_B \times 2 = 4 \times 2 = 8$ kN-m

B.M. between DE is equal to 8 kN-m.

5. (a) (i) Page 295 Art. 9.10.
(ii) Page 311 Art. 9.13.
(iii) Page 269 Art. 9.1.
- (b) Art. 9.2 on page 269.
- (c) Page 316 Art. 9.15.3. Consider circular plate of thickness ' t ' as circular cylinder of length L . Hence, instead of ' t ' the length of cylinder L should be taken in the derivation. Then

$$I_{zz} = \frac{MR^2}{2}, \text{ where } M = \text{Mass of cylinder} = \rho \times \pi R^2 \times L$$

6. (a) Page 329 Art. 10.1.
(b) Solved problem 10.17 on page 352.
(c) Page 448 Art. 11.9.6 and 11.9.7.
7. (a) Solved problem 15.7 on page 540.
(b) (i) **Assumptions** in the theory of pure torsion. Page 546 Art. 16.2.1.
(ii) **Numerical Part.** Solved problem 16.9 on page 555.

ABOUT THE BOOK

The course contents of the revised eighth edition of the book entitled, '**A Textbook of Engineering Mechanics**' are planned in such a way that the book covers the complete course of first year students of all disciplines of Gautam Buddha Technical University, Lucknow according to the latest syllabus. This edition has been thoroughly revised and made up-to-date. Also the solved papers of Gautam Buddha Technical University upto May 2009 have been added at proper places. These additions makes the book more useful to the students.

The book is written in a simple and easy-to-follow languages, so that even an average student can grasp the subject by self-study. At the end of each chapter, highlights, theoretical questions and many unsolved problems with answers are given for the students to solve them.

ABOUT THE AUTHOR

Dr. R. K. Bansal graduated in 1966. He obtained his Master's Degree in 1975 with honours from I.I.T., Delhi and Ph.D. in 1981 from University of Delhi. He joined Delhi College of Engineering as a Lecturer and became Assistant Professor in 1984. He was selected for the post of Professor in 1995. He also held the posts of Head of Mech. Engineering Department (Faculty of Technology, University of Delhi), Dean (P.G.) Studies and Dean (U.G.) Studies, Delhi College of Engineering. During his teaching career of about thirty-five years, he guided a large number of research students and got published many papers. He is the author of many books which are followed as textbooks.



LAXMI PUBLICATIONS (P) LTD

ISBN 978-81-318-0855-9



9 788131 808559